Products and h-homogeneity

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June 27, 2010

All spaces we consider are *Tychonoff* (so that we can take the Stone-Čech compactification) and infinite.
A space X is *zero-dimensional* if it has a T_1 basis consisting of clopen sets. Every such space is Tychonoff.

Definition (Ostrovskii, 1981; van Mill, 1981)

A topological space X is *h-homogeneous* (or *strongly homogeneous*) if all non-empty clopen sets in X are homeomorphic.

Examples:

- The Cantor set 2^ω, the rationals $\mathbb{Q}$, the irrationals ω^ω. (Use the respective characterizations.)
- Any connected space.
- The Knaster-Kuratowski fan.

Main results

- In the class of zero-dimensional spaces, h-homogeneity is productive.
- If the product is pseudocompact, then the zero-dimensionality requirement can be dropped.
- Clopen sets in pseudocompact products depend on finitely many coordinates.
- Partial answers to Terada's question: is the infinite power X^ω h-homogeneous for every zero-dimensional first-countable X?

A useful base for βX

Definition

Given U open in X, define $\mathrm{Ex}(U) = \beta X \setminus \mathrm{cl}_{\beta X}(X \setminus U)$.

Basic facts:

- $\mathrm{Ex}(U)$ is the biggest open set in βX such that its intersection with X is U.
- The collection $\{\mathrm{Ex}(U) : U \text{ is open in } X\}$ is a base for βX.
- If C is clopen in X then $\mathrm{Ex}(C) = \mathrm{cl}_{\beta X}(C)$, hence $\mathrm{Ex}(C)$ is clopen in βX.

☢ It is not true that βX is zero-dimensional whenever X is zero-dimensional. (Dowker, 1957.) ☢

When does β commute with $\prod$?

Theorem (Glicksberg, 1959)

The product $\prod_{i\in I} X_i$ is C^*-embedded in $\prod_{i\in I}\beta X_i$ if and only if $\prod_{i\in I} X_i$ is pseudocompact.

In that case,

$$\prod_{i\in I}\beta X_i \cong \beta\left(\prod_{i\in I} X_i\right).$$

More precisely, there exists a homeomorphism

$$h:\prod_{i\in I}\beta X_i \longrightarrow \beta\left(\prod_{i\in I} X_i\right)$$

such that $h \restriction \prod_{i\in I} X_i = \mathrm{id}$.

The productivity of h-homogeneity

Theorem (Terada, 1993)

If X_i is h-homogeneous and zero-dimensional for every $i \in I$ and $P = \prod_{i \in I} X_i$ is compact or non-pseudocompact, then P is h-homogeneous.

Proof of the compact case, for $P = X \times Y$:
Observe that $n \times X \cong X$ whenever $1 \leq n < \omega$.
So $n \times X \times Y \cong X \times Y$ whenever $1 \leq n < \omega$.
Let C be non-empty and clopen in $X \times Y$. By compactness, zero-dimensionality and ✂, find clopen rectangles C_i such that

$$C = C_1 \oplus \cdots \oplus C_n.$$

By h-homogeneity, $C \cong n \times X \times Y \cong X \times Y$.

Theorem

If $X \times Y$ is pseudocompact, then every clopen set C can be written as a finite union of open rectangles.

Proof: By Glicksberg's theorem, there exists a homeomorphism

$$h : \beta X \times \beta Y \longrightarrow \beta(X \times Y)$$

such that $h(x, y) = (x, y)$ whenever $(x, y) \in X \times Y$.
Since $\{\text{Ex}(U) : U \text{ is open in } X\}$ is a base for βX and $\{\text{Ex}(V) : V \text{ is open in } Y\}$ is a base for βY, the collection

$$\mathcal{B} = \{\text{Ex}(U) \times \text{Ex}(V) : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

is a base for $\beta X \times \beta Y$.

Therefore $\{h[B] : B \in \mathcal{B}\}$ is a base for $\beta(X \times Y)$.
Hence we can write $\mathrm{Ex}(C) = h[B_1] \cup \cdots \cup h[B_n]$ for some $B_1, \ldots, B_n \in \mathcal{B}$ by compactness.
Finally, if we let $B_i = \mathrm{Ex}(U_i) \times \mathrm{Ex}(V_i)$ for each i, we get

$$\begin{aligned}
C &= \mathrm{Ex}(C) \cap X \times Y \\
&= (h[B_1] \cup \cdots \cup h[B_n]) \cap h[X \times Y] \\
&= h[B_1 \cap X \times Y] \cup \cdots \cup h[B_n \cap X \times Y] \\
&= (B_1 \cap X \times Y) \cup \cdots \cup (B_n \cap X \times Y) \\
&= (U_1 \times V_1) \cup \cdots \cup (U_n \times V_n).
\end{aligned}$$

But we would like *clopen* rectangles... ☹
Why? Because then we could prove the following.
(Notice that zero-dimensionality is not needed.)

Theorem

Assume that $X \times Y$ *is pseudocompact. If* X *and* Y *are h-homogeneous then* $X \times Y$ *is h-homogeneous.*

Proof: If X and Y are both connected then $X \times Y$ is connected, so assume without loss of generality that X is not connected. It follows that $X \cong n \times X$ whenever $1 \leq n < \omega$.

...then finish the proof as in the compact case.

Lemma

Let $C \subseteq X \times Y$ be a clopen set that can be written as the union of finitely many rectangles. Then C can be written as the union of finitely many pairwise disjoint clopen rectangles. ☺

[✎ *Draws an enlightening picture on the board.*]

Proof: For every $x \in X$, let $C_x = \{y \in Y : (x, y) \in C\}$ be the corresponding vertical cross-section. For every $y \in Y$, let $C^y = \{x \in X : (x, y) \in C\}$ be the corresponding horizontal cross-section. Since C is clopen, each cross-section is clopen.

Let $\mathcal{A}$ be the Boolean subalgebra of the clopen algebra of X generated by $\{C^y : y \in Y\}$. Since $\mathcal{A}$ is finite, it must be atomic. Let $P_1, \ldots, P_m$ be the atoms of $\mathcal{A}$. Similarly, let $\mathcal{B}$ be the Boolean subalgebra of the clopen algebra of Y generated by $\{C_x : x \in X\}$, and let $Q_1, \ldots, Q_n$ be the atoms of $\mathcal{B}$.

Observe that the rectangles $P_i \times Q_j$ are clopen and pairwise disjoint. Furthermore, given any i, j, either $P_i \times Q_j \subseteq C$ or $P_i \times Q_j \cap C = \varnothing$. Hence C is the union of a (finite) collection of such rectangles.

Corollary

Assume that $X = X_1 \times \cdots \times X_n$ is pseudocompact. If each X_i is h-homogeneous then X is h-homogeneous.

An obvious modification of the proof of the theorem yields:

Theorem

Assume that $X = \prod_{i \in I} X_i$ is pseudocompact. If $C \subseteq X$ is clopen then C can be written as the union of finitely many open rectangles.

Corollary

Assume that $X = \prod_{i \in I} X_i$ is pseudocompact. If $C \subseteq X$ is clopen then C depends on finitely many coordinates.

[*The speaker takes a walk down memory lane...*]

Theorem

Assume that $X = \prod_{i \in I} X_i$ is pseudocompact. If X_i is h-homogeneous for every $i \in I$ then X is h-homogeneous.

Proof: Let $C \subseteq X$ be clopen and non-empty.
Then there exists a finite subset F of I such that C is homeomorphic to $C' \times \prod_{i \in I \setminus F} X_i$, where C' is a clopen subset of $\prod_{i \in F} X_i$.
But $\prod_{i \in F} X_i$ is h-homogeneous, so

$$C \cong C' \times \prod_{i \in I \setminus F} X_i \cong \prod_{i \in F} X_i' \times \prod_{i \in I \setminus F} X_i \cong X.$$

☕

Conclusions

Putting together our results with Terada's theorem, we obtain the following.

Theorem

If X_i is h-homogeneous and zero-dimensional for every $i \in I$ and $X = \prod_{i \in I} X_i$ then X is h-homogeneous.

After all this work...

Problem

Is h-homogeneity productive?

Some applications

The following result was first proved by Motorov in the compact case.

Theorem

Assume that X has a π-base $\mathcal{B}$ consisting of clopen sets. Then $(X \times 2 \times \prod \mathcal{B})^\kappa$ is h-homogeneous for every infinite cardinal κ.

Corollary

For every zero-dimensional space X there exists a zero-dimensional space Y such that $X \times Y$ is h-homogeneous.

Problem

Is it true that for every space X there exists a space Y such that $X \times Y$ is h-homogeneous?

The case $\kappa = \omega$ of the following result is an easy consequence of a result of Matveev. Motorov first proved it under the additional assumption that X is first-countable and compact. Terada proved it for an arbitrary infinite κ, under the additional assumption that X is non-pseudocompact.

Theorem

Assume that X is a space such that the isolated points are dense in X. Then X^κ is h-homogeneous for every infinite cardinal κ.

For example, if α is an ordinal with the order topology and κ is an infinite cardinal then α^κ is h-homogeneous.

Homogeneity vs h-homogeneity

All spaces are assumed to be first-countable and zero-dimensional from now on.

Definition

A space X is *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $f : X \longrightarrow X$ such that $f(x) = y$.

By a picture-proof, h-homogeneity implies homogeneity. ✎
Erik van Douwen constructed a compact homogeneous space that is not h-homogeneous.

Theorem (Motorov, 1989)

If X is a compact homogeneous space of uncountable cellularity then X is h-homogeneous.

Infinite powers

Problem (Terada, 1993)

Is X^ω always h-homogeneous?

The following remarkable theorem is based on work by Motorov and Lawrence.

Theorem (Dow and Pearl, 1997)

X^ω is homogeneous.

However, Terada's question remains open.

Motorov's main result

Theorem (Motorov, 1989)

If X has a π-base consisting of clopen sets that are homeomorphic to X then X is h-homogeneous.

Proof: Let C be a non-empty clopen set in X. By first-countability, write

$$X = \{x\} \cup \bigcup_{n \in \omega} X_n \qquad \text{and} \qquad C = \{y\} \cup \bigcup_{n \in \omega} C_n$$

where the X_n are disjoint, clopen, they converge to x but do not contain x, and the C_n are disjoint, clopen, they converge to y but do not contain y.
[✎ *Finishes the proof by juggling with clopen sets.*]

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Divisibility

Definition

A space F is a *factor* of X (or X is *divisible* by F) if there exists Y such that $F \times Y \cong X$. If $F \times X \cong X$ then F is a *strong factor* of X (or X is *strongly divisible* by F).

Problem (Motorov, 1989)

Is X^ω always divisible by 2?

As we observed already, h-homogeneity implies divisibility by 2. We will show that Terada's question is equivalent to Motorov's question. Actually, even weaker conditions suffice.

Lemma

The following are equivalent.

1. *F is a factor of X^ω.*
2. $F \times X^\omega \cong X^\omega$.
3. $F^\omega \times X^\omega \cong X^\omega$.

The implications $2 \to 1$ and $3 \to 1$ are clear.
Assume 1. Then there exists Y such that $F \times Y \cong X^\omega$, hence

$$X^\omega \cong (X^\omega)^\omega \cong (F \times Y)^\omega \cong F^\omega \times Y^\omega.$$

Since multiplication by F or by F^ω does not change the right hand side, it follows that 2 and 3 hold.

The key lemma

Lemma

$X = (Y \oplus 1)^\omega$ is h-homogeneous.

Proof: Recall that $1 = \{0\}$. For each $n \in \omega$, define

$$U_n = \underbrace{\{0\} \times \{0\} \times \cdots \times \{0\}}_{n \text{ times}} \times (Y \oplus 1) \times (Y \oplus 1) \times \cdots$$

Observe that $\{U_n : n \in \omega\}$ is a local base for X at $(0, 0, \ldots)$ consisting of clopen sets that are homeomorphic to X.
But X is homogeneous by the Dow-Pearl theorem, therefore it has a base (hence a π-base) consisting of clopen sets that are homeomorphic to X.
It follows from Motorov's result that X is h-homogeneous.

Lemma

Let $X = (Y \oplus 1)^\omega$. *Then*

$$X \cong Y^\omega \times (Y \oplus 1)^\omega \cong 2^\omega \times Y^\omega.$$

Proof: Observe that

$$X \cong (Y \oplus 1) \times X \cong (Y \times X) \oplus X,$$

hence $X \cong Y \times X$ by h-homogeneity. It follows that $X \cong Y^\omega \times (Y \oplus 1)^\omega$. Finally,

$$Y^\omega \times (Y \oplus 1)^\omega \cong (Y^\omega \times (Y \oplus 1))^\omega \cong (Y^\omega \oplus Y^\omega)^\omega \cong 2^\omega \times Y^\omega,$$

that concludes the proof.

Theorem

The following are equivalent.

1. $X^\omega \cong (X \oplus 1)^\omega$.
2. $X^\omega \cong Y^\omega$ *for some* Y *with at least one isolated point.*
3. X^ω *is h-homogeneous.*
4. X^ω *has a clopen subset that is strongly divisible by* 2.
5. X^ω *has a proper clopen subspace homeomorphic to* X^ω.
6. X^ω *has a proper clopen subspace as a factor.*

Proof: The implication $1 \to 2$ is trivial; the implication $2 \to 3$ follows from the lemma; the implications $3 \to 4 \to 5 \to 6$ are trivial.

Assume that 6 holds. Let C be a proper clopen subset of X^ω that is also a factor of X^ω and let $D = X^\omega \setminus C$. Then

$$\begin{aligned} X^\omega &\cong (C \oplus D) \times X^\omega \\ &\cong (C \times X^\omega) \oplus (D \times X^\omega) \\ &\cong X^\omega \oplus (D \times X^\omega) \\ &\cong (1 \oplus D) \times X^\omega, \end{aligned}$$

hence $X^\omega \cong (1 \oplus D)^\omega \times X^\omega$. Since $(1 \oplus D)^\omega \cong 2^\omega \times D^\omega$ by the lemma, it follows that $X^\omega \cong 2^\omega \times X^\omega$. Therefore 1 holds by the lemma.

The pseudocompact case

The next two theorems show that in the pseudocompact case we can say something more.

Theorem

Assume that X^ω is pseudocompact. Then $C^\omega \cong (X \oplus 1)^\omega$ for every non-empty proper clopen subset C of X^ω.

Theorem

Assume that X^ω is pseudocompact. Then the following are equivalent.

1. *X^ω is h-homogeneous.*
2. *X^ω has a proper clopen subspace C such that $C \cong Y^\omega$ for some Y.*

Ultraparacompactness

The following notion allows us to give us a positive answer to Terada's question for a certain class of spaces.

Definition

A space X is *ultraparacompact* if every open cover of X has a refinement consisting of pairwise disjoint clopen sets.

A metric space X is ultraparacompact if and only if $\dim X = 0$.

Theorem

If X^ω is ultraparacompact and non-Lindelöf then X^ω is h-homogeneous.