Topological homogeneity and infinite powers

Andrea Medini

Kurt Gödel Research Center
University of Vienna

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Homogeneity an ubiquitous

notion in mathematics is...

Topological homogeneity

Let $\mathcal{H}(X)$ denote the group of homeomorphisms of X.
A space is homogeneous if all points "look alike":

Definition

A topological space X is *homogeneous* if for every $x, y \in X$ there exists $h \in \mathcal{H}(X)$ such that $h(x) = y$.

Non-examples:

- $[0, 1]^n$ whenever $1 \leq n < \omega$. (Points on the boundary are different from points in the interior.)
- The Stone-Čech compactification $\beta\omega$ of the natural numbers. (Trivial: the points of ω are isolated!)
- The Stone-Čech remainder $\omega^* = \beta\omega \setminus \omega$.
  (Frolík, 1967, using a cardinality argument.)
  (Kunen, 1978, by proving the existence of weak P-points.)

Examples:

- $\mathbb{R}^\kappa$, 2^κ, ω^κ, $\mathbb{Q}^\kappa$. (Actually, any topological group: just translate!)
- Any product of homogeneous spaces. (Just take the product homeomorphism.)
- The Hilbert cube $[0,1]^\omega$ (Keller, 1931).
  Notice that $[0,1]^\omega$ is not a topological group because it has the fixed point property (by Brouwer's theorem).

Homogeneous spaces are decently understood.
Compact homogeneous spaces are shrouded in mystery:

Question (Van Douwen, 1970s)

Is there a compact homogeneous space with more than $\mathfrak{c}$ pairwise disjoint non-empty open sets?

Question (W. Rudin, 1958)

Is there a compact homogeneous space with no non-trivial convergent ω-sequences?

h-Homogeneity

Definition

A space X is *h-homogeneous* if every non-empty clopen subset of X is homeomorphic to X.

Examples:

- Any connected space.
- $\mathbb{Q}$, 2^ω, ω^ω. (Use their characterizations.)
- Any product of zero-dimensional h-homogeneous spaces (Medini, 2011, building on work of Terada, 1993).

Non-zero-dimensional but “very” disconnected examples:

- Erdős space $\mathfrak{E} = \{x \in \ell^2 : x_n \in \mathbb{Q} \text{ for all } n \in \omega\}$ (Deep result due to Dijkstra and van Mill, 2010).
- The Knaster-Kuratowski fan.

Homogeneity vs. h-homogeneity

To see that h-homogeneity does not imply homogeneity, not even for zero-dimensional spaces, consider ω^* or $(\omega_1 + 1)^\omega$.

Theorem (folklore)

Let X be a first-countable zero-dimensional space. If X is h-homogeneous then X is homogeneous.

The following remarkable example of van Douwen shows that the reverse implication need not hold.

Theorem (van Douwen, 1984)

There exists a first-countable zero-dimensional compact homogeneous space that is not h-homogeneous.

Actually, his space has “a measure that knows which sets are homeomorphic”.

Countable dense homogeneity

Definition (Bennett, 1972)

A space X is *countable dense homogeneous* (briefly, CDH) if for every pair (D, E) of countable dense subsets of X there exists $h \in \mathcal{H}(X)$ such that $h[D] = E$.

Examples:

- $\mathbb{R}$ (Cantor, 1895), $\mathbb{R}^n$ (Brouwer, 1913), 2^ω, ω^ω.
- Any separable euclidean manifold of weight less than $\mathfrak{b}$ (Steprāns and Zhou, 1988).
- The Hilbert cube $[0, 1]^\omega$ (Fort, 1962).
- Any countable disjoint sum of CDH spaces.

Non-examples:

- $\mathbb{Q}$ (Trivial! Less trivially, it actually has $\mathfrak{c}$ types of countable dense subsets...)
- $\mathbb{Q}^\omega$ (Fitzpatrick and Zhou, 1992).

Homogeneity vs. countable dense homogeneity

To see that countable dense homogeneity does not imply homogeneity, consider $S^1 \oplus S^2$.

Theorem (Fitzpatrick and Lauer, 1987)

Let X be a connected CDH *space. Then X is homogeneous.*

For an example of a homogeneous space that is not CDH, consider $\mathbb{Q}$ or any non-P-point ultrafilter $\mathcal{U} \subseteq 2^\omega$. However, adding a completeness assumption fixes the problem:

Theorem (Curtis, Anderson and van Mill, 1985)

Let X be Polish and strongly locally homogeneous. Then X is CDH*.*

Corollary

Let $X \subseteq 2^\omega$ be Polish and homogeneous. Then X is CDH*.*

Non-Polish CDH spaces

Question (Fitzpatrick and Zhou, 1990)

Is there a non-Polish CDH *space?*

Theorem (Baldwin and Beaudoin, 1989)

Assume MA(σ-centered). *Then there exists a* CDH *Bernstein set* $X \subseteq 2^\omega$.

Theorem (Farah, Hrušák and Martínez Ranero, 2005)

There exists a λ*-set* $X \subseteq 2^\omega$ *of size* ω_1 *that is* CDH.

In 2014, Hernández-Gutiérrez, Hrušák and van Mill gave a new, simpler proof of the above result, using the technique of Knaster-Reichbach covers. In 2015, Medvedev further developed their methods. Our main theorem (see a later slide!) relies heavily on their work.

Infinite powers, part I: homogeneity

It is an interesting theme in general topology that taking infinite powers improves the homogeneity properties of a space. Remember the Hilbert cube $[0,1]^\omega$! But this phenomenon is particulary striking for zero-dimensional spaces. For example, if $X \subseteq 2^\omega$ is a Polish space (equivalently, a G_δ), then X^ω will be homogeneous, CDH, and h-homogeneous because

- $X^\omega \approx 2^\omega$ if X is compact and $|X| \geq 2$,
- $X^\omega \approx \omega^\omega$ if X is not compact.

Question (Fitzpatrick and Zhou, 1990)

Which $X \subseteq 2^\omega$ are such that X^ω is homogeneous? Countable dense homogeneous?

Theorem (Lawrence, 1998)

Let $X \subseteq 2^\omega$. Then X^ω is homogeneous.

The Dow-Pearl theorem

Question (Gruenhage, 1990)

Is X^ω homogeneous for all zero-dimensional first-countable X?

By combining the methods of Lawrence with the technique of elementary submodels, it is possible to give an affirmative answer to Gruenhage's question.

Theorem (Dow and Pearl, 1997)

Let X be first-countable and zero-dimensional. Then X^ω is homogeneous.

The following result is an interesting application of the Dow-Pearl theorem. Since a compact S-space cannot be a topological group, it is in a sense best possible.

Theorem (De la Vega and Kunen, 2004)

Under CH*, there exists a compact homogeneous* S*-space.*

Infinite powers, part II: h-homogeneity

Question (Terada, 1993)

Is X^ω h-homogeneous for every zero-dimensional first-countable X?

A positive answer would give a strengthening of the Dow-Pearl theorem. The above question is open, even for separable metrizable spaces! However, partial results are available:

Theorem (van Engelen, 1992; Medvedev, 2012)

Let X be a metrizable space such that $\dim(X) = 0$. If X is meager or X has a completely metrizable dense subspace then X^ω is h-homogeneous.

Theorem (Medini, 2011)

Let X be a non-separable metrizable space such that $\dim(X) = 0$. Then X^ω is h-homogeneous.

Infinite powers, part III: CDH spaces

We are looking for a quotable property $\mathcal{P}$ such that the following are equivalent for every $X \subseteq 2^\omega$.

- X^ω is CDH.
- X has property $\mathcal{P}$.

The first breakthrough was the following.

Theorem (Hrušák and Zamora Avilés, 2005)

*For a **Borel** $X \subseteq 2^\omega$, the following conditions are equivalent.*

- X^ω *is* CDH*.*
- X *is Polish.*

It is natural to ask: is $\mathcal{P}$ $=$ Polish the characterization that we're looking for? In other words:

Question (Hrušák and Zamora Avilés, 2005)

Is there a non-G_δ subset X of 2^ω such that X^ω is CDH*?*

A consistent answer

Theorem (Medini and Milovich, 2012)

Assume MA(countable). *Then there exists an ultrafilter* $\mathcal{U}$ *such that* $\mathcal{U}^\omega$ *is* CDH.

Theorem (Hernández-Gutiérrez and Hrušák, 2013)

Let $\mathcal{F}$ *be a non-meager* P*-filter. Then* $\mathcal{F}$ *and* $\mathcal{F}^\omega$ *are both* CDH.

Theorem (Kunen, Medini and Zdomskyy, 2015)

Let $\mathcal{F}$ *filter. Then the following are equivalent.*

- $\mathcal{F}$ *is a non-meager* P*-filter.*
- $\mathcal{F}$ *is* CDH.
- $\mathcal{F}^\omega$ *is* CDH.

Great! ☺ Do non-meager P-filters exist?

It's a long-standing open problem... ☹

A ZFC answer

It turns out that the result of Hrušák and Zamora Avilés can be slightly improved by weakening “Borel” to “coanalytic”:

Theorem (Medini, 2015)

*For a **coanalytic** $X \subseteq 2^\omega$, the following conditions are equivalent.*

- X^ω *is* CDH.
- *X is Polish.*

More importantly, the improved version is sharp:

Theorem (Medini, 2015)

There exists $X \subseteq 2^\omega$ with the following properties.

- X^ω *is* CDH.
- *X is not Polish.*
- *If* $\mathrm{MA} + \neg\mathrm{CH} + \omega_1 = \omega_1^{\mathrm{L}}$ *holds then X is analytic.*

Two plausible candidates for the property $\mathcal{P}$

From results of Hrušák and Zamora Avilés, it follows that

$$\text{Polish} \to \mathcal{P} \to \text{Baire}$$

The following three properties (in strictly descreasing order of strength) satisfy this requirement.

1. $\mathcal{P} =$ Miller property (every countable crowded subspace has a crowded subspace with compact closure).
2. $\mathcal{P} =$ Cantor-Bendixson property (every closed subspace is either scattered or contains a copy of 2^ω).
3. $\mathcal{P} =$ completely Baire (every closed subspace is Baire).

Furthermore, they are all equivalent for filters or coanalytic sets. However, property (3) must be discarded by the following result.

Theorem (Hernández-Gutiérrez, 2013)

If $X \subseteq 2^\omega$ is a Bernstein set then X^ω is not CDH.

How do we construct the ZFC example?

Recall that a λ-*set* is a space where all countable sets are G_δ. Since a λ-set cannot contain copies of 2^ω, no uncountable λ-set can be Polish (or even Borel).
Recall that a λ'-*set* is a subspace X of 2^ω such that $X \cup D$ is a λ-set for every countable $D \subseteq 2^\omega$.

Theorem (Sierpiński, 1945)

There exists a λ'-set of size ω_1.

Our example will be the **complement** of a λ'-set of size ω_1. In particular, it will be consistently analytic by the following classical theorem.

Theorem (Martin and Solovay, 1970)

Assume $\mathrm{MA} + \neg\mathrm{CH} + \omega_1 = \omega_1^{\mathrm{L}}$. Then every subspace of 2^ω of size ω_1 is coanalytic.

The main theorem

Let Y be a λ'-set and $X = 2^\omega \setminus Y$. Notice that every countable subset of X is included in a Polish subspace of X (we will say that X is *countably controlled*).

We will say that $X \subseteq 2^\omega$ is *h-homogeneously embedded* in 2^ω if there exists a (countable) π-base $\mathcal{B}$ for 2^ω consisting of clopen sets and homeomorphisms $\varphi_U : 2^\omega \longrightarrow U$ for $U \in \mathcal{B}$ such that $\varphi_U[X] = X \cap U$.

Theorem (Medini, 2015)

Assume that X is countably controlled and h-homogeneously embedded in 2^ω. Then X is CDH*.*

Notice that both properties are preserved by taking the ω-power, so it will be enough to construct a λ'-set of size ω_1 that is h-homogeneously embedded in 2^ω. This is easy enough, but it turns out that **any** λ'-set would work!

A flashback from a few slides ago...

Theorem (van Engelen, 1992; Medvedev, 2012)

Assume that $X \subseteq 2^\omega$ has a dense Polish subspace. Then X^ω is h-homogeneous.

Proposition (Medini, 2015)

Let $X \subseteq 2^\omega$ and $|X| \geq 2$. Then the following are equivalent.

- *X^ω is h-homogeneous.*
- *X^ω can be h-homogeneously embedded in 2^ω.*

Corollary (Medini, 2015)

Assume that $X \subseteq 2^\omega$ has a dense Polish subspace and $|X| \geq 2$. Then X^ω can be h-homogeneously embedded in 2^ω.

Knaster-Reichbach covers

Fix a homeomorphism $h : E \longrightarrow F$ between closed nowhere dense subsets of 2^ω. We will say that $\langle \mathcal{V}, \mathcal{W}, \psi \rangle$ is a *Knaster-Reichbach cover* (briefly, a KR-cover) for $\langle 2^\omega \setminus E, 2^\omega \setminus F, h \rangle$ if the following conditions hold.

- $\mathcal{V}$ is a cover of $2^\omega \setminus E$ consisting of pairwise disjoint non-empty clopen subsets of 2^ω.
- $\mathcal{W}$ is a cover of $2^\omega \setminus F$ consisting of pairwise disjoint non-empty clopen subsets of 2^ω.
- $\psi : \mathcal{V} \longrightarrow \mathcal{W}$ is a bijection.
- If $f : 2^\omega \longrightarrow 2^\omega$ is a bijection such that $h \subseteq f$ and $f[V] = \psi(V)$ for every $V \in \mathcal{V}$ (we say that f *respects* ψ), then f is continuous on E and f^{-1} is continuous on F.

Lemma

Let $h : E \longrightarrow F$ be a homeomorphism between closed nowhere dense subsets of 2^ω. Then there exists a KR*-cover for $\langle 2^\omega \setminus E, 2^\omega \setminus F, h \rangle$.*

Knaster-Reichbach systems

Fix an admissible metric on 2^ω. We will say that a sequence $\langle\langle h_n, \mathcal{K}_n\rangle : n \in \omega\rangle$ is a *Knaster-Reichbach system* (briefly, a KR-system) if the following conditions are satisfied.

(1) Each $h_n : E_n \longrightarrow F_n$ is a homeomorphism between closed nowhere dense subsets of 2^ω.

(2) $h_m \subseteq h_n$ whenever $m \leq n$.

(3) Each $\mathcal{K}_n = \langle \mathcal{V}_n, \mathcal{W}_n, \psi_n\rangle$ is a KR-cover for $\langle 2^\omega \setminus E_n, 2^\omega \setminus F_n, h_n\rangle$.

(4) $\mathrm{mesh}(\mathcal{V}_n) \leq 2^{-n}$ and $\mathrm{mesh}(\mathcal{W}_n) \leq 2^{-n}$ for each n.

(5) $\mathcal{V}_m$ refines $\mathcal{V}_n$ and $\mathcal{W}_m$ refines $\mathcal{W}_n$ whenever $m \geq n$.

(6) Given $U \in \mathcal{V}_m$ and $V \in \mathcal{V}_n$ with $m \geq n$, then $U \subseteq V$ if and only if $\psi_m(U) \subseteq \psi_n(V)$.

Why do we care about Knaster-Reichbach systems?

Theorem

Assume that $\langle\langle h_n, \mathcal{K}_n\rangle : n \in \omega\rangle$ *is a* KR*-system. Then there exists* $h \in \mathcal{H}(2^\omega)$ *such that* $h \supseteq \bigcup_{n\in\omega} h_n$*.*

Corollary

Let X *be a subspace of* 2^ω*. Assume that* $\langle\langle h_n, \mathcal{K}_n\rangle : n \in \omega\rangle$ *is a* KR*-system satisfying the following additional conditions.*

(7) $2^\omega \setminus \bigcup_{n\in\omega} E_n \subseteq X$.

(8) $2^\omega \setminus \bigcup_{n\in\omega} F_n \subseteq X$.

(9) $h_n[X \cap E_n] = X \cap F_n$ *for each* n.

Then there exists $h \in \mathcal{H}(2^\omega)$ *such that* $h \supseteq \bigcup_{n\in\omega} h_n$ *and* $h[X] = X$*.*

Proof of the main result: the setup, part I

Let $X \neq \varnothing$ be h-homogeneously embedded in 2^ω and countably controlled. We will show that X is CDH.

Fix a (countable) π-base $\mathcal{B}$ for 2^ω consisting of clopen sets and homeomorphisms $\varphi_U : 2^\omega \longrightarrow U$ for $U \in \mathcal{B}$ such that $\varphi_U[X] = X \cap U$. In particular, X is dense in 2^ω.

Fix a pair (A, B) of countable dense subsets of X. It is easy to construct a countable dense subset D of 2^ω such that

- $A \cup B \subseteq D \subseteq X$,
- $\varphi_U^{-1}(x) \in D$ whenever $x \in D \cap U$ for some $U \in \mathcal{B}$.

(Start with $A \cup B$, then repeatedly close-off under all the functions φ_U^{-1} in ω steps.)

Proof: the setup, part II

Since X is countably controlled, it is possible to find a G_δ subset G of 2^ω such that $D \subseteq G \subseteq X$. Without loss of generality, assume that $2^\omega \setminus G$ is dense in 2^ω. Fix closed nowhere dense subsets K_ℓ of 2^ω for $\ell \in \omega$ such that $2^\omega \setminus G = \bigcup_{\ell \in \omega} K_\ell$.

Fix the following injective enumerations.

- $A = \{a_i : i \in \omega\}$.
- $B = \{b_j : j \in \omega\}$.

Fix an admissible metric on 2^ω such that $\mathrm{diam}(2^\omega) \leq 1$.

Our strategy is to construct a suitable KR-system $\langle\langle h_n, \mathcal{K}_n\rangle : n \in \omega\rangle$, where each $h_n : E_n \longrightarrow F_n$ and $\mathcal{K}_n = \langle \mathcal{V}_n, \mathcal{W}_n, \psi_n\rangle$.

Proof: the setup, part III

Of course, we want to satisfy conditions (1)-(6) in the definition of a KR-system. But we will also make sure that the following conditions are satisfied for every $n \in \omega$.

(I) $\bigcup_{\ell<n} K_\ell \subseteq E_n$.

(II) $\bigcup_{\ell<n} K_\ell \subseteq F_n$.

(III) $h_n[X \cap E_n] = X \cap F_n$.

(IV) $\{a_i : i < n\} \subseteq E_n$.

(V) $\{b_j : j < n\} \subseteq F_n$.

(VI) $h_n[A \cap E_n] = B \cap F_n$.

Conditions (I)-(III) will guarantee that the additional conditions (7)-(9) in the Corollary hold.

On the other hand, conditions (IV)-(VI) will guarantee that $h[A] = B$.

Proof: the construction, part I

Start by letting $h_0 = \varnothing$ and $\mathcal{K}_0 = \langle\{2^\omega\}, \{2^\omega\}, \{\langle 2^\omega, 2^\omega\rangle\}\rangle$.

Now assume that $\langle h_n, \mathcal{K}_n\rangle$ is given. First, for any given $V \in \mathcal{V}_n$, we will define a homeomorphism $h_V : E_V \longrightarrow F_V$, where E_V will be a closed nowhere dense subset of V and F_V will be a closed nowhere dense subset of $\psi_n(V)$.

So fix $V \in \mathcal{V}_n$, and let $W = \psi_n(V)$. Define the following indices.

- $\ell(V) = \min\{\ell \in \omega : K_\ell \cap V \neq \varnothing\}$.
- $\ell(W) = \min\{\ell \in \omega : K_\ell \cap W \neq \varnothing\}$.
- $i(V) = \min\{i \in \omega : a_i \in V \setminus K_{\ell(V)}\}$.
- $j(W) = \min\{j \in \omega : b_j \in W \setminus K_{\ell(W)}\}$.

Notice that the indices $\ell(V)$ and $\ell(W)$ are well-defined because $\bigcup_{\ell\in\omega} K_\ell = 2^\omega \setminus G$ is dense in 2^ω.

Proof: the construction, part II

Let $S = V \cap K_{\ell(V)}$. Since $K_{\ell(V)}$ is a closed nowhere dense subset of 2^ω, we can fix $U(S) \in \mathcal{B}$ such that $U(S) \subseteq V \setminus (S \cup \{a_{i(V)}\})$.

Let $T = W \cap K_{\ell(W)}$. Similarly, we can fix $U(T) \in \mathcal{B}$ such that $U(T) \subseteq W \setminus (T \cup \{b_{j(W)}\})$.

Define the following closed nowhere dense sets:

- $E_V = \{a_{i(V)}\} \cup S \cup \varphi_{U(S)}[T]$,
- $F_V = \{b_{j(W)}\} \cup T \cup \varphi_{U(T)}[S]$.

Define $h_V : E_V \longrightarrow F_V$ by setting

$$h_V(x) = \begin{cases} b_{j(W)} & \text{if } x = a_{i(V)}, \\ \varphi_{U(T)}(x) & \text{if } x \in S, \\ (\varphi_{U(S)})^{-1}(x) & \text{if } x \in \varphi_{U(S)}[T]. \end{cases}$$

It is clear that h_V is a homeomorphism.

Proof: the construction, part III

By the Lemma, there exists a KR-cover $\langle \mathcal{V}_V, \mathcal{W}_V, \psi_V \rangle$ for $\langle V \setminus E_V, W \setminus F_V, h_V \rangle$.

Furthermore, it is easy to realize that

$$h_V[X \cap E_V] = X \cap F_V.$$

This will allow us to mantain condition (III).

Notice that $\phi_{U(S)}[T] \cap D = \varnothing$, because $\phi_U[K_\ell] \cap D = \varnothing$ for every $U \in \mathcal{B}$ and $\ell \in \omega$ by the choice of D. Similarly, one sees that $\phi_{U(T)}[S] \cap D = \varnothing$. Since $A \cup B \subseteq D$, it follows that

$$h_V[A \cap E_V] = h_V[\{a_{i(V)}\}] = \{b_{j(W)}\} = B \cap F_V.$$

This will allow us to mantain condition (VI).

Proof: the construction, part IV

Repeat this construction for every $V \in \mathcal{V}_n$, then let $E_{n+1} = E_n \cup \bigcup\{E_V : V \in \mathcal{V}_n\}$ and $F_{n+1} = F_n \cup \bigcup\{F_V : V \in \mathcal{V}_n\}$.

Define

$$h_{n+1} = h_n \cup \bigcup_{V \in \mathcal{V}_n} h_V.$$

It is straightforward to check that $h_{n+1} : E_{n+1} \longrightarrow F_{n+1}$ is a homeomorphism.

Finally, we define $\mathcal{K}_{n+1} = \langle \mathcal{V}_{n+1}, \mathcal{W}_{n+1}, \psi_{n+1} \rangle$. Let $\mathcal{V}_{n+1} = \bigcup\{\mathcal{V}_V : V \in \mathcal{V}_n\}$ and $\mathcal{W}_{n+1} = \bigcup\{\mathcal{W}_V : V \in \mathcal{V}_n\}$.

By further refining $\mathcal{V}_{n+1}$ and $\mathcal{W}_{n+1}$, we can assume that $\mathrm{mesh}(\mathcal{V}_{n+1}) \leq 2^{-(n+1)}$ and $\mathrm{mesh}(\mathcal{W}_{n+1}) \leq 2^{-(n+1)}$.

Thank you for your attention

and good night!