Products and h-homogeneity

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All spaces we consider are *Tychonoff* (so that we can take the Stone-Čech compactification) and infinite. A space $X$ is *zero-dimensional* if it has a $T_1$ basis consisting of clopen sets. Every such space is Tychonoff.

**Definition (Ostrovskii, 1981; van Mill, 1981)**

A topological space $X$ is *h-homogeneous* (or *strongly homogeneous*) if all non-empty clopen sets in $X$ are homeomorphic.

**Examples:**

- The Cantor set $2^\omega$, the rationals $\mathbb{Q}$, the irrationals $\omega^\omega$. (Use the respective characterizations.)
- Any connected space.
- The Knaster-Kuratowski fan.
Main results

- In the class of zero-dimensional spaces, h-homogeneity is productive.
- If the product is pseudocompact, then the zero-dimensionality requirement can be dropped.
- Clopen sets in pseudocompact products depend on finitely many coordinates.
- Partial answers to Terada’s question: is the infinite power \( X^\omega \) h-homogeneous for every zero-dimensional first-countable \( X \)?
A useful base for $\beta X$

Definition

Given $U$ open in $X$, define $\text{Ex}(U) = \beta X \setminus \text{cl}_{\beta X}(X \setminus U)$.

Basic facts:

- $\text{Ex}(U)$ is the biggest open set in $\beta X$ such that its intersection with $X$ is $U$.
- The collection $\{\text{Ex}(U) : U \text{ is open in } X\}$ is a base for $\beta X$.
- If $C$ is clopen in $X$ then $\text{Ex}(C) = \text{cl}_{\beta X}(C)$, hence $\text{Ex}(C)$ is clopen in $\beta X$.

⚠️ It is not true that $\beta X$ is zero-dimensional whenever $X$ is zero-dimensional. (Dowker, 1957.) ⚠️
When does $\beta$ commute with $\prod$?

**Theorem (Glicksberg, 1959)**

The product $\prod_{i \in I} X_i$ is $C^*$-embedded in $\prod_{i \in I} \beta X_i$ if and only if $\prod_{i \in I} X_i$ is pseudocompact.

In that case,

$$\prod_{i \in I} \beta X_i \cong \beta \left( \prod_{i \in I} X_i \right).$$

More precisely, there exists a homeomorphism

$$h : \prod_{i \in I} \beta X_i \to \beta \left( \prod_{i \in I} X_i \right)$$

such that $h \upharpoonright \prod_{i \in I} X_i = \text{id}$. 
The productivity of h-homogeneity

Theorem (Terada, 1993)

If $X_i$ is h-homogeneous and zero-dimensional for every $i \in I$ and $P = \prod_{i \in I} X_i$ is compact or non-pseudocompact, then $P$ is h-homogeneous.

Proof of the compact case, for $P = X \times Y$:
Observe that $n \times X \cong X$ whenever $1 \leq n < \omega$.
So $n \times X \times Y \cong X \times Y$ whenever $1 \leq n < \omega$.
Let $C$ be non-empty and clopen in $X \times Y$. By compactness, zero-dimensionality and $\prec$, find clopen rectangles $C_i$ such that

$$C = C_1 \oplus \cdots \oplus C_n.$$  

By h-homogeneity, $C \cong n \times X \times Y \cong X \times Y$. 

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Products and h-homogeneity
Theorem

If $X \times Y$ is pseudocompact, then every clopen set $C$ can be written as a finite union of open rectangles.

Proof: By Glicksberg’s theorem, there exists a homeomorphism

$$h : \beta X \times \beta Y \rightarrow \beta(X \times Y)$$

such that $h(x, y) = (x, y)$ whenever $(x, y) \in X \times Y$.

Since $\{\text{Ex}(U) : U \text{ is open in } X\}$ is a base for $\beta X$ and $\{\text{Ex}(V) : V \text{ is open in } Y\}$ is a base for $\beta Y$, the collection

$$\mathcal{B} = \{\text{Ex}(U) \times \text{Ex}(V) : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

is a base for $\beta X \times \beta Y$. 
Therefore $\{h[B] : B \in \mathcal{B}\}$ is a base for $\beta(X \times Y)$.
Hence we can write $\text{Ex}(C) = h[B_1] \cup \cdots \cup h[B_n]$ for some $B_1, \ldots, B_n \in \mathcal{B}$ by compactness.
Finally, if we let $B_i = \text{Ex}(U_i) \times \text{Ex}(V_i)$ for each $i$, we get

\[
C = \text{Ex}(C) \cap X \times Y
= (h[B_1] \cup \cdots \cup h[B_n]) \cap h[X \times Y]
= h[B_1 \cap X \times Y] \cup \cdots \cup h[B_n \cap X \times Y]
= (B_1 \cap X \times Y) \cup \cdots \cup (B_n \cap X \times Y)
= (U_1 \times V_1) \cup \cdots \cup (U_n \times V_n).
\]
But we would like *clopen* rectangles... 😞

Why? Because then we could prove the following. (Notice that zero-dimensionality is not needed.)

**Theorem**

Assume that $X \times Y$ is pseudocompact. If $X$ and $Y$ are $h$-homogeneous then $X \times Y$ is $h$-homogeneous.

Proof: If $X$ and $Y$ are both connected then $X \times Y$ is connected, so assume without loss of generality that $X$ is not connected. It follows that $X \cong n \times X$ whenever $1 \leq n < \omega$.

...then finish the proof as in the compact case.
Lemma

Let $C \subseteq X \times Y$ be a clopen set that can be written as the union of finitely many rectangles. Then $C$ can be written as the union of finitely many pairwise disjoint clopen rectangles. 😊

[ 🎨 Draws an enlightening picture on the board.]

Proof: For every $x \in X$, let $C_x = \{y \in Y : (x, y) \in C\}$ be the corresponding vertical cross-section. For every $y \in Y$, let $C^y = \{x \in X : (x, y) \in C\}$ be the corresponding horizontal cross-section. Since $C$ is clopen, each cross-section is clopen.
Let $\mathcal{A}$ be the Boolean subalgebra of the clopen algebra of $X$ generated by $\{C^y : y \in Y\}$. Since $\mathcal{A}$ is finite, it must be atomic. Let $P_1, \ldots, P_m$ be the atoms of $\mathcal{A}$. Similarly, let $\mathcal{B}$ be the Boolean subalgebra of the clopen algebra of $Y$ generated by $\{C_x : x \in X\}$, and let $Q_1, \ldots, Q_n$ be the atoms of $\mathcal{B}$.

Observe that the rectangles $P_i \times Q_j$ are clopen and pairwise disjoint. Furthermore, given any $i, j$, either $P_i \times Q_j \subseteq C$ or $P_i \times Q_j \cap C = \emptyset$. Hence $C$ is the union of a (finite) collection of such rectangles.
Corollary

Assume that $X = X_1 \times \cdots \times X_n$ is pseudocompact. If each $X_i$ is $h$-homogeneous then $X$ is $h$-homogeneous.

An obvious modification of the proof of the theorem yields:

Theorem

Assume that $X = \prod_{i \in I} X_i$ is pseudocompact. If $C \subseteq X$ is clopen then $C$ can be written as the union of finitely many open rectangles.

Corollary

Assume that $X = \prod_{i \in I} X_i$ is pseudocompact. If $C \subseteq X$ is clopen then $C$ depends on finitely many coordinates.

[The speaker takes a walk down memory lane...]
Theorem

Assume that $X = \prod_{i \in I} X_i$ is pseudocompact. If $X_i$ is $h$-homogeneous for every $i \in I$ then $X$ is $h$-homogeneous.

Proof: Let $C \subseteq X$ be clopen and non-empty. Then there exists a finite subset $F$ of $I$ such that $C$ is homeomorphic to $C' \times \prod_{i \in I \setminus F} X_i$, where $C'$ is a clopen subset of $\prod_{i \in F} X_i$. But $\prod_{i \in F} X_i$ is $h$-homogeneous, so

$$C \cong C' \times \prod_{i \in I \setminus F} X_i \cong \prod_{i \in F} X'_i \times \prod_{i \in I \setminus F} X_i \cong X.$$
Conclusions
Putting together our results with Terada’s theorem, we obtain the following.

**Theorem**

If $X_i$ is $h$-homogeneous and zero-dimensional for every $i \in I$ and $X = \prod_{i \in I} X_i$ then $X$ is $h$-homogeneous.

After all this work...

**Problem**

*Is $h$-homogeneity productive?*
Some applications
The following result was first proved by Motorov in the compact case.

**Theorem**
Assume that $X$ has a $\pi$-base $B$ consisting of clopen sets. Then $(X \times 2 \times \prod B)^\kappa$ is $h$-homogeneous for every infinite cardinal $\kappa$.

**Corollary**
For every zero-dimensional space $X$ there exists a zero-dimensional space $Y$ such that $X \times Y$ is $h$-homogeneous.

**Problem**
Is it true that for every space $X$ there exists a space $Y$ such that $X \times Y$ is $h$-homogeneous?
The case $\kappa = \omega$ of the following result is an easy consequence of a result of Matveev. Motorov first proved it under the additional assumption that $X$ is first-countable and compact. Terada proved it for an arbitrary infinite $\kappa$, under the additional assumption that $X$ is non-pseudocompact.

**Theorem**

*Assume that $X$ is a space such that the isolated points are dense in $X$. Then $X^\kappa$ is $h$-homogeneous for every infinite cardinal $\kappa$.*

For example, if $\alpha$ is an ordinal with the order topology and $\kappa$ is an infinite cardinal then $\alpha^\kappa$ is $h$-homogeneous.
Homogeneity vs h-homogeneity
All spaces are assumed to be first-countable and zero-dimensional from now on.

Definition
A space $X$ is **homogeneous** if for every $x, y \in X$ there exists a homeomorphism $f : X \to X$ such that $f(x) = y$.

By a picture-proof, h-homogeneity implies homogeneity. 🎨 Erik van Douwen constructed a compact homogeneous space that is not h-homogeneous.

Theorem (Motorov, 1989)
*If $X$ is a compact homogeneous space of uncountable cellularity then $X$ is h-homogeneous.*
Infinite powers

Problem (Terada, 1993)

Is $X^\omega$ always h-homogeneous?

The following remarkable theorem is based on work by Motorov and Lawrence.

Theorem (Dow and Pearl, 1997)

$X^\omega$ is homogeneous.

However, Terada’s question remains open.
Motorov’s main result

**Theorem (Motorov, 1989)**

*If $X$ has a $\pi$-base consisting of clopen sets that are homeomorphic to $X$ then $X$ is h-homogeneous.*

Proof: Let $C$ be a non-empty clopen set in $X$. By first-countability, write

$$X = \{x\} \cup \bigcup_{n \in \omega} X_n \quad \text{and} \quad C = \{y\} \cup \bigcup_{n \in \omega} C_n$$

where the $X_n$ are disjoint, clopen, they converge to $x$ but do not contain $x$, and the $C_n$ are disjoint, clopen, they converge to $y$ but do not contain $y$.

[ ☕️ *Finishes the proof by juggling with clopen sets.*]
Divisibility

Definition

A space $F$ is a factor of $X$ (or $X$ is divisible by $F$) if there exists $Y$ such that $F \times Y \cong X$. If $F \times X \cong X$ then $F$ is a strong factor of $X$ (or $X$ is strongly divisible by $F$).

Problem (Motorov, 1989)

Is $X^\omega$ always divisible by 2?

As we observed already, h-homogeneity implies divisibility by 2. We will show that Terada’s question is equivalent to Motorov’s question. Actually, even weaker conditions suffice.
Lemma

The following are equivalent.

1. $F$ is a factor of $X^\omega$.
2. $F \times X^\omega \cong X^\omega$.
3. $F^\omega \times X^\omega \cong X^\omega$.

The implications $2 \rightarrow 1$ and $3 \rightarrow 1$ are clear.
Assume 1. Then there exists $Y$ such that $F \times Y \cong X^\omega$, hence

$$X^\omega \cong (X^\omega)^\omega \cong (F \times Y)^\omega \cong F^\omega \times Y^\omega.$$  

Since multiplication by $F$ or by $F^\omega$ does not change the right hand side, it follows that 2 and 3 hold.
The key lemma

Lemma

\( X = (Y \oplus 1)^\omega \) is h-homogeneous.

Proof: Recall that 1 = \{0\}. For each \( n \in \omega \), define

\[ U_n = \{0\} \times \{0\} \times \cdots \times \{0\} \times (Y \oplus 1) \times (Y \oplus 1) \times \cdots \]

\( n \) times

Observe that \( \{U_n : n \in \omega\} \) is a local base for \( X \) at \((0, 0, \ldots)\) consisting of clopen sets that are homeomorphic to \( X \).

But \( X \) is homogeneous by the Dow-Pearl theorem, therefore it has a base (hence a \( \pi \)-base) consisting of clopen sets that are homeomorphic to \( X \).

It follows from Motorov’s result that \( X \) is h-homogeneous.
Lemma

Let $X = (Y \oplus 1)^\omega$. Then

$$X \cong Y^\omega \times (Y \oplus 1)^\omega \cong 2^\omega \times Y^\omega.$$ 

Proof: Observe that

$$X \cong (Y \oplus 1) \times X \cong (Y \times X) \oplus X,$$

hence $X \cong Y \times X$ by h-homogeneity. It follows that $X \cong Y^\omega \times (Y \oplus 1)^\omega$. Finally,

$$Y^\omega \times (Y \oplus 1)^\omega \cong (Y^\omega \times (Y \oplus 1))^\omega \cong (Y^\omega \oplus Y^\omega)^\omega \cong 2^\omega \times Y^\omega,$$

that concludes the proof.
Theorem

The following are equivalent.

1. $X^\omega \cong (X \oplus 1)^\omega$.
2. $X^\omega \cong Y^\omega$ for some $Y$ with at least one isolated point.
3. $X^\omega$ is $h$-homogeneous.
4. $X^\omega$ has a clopen subset that is strongly divisible by 2.
5. $X^\omega$ has a proper clopen subspace homeomorphic to $X^\omega$.
6. $X^\omega$ has a proper clopen subspace as a factor.

Proof: The implication 1 $\rightarrow$ 2 is trivial; the implication 2 $\rightarrow$ 3 follows from the lemma; the implications 3 $\rightarrow$ 4 $\rightarrow$ 5 $\rightarrow$ 6 are trivial.
Assume that 6 holds. Let $C$ be a proper clopen subset of $X^\omega$ that is also a factor of $X^\omega$ and let $D = X^\omega \setminus C$. Then

$$
X^\omega \cong (C \oplus D) \times X^\omega \\
\cong (C \times X^\omega) \oplus (D \times X^\omega) \\
\cong X^\omega \oplus (D \times X^\omega) \\
\cong (1 \oplus D) \times X^\omega,
$$

hence $X^\omega \cong (1 \oplus D)^\omega \times X^\omega$. Since $(1 \oplus D)^\omega \cong 2^\omega \times D^\omega$ by the lemma, it follows that $X^\omega \cong 2^\omega \times X^\omega$. Therefore 1 holds by the lemma.
The pseudocompact case
The next two theorems show that in the pseudocompact case we can say something more.

Theorem
Assume that $X^\omega$ is pseudocompact. Then $C^\omega \cong (X \oplus 1)^\omega$ for every non-empty proper clopen subset $C$ of $X^\omega$.

Theorem
Assume that $X^\omega$ is pseudocompact. Then the following are equivalent.

1. $X^\omega$ is $h$-homogeneous.
2. $X^\omega$ has a proper clopen subspace $C$ such that $C \cong Y^\omega$ for some $Y$. 
Ultraparacompactness
The following notion allows us to give us a positive answer to Terada’s question for a certain class of spaces.

**Definition**
A space $X$ is *ultraparacompact* if every open cover of $X$ has a refinement consisting of pairwise disjoint clopen sets.

A metric space $X$ is ultraparacompact if and only if $\dim X = 0$.

**Theorem**
*If $X^\omega$ is ultraparacompact and non-Lindelöf then $X^\omega$ is $h$-homogeneous.*