APPLICATIONS OF THE OPEN GRAPH DICHOTOMY

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Abstract. We use an infinite-dimensional analog of the open graph dichotomy to investigate functions, graphs, and sets at the second level of the Borel hierarchy.

Introduction

A topological space X is *Polish* if it is second countable and completely metrizable. A set $U \subseteq X$ is $\mathbf{\Sigma}^0_1$ if it is open. More generally, a set $B \subseteq X$ is $\mathbf{\Sigma}^0_\alpha$ if there are non-zero ordinals $\alpha_n < \alpha$ and sets $B_n \subseteq X$ such that each $\sim B_n$ is $\mathbf{\Sigma}^0_{\alpha_n}$ and $B = \bigcup_{n \in \mathbb{N}} B_n$.

A *homomorphism* from a set $A \subseteq X^I$ to a set $B \subseteq Y^I$ is a function $\pi \colon X \to Y$ for which $\pi^I(A) \subseteq B$. A *reduction* of a set $A \subseteq X^I$ to a set $B \subseteq Y^I$ is a function $\pi \colon X \to Y$ that is both a homomorphism from A to B and a homomorphism from $\sim A$ to $\sim B$. The following fact is the starting point for the work we discuss here.

Theorem (Wadge). *Suppose that $\alpha > 0$ is a countable ordinal. Then under continuous reducibility there is a minimal non-$\mathbf{\Sigma}^0_\alpha$ Borel subset of a Polish space.*

While Wadge actually proved much more than this (see [Wad12]), his approach was nevertheless insufficient to obtain analogous parametrized results for Borel subsets of the plane. This problem was eventually rectified by Louveau-Saint Raymond (see [LSR87]), and their ideas were subsequently employed to obtain various further generalizations, such as Lecomte's characterization of potentially $\mathbf{\Sigma}^0_\alpha$ subsets of the plane (see [Lec13]).

Unfortunately, the underlying arguments tend to be quite technical, particularly in Lecomte's work. For instance, even in the special case that $\alpha = 2$, while not particularly complicated, the proof nevertheless relies upon a finite injury argument, and does not produce the corresponding theorem in its natural generality. It is therefore natural to

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ask whether there is another approach to this special case, rectifying these issues and leading to a more natural proof of the full result.

Far from merely providing a new proof of an old theorem, there is good reason to believe that a positive answer to this question could yield entirely new results.

One example comes from the study of functions lying on the border between descriptive set theory and topology. A function $\phi \colon X \to Y$ between topological spaces is *σ-continuous with closed witnesses* if there are closed sets $C_n \subseteq X$ with the property that each $\phi \upharpoonright C_n$ is continuous and $X = \bigcup_{n \in \mathbb{N}} C_n$. A set is F_σ if it is a union of countably-many closed sets, and G_δ if it is an intersection of countably-many open sets. A non-empty topological space is *analytic* if it is empty or a continuous image of $\mathbb{N}^\mathbb{N}$.

Theorem (Jayne-Rogers)**.** *Suppose that X is an analytic metric space, Y is a separable metric space, and $\phi \colon X \to Y$. Then ϕ is G_δ-measurable if and only if it is σ-continuous with closed witnesses.*

While the original proof of this result (see [JR82]) was quite involved, a simpler proof of a stronger theorem eventually appeared in [Sol98]. A *topological embedding* of a function $\phi \colon X \to Y$ into a function $\phi' \colon X' \to Y'$ is a pair of topological embeddings $\pi_X \colon X \to X'$ and $\pi_Y \colon \phi(X) \to \phi'(X')$ such that $\pi_Y \circ \phi = \phi' \circ \pi_X$. A function is *Baire class one* if it is F_σ-measurable. A *quasi-order* on a set Z is a reflexive transitive binary relation $\leq$ on Z. A set $B \subseteq Z$ is a *basis* under $\leq$ for Z if $\forall z \in Z \exists b \in B \; b \leq z$.

Theorem (Solecki)**.** *There is a two-element basis under topological embeddability for the class of non-σ-continuous-with-closed-witnesses Baire-class-one functions from analytic metric spaces to separable metric spaces.*

As G_δ-measurable functions are trivially Baire class one, this reduces the problem of verifying any result similar to the Jayne-Rogers theorem to simply checking that the two basis elements have the relevant property. However, it also leads to several natural questions. For instance, can the notion of topological embeddability be substantially strengthened? And can the class of functions be substantially broadened beyond the first Baire class? It is natural to hope that a new approach to the Hurewicz theorem could yield answers to these questions, and that subsequent generalizations to $\alpha > 2$ could shed light on recent generalizations of the Jayne-Rogers theorem.

Another example comes from the search for finer versions of the $\mathbb{G}_0$ dichotomy concerning colorings of bounded complexity. A *digraph* on a set X is an irreflexive set $G \subseteq X \times X$, and a Y-*coloring* of G is a function $c \colon X \to Y$ such that $\forall (w, x) \in G \; c(w) \neq c(x)$. Given a family Γ of subsets of X, a function $\phi \colon X \to Y$ is Γ-*measurable* if the ϕ-preimage of every open subset of Y is in Γ. A subset of a Polish space is $\mathbf{\Delta}^0_2$ if both it and its complement are $\mathbf{\Sigma}^0_2$.

Theorem (Lecomte-Zeleny)**.** *Suppose that $\alpha \in \{2, 3\}$. Then under continuous homomorphism there is a minimal Borel digraph on a Polish space that does not admit a $\mathbf{\Delta}^0_\alpha$-measurable $\mathbb{N}$-coloring.*

The Lecomte-Zeleny argument for the case $\alpha = 2$ is based on the proof of Hurewicz theorem mentioned above (see [LZ14]), and is therefore susceptible to the same objections. A rather different remark in this case is that since the corresponding minimal graph depends on an arbitrary parameter, one wonders whether it has a more canonical representation. But most importantly, it is not clear whether the case $\alpha > 3$ has resisted solution because of genuine mathematical difficulty, or simply because of limitations inherent in their approach.

Our goal here is to present a new graph-theoretic approach to the Hurewicz theorem, positively answering all of the above questions in the case $\alpha = 2$. The case $\alpha > 2$ remains a subject of future work, although some initial progress has already been made in this direction.

1. The open graph dichotomy

The best-known and easiest-to-prove descriptive-set-theoretic dichotomy theorem is Souslin's perfect set theorem (see [Sou17]), yielding a continuous injection of $2^{\mathbb{N}}$ into every uncountable analytic Hausdorff space. A somewhat less well-known generalization is Feng's open graph dichotomy (see [Fen93]), yielding a continuous homomorphism from the complete graph on $2^{\mathbb{N}}$ to every open graph on an analytic Hausdorff space with no $\mathbb{N}$-coloring. The proof of the former result easily adapts to yield a proof of the latter. By examining this proof, one can easily extract a higher-dimensional analog of the complete graph yielding a higher-dimensional analog of the open graph dichotomy.

An $\mathbb{N}$-*dimensional dihypergraph* on X is a set $H \subseteq X^{\mathbb{N}}$ of non-constant sequences, and a Y-*coloring* of H is a function $c \colon X \to Y$ with the property that $\forall (x_n)_{n \in \mathbb{N}} \in H \; c \upharpoonright \{x_n \mid n \in \mathbb{N}\}$ is not constant. Let $\mathbb{H}$ denote the $\mathbb{N}$-dimensional dihypergraph on $\mathbb{N}^{\mathbb{N}}$ given by $\mathbb{H} = \{(t \frown (i) \frown b(i))_{i \in \mathbb{N}} \mid b \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \text{ and } t \in \mathbb{N}^{<\mathbb{N}}\}$.

Given a sequence $(X_i)_{i \in I}$ of topological spaces, the *box topology* on $\prod_{i \in I} X_i$ is the topology generated by the sets of the form $\prod_{i \in I} U_i$, where $U_i \subseteq X_i$ is open for all $i \in I$. The real strength of the following generalization of the open graph dichotomy comes from the fact that it not only holds for product-open $\mathbb{N}$-dimensional dihypergraphs, but for box-open ones as well.

Theorem 1.1. *Suppose that X is an analytic Hausdorff space and H is a box-open $\mathbb{N}$-dimensional dihypergraph on X. Then exactly one of the following holds:*

1. *There is an $\mathbb{N}$-coloring of H.*
2. *There is a continuous homomorphism from $\mathbb{H}$ to H.*

Proof. To see that conditions (1) and (2) are mutually exclusive, suppose that $c \colon X \to \mathbb{N}$ is a coloring of H and $\phi \colon \mathbb{N}^\mathbb{N} \to X$ is a homomorphism from $\mathbb{H}$ to H, recursively construct $b \in \mathbb{N}^\mathbb{N}$ such that for no $n \in \mathbb{N}$ is there an extension a of $b \upharpoonright (n+1)$ for which $(c \circ \phi)(a) = n$, and observe that $(c \circ \phi)(b) \neq n$ for all $n \in \mathbb{N}$, a contradiction.

To see $\neg(1) \implies (2)$, suppose that $\phi \colon \mathbb{N}^\mathbb{N} \to X$ is a continuous surjection, let T be the set of all $t \in \mathbb{N}^{<\mathbb{N}}$ for which the restriction of H to the ϕ-image of the basic open set determined by t does not have an $\mathbb{N}$-coloring, and set $[T] = \{b \in \mathbb{N}^\mathbb{N} \mid \forall n \in \mathbb{N} \ b \upharpoonright n \in T\}$. Then for no $t \in T$ is there an $\mathbb{N}$-coloring of the restriction of H to the ϕ-image of the basic open subset of $[T]$ determined by t. In particular, it follows that if $t \in T$, then there is a sequence $(b_n)_{n \in \mathbb{N}}$ of extensions of t in $[T]$ such that $(\phi(b_n))_{n \in \mathbb{N}} \in H$, in which case the continuity of ϕ and openness of H yield proper extensions $t_n \sqsubset b_n$ of t such that the $\phi^\mathbb{N}$-image of the basic box-open set determined by $(t_n)_{n \in \mathbb{N}}$ is contained in H. By recursively applying this observation, we obtain a function $\pi \colon \mathbb{N}^{<\mathbb{N}} \to T$ such that the ϕ-image of the basic box-open set determined by $(\pi(t \frown (n)))_{n \in \mathbb{N}}$ is contained in H for all $t \in \mathbb{N}^{<\mathbb{N}}$ and $\pi(t) \sqsubset \pi(t \frown (n))$ for all $n \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$. Define $\psi \colon \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $\psi(b) = \bigcup_{n \in \mathbb{N}} \pi(b \upharpoonright n)$, and observe that the function $\phi \circ \psi$ is a continuous homomorphism from $\mathbb{H}$ to H. ⊠

A set $Y \subseteq X$ is *H-independent* if $H \upharpoonright Y = \emptyset$. The following fact easily yields the strengthening of Theorem 1.1 in which the coloring is $\mathbf{\Delta}^0_2$-measurable.

Proposition 1.2. *Suppose that X is a topological space and H is a box-open $\mathbb{N}$-dimensional dihypergraph on X. If H admits an $\mathbb{N}$-coloring, then X is the union of countably-many H-independent closed sets.*

Proof. It is sufficient to show that if the closure of a set $Y \subseteq X$ is H-dependent, then so too is Y itself. Towards this end, fix a sequence $(\overline{y}_n)_{n \in \mathbb{N}} \in H$ of points in the closure of Y, as well as open neighborhoods U_n of $\overline{y}_n$ whose product is contained in H and points $y_n \in U_n \cap Y$, and observe that $(y_n)_{n \in \mathbb{N}} \in H$. ⊠

Remark 1.3. The proof of Theorem 1.1 can also be used to establish the analogous result for κ-colorability of box-open dihypergraphs on κ-Souslin Hausdorff spaces. While this argument uses uncountable choice for uncountable κ, one can easily introduce a derivative to eliminate such concerns. As $\mathtt{AD}_{\mathbb{R}}$ ensures that all subsets of analytic Hausdorff spaces are κ-Souslin for some aleph κ, and $\mathtt{AD}$ ensures that there is no injection of ω_1 into the family of closed subsets of an analytic Hausdorff space, the generalization of Theorem 1.1 to $\mathbb{N}$-colorability of box-open dihypergraphs on arbitrary subsets of analytic Hausdorff spaces follows from Proposition 1.2 under $\mathtt{AD}_{\mathbb{R}}$. As a consequence, all of our applications of Theorem 1.1 admit analogous generalizations.

In addition, one often automatically obtains the strengthening of Theorem 1.1 in which the homomorphism is injective.

Proposition 1.4. *Suppose that ϕ is a homomorphism from $\mathbb{H}$ to an $\mathbb{N}$-dimensional dihypergraph H consisting solely of injective sequences. Then ϕ is injective.*

Proof. Suppose that $a, b \in \mathbb{N}^{\mathbb{N}}$ are distinct, let $t \in \mathbb{N}^{<\mathbb{N}}$ be their maximal common initial segment, and fix a sequence $(c_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{N}^{\mathbb{N}}$ such that $a, b \in \{c_n \mid n \in \mathbb{N}\}$ and $t \frown (n) \sqsubseteq c_n$, for all $n \in \mathbb{N}$. As $(c_n)_{n\in\mathbb{N}} \in \mathbb{H}$, it follows that $(\phi(c_n))_{n\in\mathbb{N}}$ is in H, and is therefore injective, thus $\phi(a) \neq \phi(b)$. ⊠

2. A compactification of $\mathbb{N}^{\leq\mathbb{N}}$

The strength of Theorem 1.1 lies in the fact that the dihypergraph H need only be box-open, thereby allowing us to express facts about convergence. As a consequence, our applications of Theorem 1.1 will often concern a natural compactification of $\mathbb{N}^{\mathbb{N}}$, which we describe here.

Endow the set $\mathbb{N}^{\leq\mathbb{N}}_* = \mathbb{N}^{\leq\mathbb{N}} \cup \{t \frown (\infty) \mid t \in \mathbb{N}^{<\mathbb{N}}\}$ with the smallest topology making the sets $\{t\}$ and $\mathcal{N}_t = \{c \in \mathbb{N}^{\leq\mathbb{N}}_* \mid t \sqsubseteq c\}$ clopen for all $t \in \mathbb{N}^{<\mathbb{N}}$.

Proposition 2.1. *The family $\mathcal{B}$ of sets of the form $\{t\}$ and $\mathcal{N}_t \setminus (\{t\} \cup \bigcup_{j<i} \mathcal{N}_{t\frown(j)})$, where $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$, is a clopen basis for $\mathbb{N}^{\leq\mathbb{N}}_*$.*

Proof. Let τ be the topology generated by $\mathcal{B}$. As every set in $\mathcal{B}$ is clearly clopen, it is sufficient to show that the sets $\{t\}$ and $\mathcal{N}_t$ are τ-clopen for all $t \in \mathbb{N}^{<\mathbb{N}}$. As these sets are clearly τ-open, we need only show that they are τ-closed. As $\mathcal{N}_{t\frown(i)}$ is τ-closed in $\mathcal{N}_t$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$, a straightforward induction shows that $\mathcal{N}_t$ is τ-closed for all $t \in \mathbb{N}^{<\mathbb{N}}$. As $\{t\}$ is τ-closed in $\mathcal{N}_t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$, it follows that $\{t\}$ is τ-closed for all $t \in \mathbb{N}^{<\mathbb{N}}$. ⊠

Proposition 2.2. *The space $\mathbb{N}^{\leq\mathbb{N}}_*$ is compact.*

Proof. Suppose, towards a contradiction, that there is an open cover $\mathcal{U}$ of $\mathbb{N}^{\leq\mathbb{N}}_*$ with no finite subcover.

Lemma 2.3. *Suppose that $t \in \mathbb{N}^{<\mathbb{N}}$ and no finite set $\mathcal{V} \subseteq \mathcal{U}$ covers $\mathcal{N}_t$. Then there exists $j \in \mathbb{N}$ such that no finite set $\mathcal{V} \subseteq \mathcal{U}$ covers $\mathcal{N}_{t \frown (j)}$.*

Proof. Fix $U \in \mathcal{U}$ containing $t \frown (\infty)$. Proposition 2.1 then yields $i \in \mathbb{N}$ with $\mathcal{N}_t \setminus (\{t\} \cup \bigcup_{j<i} \mathcal{N}_{t \frown (j)}) \subseteq U$, in which case no finite set $\mathcal{V} \subseteq \mathcal{U}$ covers $\bigcup_{j<i} \mathcal{N}_{t \frown (j)}$, and it follows that there exists $j < i$ for which no finite set $\mathcal{V} \subseteq \mathcal{U}$ covers $\mathcal{N}_{t \frown (j)}$. ⊠

By recursively applying Lemma 2.3, we obtain $b \in \mathbb{N}^\mathbb{N}$ such that for no $i \in \mathbb{N}$ is there a finite set $\mathcal{V} \subseteq \mathcal{U}$ covering $\mathcal{N}_{b \upharpoonright i}$. But Proposition 2.1 implies that every open neighborhood of b contains such a set. ⊠

Given a countable set I and a topological space X, we say that a sequence $(x_i)_{i \in I} \in X^I$ *converges* to a point $x \in X$, or $x_i \to x$, if for every open neighborhood U of x there are only finitely many $i \in I$ with $x_i \notin U$. When I and X are equipped with partial orders $\leq_I$ and $\leq_X$, we say that $(x_i)_{i \in I}$ is *decreasing* if $i \leq_I j \implies x_j \leq_X x_i$ for all $i, j \in I$.

Proposition 2.4. *The space $\mathbb{N}^{\leq\mathbb{N}}_*$ has a compatible ultrametric.*

Proof. Fix a decreasing sequence $(\epsilon_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$ of positive real numbers converging to zero. Set $d(a, a) = 0$ for all $a \in \mathbb{N}^{\leq\mathbb{N}}_*$, as well as $d(a, b) = \max\{\epsilon_t \mid t \in \{a \upharpoonright i(a, b), b \upharpoonright i(a, b)\} \cap \mathbb{N}^{<\mathbb{N}}\}$ for all distinct $a, b \in \mathbb{N}^{\leq\mathbb{N}}_*$, where $i(a, b) = \min\{i \in \mathbb{N} \mid a \upharpoonright i \neq b \upharpoonright i\}$. We also set $i(a, a) = \infty$ for all $a \in \mathbb{N}^{\leq\mathbb{N}}_*$.

To see that d is an ultrametric, suppose that $a, b, c \in \mathbb{N}^{\leq\mathbb{N}}_*$. Note that if $i(a, c) < \max\{i(a, b), i(b, c)\}$, then $d(a, c) \in \{d(b, c), d(a, b)\}$, so $d(a, c) \leq \max\{d(a, b), d(b, c)\}$. And if $i(a, c) = \max\{i(a, b), i(b, c)\}$, then setting $i = i(a, b) = i(a, c) = i(b, c)$, it follows that

$$\begin{aligned} d(a, c) &= \max\{\epsilon_t \mid t \in \{a \upharpoonright i, c \upharpoonright i\} \cap \mathbb{N}^{<\mathbb{N}}\} \\ &\leq \max\{\epsilon_t \mid t \in \{a \upharpoonright i, b \upharpoonright i, c \upharpoonright i\} \cap \mathbb{N}^{<\mathbb{N}}\} \\ &= \max\{d(a, b), d(b, c)\}. \end{aligned}$$

And finally, if $i(a, c) > \max\{i(a, b), i(b, c)\}$, then setting $\epsilon = d(a, b) = d(b, c)$ and $t = a \upharpoonright i(a, b) = c \upharpoonright i(b, c)$, it follows that $d(a, c) \leq \epsilon_t \leq \epsilon$, and therefore $d(a, c) \leq \max\{d(a, b), d(b, c)\}$.

Given $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$, set $\epsilon = \min\{\epsilon_t, \min\{\epsilon_{t \frown (j)} \mid j < i\}\}$. Then $\{t\} = \mathcal{B}(t, \epsilon_t)$ and $\mathcal{N}_t \setminus (\{t\} \cup \bigcup_{j<i} \mathcal{N}_{t \frown (j)}) = \mathcal{B}(\mathcal{N}_t \setminus (\{t\} \cup \bigcup_{j<i} \mathcal{N}_{t \frown (j)}), \epsilon)$, so Proposition 2.1 ensures that every open subset of $\mathbb{N}^{\leq\mathbb{N}}_*$ is d-open.

Given $b \in \mathbb{N}^{\mathbb{N}}$ and $\epsilon > 0$, fix $i \in \mathbb{N}$ with $\epsilon_{b \upharpoonright i} < \epsilon$, set $t = b \upharpoonright i$, and note that $\mathcal{N}_t \subseteq \mathcal{B}(b, \epsilon)$. Given $t \in \mathbb{N}^{<\mathbb{N}}$ and $\epsilon > 0$, fix $i \in \mathbb{N}$ with $\epsilon_{t \frown (j)} < \epsilon$ for all $j \geq i$, and observe that $\mathcal{N}_t \setminus (\{t\} \cup \bigcup_{j<i} \mathcal{N}_{t \frown (j)}) \subseteq \mathcal{B}(t \frown (\infty), \epsilon)$. Thus every d-open subset of $\mathbb{N}^{\leq \mathbb{N}}_*$ is open. ⊠

It follows that $\mathbb{N}^{\leq \mathbb{N}}_*$ is Polish. As the space $\mathbb{N}^{\mathbb{N}}_* = \mathbb{N}^{\leq \mathbb{N}}_* \setminus \mathbb{N}^{<\mathbb{N}}$ is a perfect subset of $\mathbb{N}^{\leq \mathbb{N}}_*$, a result of Brouwer's ensures that it is homeomorphic to $2^{\mathbb{N}}$ (see, for example, [Kec95, Theorem 7.4]).

3. The Hurewicz dichotomies

As our first application of Theorem 1.1, we establish a generalization of the Hurewicz dichotomy for F_σ sets.

Given a countable set I, we say that a sequence $(X_i)_{i \in I}$ of subsets of X *converges* to a point $x \in X$, or $X_i \to x$, if for every open neighborhood U of x all but finitely many $i \in I$ have the property that $X_i \subseteq U$.

Theorem 3.1 (Hurewicz, Kechris-Louveau-Woodin)**.** *Suppose that X is a metric space. Then for every analytic set $A \subseteq X$, exactly one of the following holds:*

1. *The set A is F_σ.*
2. *There is a continuous reduction $\phi \colon \mathbb{N}^{\mathbb{N}}_* \to X$ of $\mathbb{N}^{\mathbb{N}}$ to A.*

Proof. To see that conditions (1) and (2) are mutually exclusive, suppose that $(C_i)_{i \in \mathbb{N}}$ is a sequence of closed sets whose union is A and $\phi \colon \mathbb{N}^{\mathbb{N}}_* \to X$ is a continuous reduction of $\mathbb{N}^{\mathbb{N}}$ to A, recursively construct $b \in \mathbb{N}^{\mathbb{N}}$ such that for no $i \in \mathbb{N}$ is there an extension of $b \upharpoonright (i+1)$ in $\phi^{-1}(C_i) \cap \mathbb{N}^{\mathbb{N}}$, and note that $b \notin \bigcup_{i \in \mathbb{N}} \phi^{-1}(C_i)$, a contradiction.

To see that at least one of the two conditions holds, let H denote the $\mathbb{N}$-dimensional dihypergraph on A consisting of all sequences $(x_n)_{n \in \mathbb{N}}$ of points in A converging to a point in $\sim A$. As the closure of every H-independent set in X is contained in A, it follows that if there is an $\mathbb{N}$-coloring of H, then A is F_σ.

Lemma 3.2. *The $\mathbb{N}$-dimensional dihypergraph H is box open.*

Proof. Suppose that $(x_n)_{n \in \mathbb{N}} \in H$, fix a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ of positive real numbers converging to zero, and note that $\prod_{n \in \mathbb{N}} A \cap \mathcal{B}(x_n, \epsilon_n)$ is a box-open subset of $A^{\mathbb{N}}$ contained in H. ⊠

By Theorem 1.1 and Lemma 3.2, we can assume that there is a continuous homomorphism $\phi \colon \mathbb{N}^{\mathbb{N}} \to A$ from $\mathbb{H}$ to H.

Lemma 3.3. *Suppose that $t \in \mathbb{N}^{<\mathbb{N}}$. Then there exists $x_t \in \sim A$ for which $\phi(\mathcal{N}_{t \frown (i)}) \to x_t$.*

Proof. As ϕ is a homomorphism from $\mathbb{H}$ to H, it follows that if $(x_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} \phi(\mathcal{N}_{t\frown(i)})$ then there exists $x_t \in \sim A$ for which $x_i \to x_t$. If it is not the case that $\phi(\mathcal{N}_{t\frown(i)}) \to x_t$, then there is an open neighborhood U of x_t for which there is an infinite set $I \subseteq \mathbb{N}$ such that for all $i \in I$, there exists $y_i \in \phi(\mathcal{N}_{t\frown(i)}) \setminus U$. By shrinking I if necessary, we can assume that it is also co-infinite, in which case $(x_i)_{i\in\sim I} \cup (y_i)_{i\in I}$ does not converge, a contradiction. ⊠

Now let $\overline{\phi}$ denote the extension of ϕ to a function on $\mathbb{N}^{\mathbb{N}}_*$ given by $\overline{\phi}(t \frown (\infty)) = x_t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$. Clearly ϕ is a reduction of $\mathbb{N}^{\mathbb{N}}$ to A.

Lemma 3.4. *Suppose that $t \in \mathbb{N}^{<\mathbb{N}}$. Then $\overline{\phi}(\mathcal{N}_t) \subseteq \overline{\phi(\mathcal{N}_t)}$.*

Proof. Simply note that $\overline{\phi}(\mathcal{N}_t \setminus \mathbb{N}^{\mathbb{N}}) \subseteq \bigcup_{s\sqsupseteq t} \overline{\phi(\mathcal{N}_s)} \subseteq \overline{\phi(\mathcal{N}_t)}$. ⊠

To see that $\overline{\phi}$ is continuous, suppose that $c \in \mathbb{N}^{\mathbb{N}}_*$ and $U \subseteq X$ is an open neighborhood of $\overline{\phi}(c)$, and fix an open neighborhood $V \subseteq X$ of $\overline{\phi}(c)$ whose closure is contained in U. If $c \in \mathbb{N}^{\mathbb{N}}$, then the continuity of ϕ yields $i \in \mathbb{N}$ for which $\phi(\mathcal{N}_{c\restriction i}) \subseteq V$, so Lemma 3.4 ensures that $\overline{\phi}(\mathcal{N}_{c\restriction i}) \subseteq \overline{\phi(\mathcal{N}_{c\restriction i})} \subseteq \overline{V} \subseteq U$. If $c \in \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$, then there exists $t \in \mathbb{N}^{<\mathbb{N}}$ for which $c = t \frown (\infty)$, in which case Lemma 3.3 yields $i \in \mathbb{N}$ for which $\bigcup_{j\geq i} \phi(\mathcal{N}_{t\frown(j)}) \subseteq V$, so $\overline{\phi}(\mathcal{N}_t \setminus \bigcup_{j<i} \mathcal{N}_{t\frown(j)}) = \{x_t\} \cup \bigcup_{j\geq i} \overline{\phi}(\mathcal{N}_{t\frown(j)}) \subseteq \{x_t\} \cup \bigcup_{j\geq i} \overline{\phi(\mathcal{N}_{t\frown(j)})} \subseteq \overline{V} \subseteq U$, by Lemma 3.4. ⊠

Remark 3.5. While we view metric spaces as the natural generality for such results, the above argument can be easily pushed through in more general topological spaces. In fact, a similar comment applies to all of our applications of Theorem 1.1.

A set is K_σ if it is a union of countably-many compact sets. While the following fact can also be obtained by applying Theorem 3.1 in an appropriate compactification, we use Theorem 1.1 directly.

Theorem 3.6 (Hurewicz, Kechris, Saint Raymond)**.** *Suppose that X is a metric space and $A \subseteq X$ is analytic. Then exactly one of the following holds:*

1. *The set A is contained in a K_σ subset of X.*
2. *There is a closed continuous injection $\phi \colon \mathbb{N}^{\mathbb{N}} \to X$ with the property that $\phi(\mathbb{N}^{\mathbb{N}}) \subseteq A$.*

Proof. To see that conditions (1) and (2) are mutually exclusive, suppose that $(K_i)_{i\in\mathbb{N}}$ is a sequence of compact subsets of X whose union contains A and $\phi \colon \mathbb{N}^{\mathbb{N}} \to X$ is a closed continuous injection for which $\phi(\mathbb{N}^{\mathbb{N}}) \subseteq A$, observe that $(\phi^{-1}(K_i))_{i\in\mathbb{N}}$ is a sequence of compact sets whose union is $\mathbb{N}^{\mathbb{N}}$, recursively construct $b \in \mathbb{N}^{\mathbb{N}}$ such that for no

$i \in \mathbb{N}$ is there an extension of $b \upharpoonright (i+1)$ in $\phi^{-1}(K_i)$, and note that $b \notin \bigcup_{i \in \mathbb{N}} \phi^{-1}(K_i)$, a contradiction.

Let H be the $\mathbb{N}$-dimensional dihypergraph on A consisting of all injective sequences $(x_n)_{n \in \mathbb{N}}$ with no subsequence converging to a point of X. As every H-independent set has compact closure within X, it follows that if there is an $\mathbb{N}$-coloring of H, then A is contained in a K_σ subset of X.

Lemma 3.7. *The $\mathbb{N}$-dimensional dihypergraph H is box open.*

Proof. Suppose that $(x_n)_{n \in \mathbb{N}} \in H$, and observe that if $(\epsilon_n)_{n \in \mathbb{N}}$ is a sequence of positive real numbers with the property that $\epsilon_n \to 0$ and $\epsilon_n \leq d_X(x_m, x_n)/2$ for all distinct $m, n \in \mathbb{N}$, then $A \cap \prod_{n \in \mathbb{N}} \mathcal{B}(x_n, \epsilon_n)$ is a box-open subset of $A^{\mathbb{N}}$ contained in H. ⊠

By Theorem 1.1 and Lemma 3.7, we can assume that there is a continuous homomorphism $\phi \colon \mathbb{N}^{\mathbb{N}} \to A$ from $\mathbb{H}$ to H.

Lemma 3.8. *The function ϕ is closed.*

Proof. It is sufficient to show that every sequence $(b_n)_{n \in \mathbb{N}}$ of points in $\mathbb{N}^{\mathbb{N}}$ for which $(\phi(b_n))_{n \in \mathbb{N}}$ converges to some point $x \in X$ has a convergent subsequence. Suppose, towards a contradiction, that there does not exist $b \in \mathbb{N}^{\mathbb{N}}$ such that $b_n(i) < b(i)$ for all $i, n \in \mathbb{N}$. Then by passing to a subsequence, we can assume that there exists $k \in \mathbb{N}$ for which every two distinct points along the sequence differ from one another for the first time on their k^{th} coordinates. By passing to a further subsequence, we can assume that $(b_n)_{n \in \mathbb{N}}$ is a subsequence of an element of $\mathbb{H}$, so $(\phi(b_n))_{n \in \mathbb{N}}$ is a subsequence of an element of H, contradicting the fact that $\phi(b_n) \to x$. ⊠

It only remains to note that ϕ is injective, by Proposition 1.4. ⊠

4. Dichotomies for functions

Here we give three dichotomies characterizing the classes of Borel functions that are not Baire class one, Borel functions that are not G_δ-measurable, and Borel functions that are not σ-continuous with closed witnesses. These results serve a role similar to that of the $\mathbb{G}_0$ dichotomy or effective descriptive set theory in the proofs of other dichotomy theorems, in that they reduce the problem of further characterizing the corresponding properties to purely topological special cases.

Theorem 4.1. *Suppose that X is an analytic metric space, Y is a topological space, and $\phi \colon X \to Y$ is Borel. Then exactly one of the following holds:*

(1) *The function ϕ is Baire class one.*
(2) *There is a continuous function $\psi\colon \mathbb{N}^{\mathbb{N}}_* \to X$ with the property that $(\phi \circ \psi)(\mathbb{N}^{\mathbb{N}}) \cap \overline{(\phi \circ \psi)(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})} = \emptyset$.*

Proof. The fact that $\mathbb{N}^{\mathbb{N}}$ is not an F_σ subset of $\mathbb{N}^{\mathbb{N}}_*$ ensures that the two conditions are mutually exclusive. To see that at least one of them holds, note that if ϕ is not Baire class one, then there is an open set $U \subseteq Y$ for which $\phi^{-1}(U)$ is not F_σ. As $\phi^{-1}(U)$ is Borel and Borel subsets of analytic Hausdorff spaces are analytic, Theorem 3.1 yields a continuous reduction $\psi\colon \mathbb{N}^{\mathbb{N}}_* \to X$ of $\mathbb{N}^{\mathbb{N}}$ to $\phi^{-1}(U)$. But clearly any such function is as desired. ⊠

Theorem 4.2. *Suppose that X is an analytic metric space, Y is a topological space, and $\phi\colon X \to Y$ is Borel. Then exactly one of the following holds:*

(1) *The function ϕ is G_δ-measurable.*
(2) *There is a continuous function $\psi\colon \mathbb{N}^{\mathbb{N}}_* \to X$ with the property that $\overline{(\phi \circ \psi)(\mathbb{N}^{\mathbb{N}})} \cap (\phi \circ \psi)(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}) = \emptyset$.*

Proof. The fact that $\mathbb{N}^{\mathbb{N}}$ is not an F_σ subset of $\mathbb{N}^{\mathbb{N}}_*$ ensures that the two conditions are mutually exclusive. To see that at least one of them holds, note that if ϕ is not G_δ-measurable, then there is a closed set $C \subseteq Y$ for which $\phi^{-1}(C)$ is not F_σ. As $\phi^{-1}(C)$ is Borel, and Borel subsets of analytic Hausdorff spaces are analytic, Theorem 3.1 yields a continuous reduction $\psi\colon \mathbb{N}^{\mathbb{N}}_* \to X$ of $\mathbb{N}^{\mathbb{N}}$ to $\phi^{-1}(C)$. But clearly any such function is as desired. ⊠

Given a countable set I, we say that a point $x \in X$ is a *limit point* of a sequence $(X_i)_{i \in I}$ of subsets of X if for every open neighborhood U of x all but finitely many $i \in I$ have the property that $U \cap X_i \neq \emptyset$.

While the following fact can also be obtained by applying Theorem 3.1 in an appropriate compactification, we use Theorem 1.1 directly.

Theorem 4.3. *Suppose that X is an analytic metric space, Y is a separable metric space, and $\phi\colon X \to Y$ is Borel. Then exactly one of the following holds:*

(1) *The function ϕ is σ-continuous with closed witnesses.*
(2) *There is a continuous function $\psi\colon \mathbb{N}^{\mathbb{N}}_* \to X$ such that for no $t \in \mathbb{N}^{<\mathbb{N}}$ is $(\phi \circ \psi)(t \frown (\infty))$ a limit point of $((\phi \circ \psi)(\mathcal{N}_{t \frown (i)}))_{i \in \mathbb{N}}$.*

Proof. To see that conditions (1) and (2) are mutually exclusive, suppose that $(C_i)_{i \in \mathbb{N}}$ is a sequence of closed subsets of X on which ϕ is continuous whose union is X and $\psi\colon \mathbb{N}^{\mathbb{N}}_* \to X$ is a continuous function with the property that for no $t \in \mathbb{N}^{<\mathbb{N}}$ is $(\phi \circ \psi)(t \frown (\infty))$ a limit point of $((\phi \circ \psi)(\mathcal{N}_{t \frown (i)}))_{i \in \mathbb{N}}$, recursively construct $b \in \mathbb{N}^{\mathbb{N}}$ such that for no $i \in \mathbb{N}$ is there an extension of $b \upharpoonright (i+1)$ in $\psi^{-1}(C_i) \cap \mathbb{N}^{\mathbb{N}}$, and note that $b \notin \bigcup_{i \in \mathbb{N}} \psi^{-1}(C_i)$, a contradiction.

Let H denote the $\mathbb{N}$-dimensional dihypergraph on graph(ϕ) consisting of all sequences $(x_n, y_n)_{n \in \mathbb{N}}$ of points in graph(ϕ) with the property that $(x_n)_{n \in \mathbb{N}}$ converges but $\phi(\lim_{n \to \infty} x_n) \neq \lim_{n \to \infty} y_n$.

Lemma 4.4. *Suppose that $R \subseteq$ graph(ϕ) is H-independent. Then ϕ is continuous on the closure of* $\mathrm{proj}_X(R)$.

Proof. Suppose that $(x_n)_{n \in \mathbb{N}}$ is a sequence of points in the closure of $\mathrm{proj}_X(R)$ converging to some point x. For all $n \in \mathbb{N}$, fix a sequence $(x_{m,n})_{m \in \mathbb{N}}$ of points in $\mathrm{proj}_X(R)$ converging to x_n. If $(\epsilon_n)_{n \in \mathbb{N}}$ is a sequence of positive real numbers converging to zero, then the H-independence of R ensures that for all $n \in \mathbb{N}$, there exists $m_n \in \mathbb{N}$ such that $d(x_{m_n,n}, x_n) < \epsilon_n$ and $d(\phi(x_{m_n,n}), \phi(x_n)) < \epsilon_n$. But then $x_{m_n,n} \to x$, so one more appeal to the H-independence of R yields that $\phi(x_{m_n,n}) \to \phi(x)$, thus $\phi(x_n) \to \phi(x)$. ⊠

Lemma 4.4 ensures that if there is an $\mathbb{N}$-coloring of H, then ϕ is σ-continuous with closed witnesses.

Lemma 4.5. *The $\mathbb{N}$-dimensional dihypergraph H is box open.*

Proof. Suppose that $(x_n, y_n)_{n \in \mathbb{N}} \in H$, fix a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ of positive real numbers converging to zero, and observe that $\prod_{n \in \mathbb{N}} \mathrm{graph}(\phi) \cap \mathcal{B}((x_n, y_n), \epsilon_n) \subseteq H$ is a box-open subset of graph$(\phi)^{\mathbb{N}}$. ⊠

As graphs of Borel functions between analytic Hausdorff spaces are analytic, Theorem 1.1 and Lemma 4.5 yield a continuous homomorphism $\psi \colon \mathbb{N}^{\mathbb{N}} \to \mathrm{graph}(\phi)$ from $\mathbb{H}$ to H. Set $\psi_X = \mathrm{proj}_X \circ \psi$.

Lemma 4.6. *Suppose that $t \in \mathbb{N}^{<\mathbb{N}}$. Then there exists $x_t \in X$ with the property that $\psi_X(\mathcal{N}_{t \frown (i)}) \to x_t$ but $\phi(x_t)$ is not a limit point of $((\phi \circ \psi_X)(\mathcal{N}_{t \frown (i)}))_{i \in \mathbb{N}}$.*

Proof. As ψ is a homomorphism from $\mathbb{H}$ to H, it follows that if $(x_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} \psi_X(\mathcal{N}_{t \frown (i)})$ then there exists $x_t \in X$ with the property that $x_i \to x_t$ but $\phi(x_t) \neq \lim_{i \to \infty} \phi(x_i)$.

If it is not the case that $\psi_X(\mathcal{N}_{t \frown (i)}) \to x_t$, then there is an open neighborhood U of x_t for which there is an infinite set $I \subseteq \mathbb{N}$ such that for all $i \in I$, there exists $y_i \in \psi_X(\mathcal{N}_{t \frown (i)}) \setminus U$. By shrinking I if necessary, we can assume that it is also co-infinite, in which case $(x_i)_{i \in \sim I} \cup (y_i)_{i \in I}$ does not converge, a contradiction.

It follows that if $(y_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} \psi_X(\mathcal{N}_{t \frown (i)})$, then $\phi(x_t) \neq \lim_{i \to \infty} \phi(y_i)$, so $\phi(x_t)$ is not a limit point of $((\phi \circ \psi_X)(\mathcal{N}_{t \frown (i)}))_{i \in \mathbb{N}}$. ⊠

Extend ψ_X to a function on $\mathbb{N}^{\mathbb{N}}_*$ by setting $\overline{\psi_X}(t \frown (\infty)) = x_t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$. To see that $\overline{\psi_X}$ is continuous, suppose that $c \in \mathbb{N}^{\mathbb{N}}_*$ and $U \subseteq X$ is an open neighborhood of $\overline{\psi_X}(c)$, and fix an open neighborhood $V \subseteq X$ of $\overline{\psi_X}(c)$ whose closure is contained in U. If $c \in \mathbb{N}^{\mathbb{N}}$, then the continuity of ψ_X yields $i \in \mathbb{N}$ for which $\psi_X(\mathcal{N}_{c \upharpoonright i}) \subseteq V$, so Lemma 3.4 ensures that $\overline{\psi_X}(\mathcal{N}_{c\upharpoonright i}) \subseteq \overline{\psi_X(\mathcal{N}_{c\upharpoonright i})} \subseteq \overline{V} \subseteq U$. If $c \in \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$, then there exists $t \in \mathbb{N}^{<\mathbb{N}}$ for which $c = t \frown (\infty)$, in which case Lemma 3.3 yields $i \in \mathbb{N}$ for which $\bigcup_{j \geq i} \psi_X(\mathcal{N}_{t \frown (j)}) \subseteq V$, so $\overline{\psi_X}(\mathcal{N}_t \setminus \bigcup_{j<i} \mathcal{N}_{t\frown(j)}) = \{x_t\} \cup \bigcup_{j \geq i} \overline{\psi_X}(\mathcal{N}_{t\frown(j)}) \subseteq \{x_t\} \cup \bigcup_{j\geq i} \overline{\psi_X(\mathcal{N}_{t\frown(j)})} \subseteq U$, by Lemma 3.4. ⊠

5. Meet embeddings

The *meet* of sequences $s, t \in \mathbb{N}^{<\mathbb{N}}$ is the maximal common initial segment of s and t. A $\wedge$-*embedding* is an injection $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $\pi(s \wedge t) = \pi(s) \wedge \pi(t)$ for all $s, t \in \mathbb{N}^{<\mathbb{N}}$.

Proposition 5.1. *Suppose that $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$. Then π is a $\wedge$-embedding if and only if the following conditions hold:*

(1) $\forall i \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}}\ \pi(t) \sqsubset \pi(t \frown (i))$.
(2) $\forall i, j \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}}$
$$(i \neq j \implies \pi(t \frown (i))(|\pi(t)|) \neq \pi(t \frown (j))(|\pi(t)|)).$$

Proof. Suppose first that π is a $\wedge$-embedding. To see that condition (1) holds, observe that if $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$, then $\pi(t) = \pi(t) \wedge \pi(t \frown (i))$, so $\pi(t) \sqsubseteq \pi(t \frown (i))$, thus $\pi(t) \sqsubset \pi(t \frown (i))$. And to see that condition (2) holds, note that if $i, j \in \mathbb{N}$ are distinct and $t \in \mathbb{N}^{<\mathbb{N}}$, then $\pi(t) = \pi(t \frown (i)) \wedge \pi(t \frown (j))$, so $\pi(t \frown (i))(|\pi(t)|) \neq \pi(t \frown (j))(|\pi(t)|)$.

Suppose now that π satisfies conditions (1) and (2). To see that π is a $\wedge$-embedding, suppose that $s, t \in \mathbb{N}^{<\mathbb{N}}$ are distinct, and define $r = s \wedge t$. By reversing the roles of s and t if necessary, we can assume that $|s| > |r|$, so $\pi(r) \sqsubset \pi(s)$, thus either $r = t$ or $|t| > |r|$ and $\pi(s)(|\pi(r)|) \neq \pi(t)(|\pi(r)|)$, and in both cases it follows that $\pi(s) \neq \pi(t)$ and $\pi(r) = \pi(s) \wedge \pi(t)$. ⊠

There is a simple but useful means of amalgamating appropriately indexed families of $\wedge$-embeddings.

Proposition 5.2. *Suppose that $(\pi_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$ is a sequence of $\wedge$-embeddings with the property that $\pi_t(\mathbb{N}^{<\mathbb{N}}) \subseteq \mathcal{N}_t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$. Then the function $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ given by $\pi(t) = (\prod_{n \leq |t|} \pi_{t \upharpoonright n})(t)$ is also a $\wedge$-embedding.*

Proof. Note that if $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$, then $t \frown (i) \sqsubseteq \pi_{t \frown (i)}(t \frown (i))$, so Proposition 5.1 ensures that $(\prod_{n \leq |t|} \pi_{t\upharpoonright n})(t \frown (i)) \sqsubseteq \pi(t \frown (i))$, thus $\pi(t) \sqsubset (\prod_{n \leq |t|} \pi_{t \upharpoonright n})(t \frown (i)) \sqsubseteq \pi(t \frown (i))$. It also implies that if

$i \neq j$, then $(\prod_{n \leq |t|} \pi_{t \upharpoonright n})(t \frown (i))(|\pi(t)|) \neq (\prod_{n \leq |t|} \pi_{t \upharpoonright n})(t \frown (j))(|\pi(t)|)$, so $\pi(t \frown (i))(|\pi(t)|) \neq \pi(t \frown (j))(|\pi(t)|)$. One last application of Proposition 5.1 therefore ensures that π is a $\wedge$-embedding. ⊠

We next consider the connection between $\wedge$-embeddings and closed continuous embeddings.

Proposition 5.3. *Every $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ has a unique extension to a (necessarily injective) continuous map $\overline{\pi} \colon \mathbb{N}_*^{\leq \mathbb{N}} \to \mathbb{N}_*^{\leq \mathbb{N}}$, given by $\overline{\pi}(b) = \bigcup_{i \in \mathbb{N}} \pi(b \upharpoonright i)$ and $\overline{\pi}(t \frown (\infty)) = \pi(t) \frown (\infty)$ for all $b \in \mathbb{N}^\mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$.*

Proof. Suppose that $\overline{\pi} \colon \mathbb{N}_*^{\leq \mathbb{N}} \to \mathbb{N}_*^{\leq \mathbb{N}}$ is a continuous extension of π. If $b \in \mathbb{N}^\mathbb{N}$, then $b \upharpoonright i \to b$, and since $(\pi(b \upharpoonright i))_{i \in \mathbb{N}}$ is strictly increasing by Proposition 5.1, it follows that $\overline{\pi}(b) = \bigcup_{i \in \mathbb{N}} \pi(b \upharpoonright i)$. If $t \in \mathbb{N}^{<\mathbb{N}}$, then $t \frown (i) \to t \frown (\infty)$, and since $\pi(t) = \pi(t \frown (i)) \wedge \pi(t \frown (j))$ for all distinct $i, j \in \mathbb{N}$, it follows that $\overline{\pi}(t \frown (\infty)) = \pi(t) \frown (\infty)$.

To see that these constraints actually define a continuous function, note that if $t \in \mathbb{N}^{<\mathbb{N}}$, then either $\overline{\pi}^{-1}(\mathcal{N}_t) = \emptyset$ or there exists $s \in \mathbb{N}^{<\mathbb{N}}$ of minimal length with $t \sqsubseteq \pi(s)$, in which case $\overline{\pi}^{-1}(\mathcal{N}_t) = \mathcal{N}_s$.

To see that $\overline{\pi}$ is injective, it is enough to check that its restriction to $\mathbb{N}^\mathbb{N}$ is injective. Towards this end, suppose that $a, b \in \mathbb{N}^\mathbb{N}$ are distinct, fix $i \in \mathbb{N}$ least for which $a(i) \neq b(i)$, set $t = a \upharpoonright i = b \upharpoonright i$, and observe that $\pi(t \frown (a(i)))(|\pi(t)|) \neq \pi(t \frown (b(i)))(|\pi(t)|)$ by Proposition 5.1, thus $\overline{\pi}(a)$ and $\overline{\pi}(b)$ are distinct. ⊠

Remark 5.4. It follows that the extension associated with the composition of two $\wedge$-embeddings is the composition of their extensions.

Compactness ensures that if π is a $\wedge$-embedding, then $\overline{\pi}$ and $\overline{\pi} \upharpoonright \mathbb{N}_*^\mathbb{N}$ are closed continuous embeddings. The following observations show that the two-sided restrictions $\overline{\pi} \upharpoonright \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and $\overline{\pi} \upharpoonright \mathbb{N}_*^\mathbb{N} \setminus \mathbb{N}^\mathbb{N} \to \mathbb{N}_*^\mathbb{N} \setminus \mathbb{N}^\mathbb{N}$ are also closed continuous embeddings.

Proposition 5.5. *Suppose that $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ is a $\wedge$-embedding. Then $\overline{\pi} \upharpoonright \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is closed.*

Proof. It is sufficient to show that every sequence $(b_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{N}^\mathbb{N}$ for which $(\overline{\pi}(b_n))_{n \in \mathbb{N}}$ converges to an element of $\mathbb{N}^\mathbb{N}$ is itself convergent to an element of $\mathbb{N}^\mathbb{N}$. As $(\overline{\pi}(b_n) \upharpoonright i)_{n \in \mathbb{N}}$ is eventually constant for all $i \in \mathbb{N}$, a simple induction shows that $(b_n \upharpoonright i)_{n \in \mathbb{N}}$ is also eventually constant for all $i \in \mathbb{N}$, so $(b_n)_{n \in \mathbb{N}}$ converges to an element of $\mathbb{N}^\mathbb{N}$. ⊠

Proposition 5.6. *Suppose that $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ is a $\wedge$-embedding. Then $\overline{\pi} \upharpoonright \mathbb{N}_*^\mathbb{N} \setminus \mathbb{N}^\mathbb{N} \to \mathbb{N}_*^\mathbb{N} \setminus \mathbb{N}^\mathbb{N}$ is closed.*

Proof. It is sufficient to show that every sequence $(s_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{N}^{<\mathbb{N}}$ such that $(\pi(s_n))_{n \in \mathbb{N}}$ converges to $t \frown (\infty)$ for some $t \in \mathbb{N}^{<\mathbb{N}}$ has a subsequence converging to an element of $\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$. By passing to a subsequence, we can assume that $\pi(s_m) \wedge \pi(s_n) = t$ for all distinct $m, n \in \mathbb{N}$. Let s be the $\sqsubseteq$-minimal element of $\mathbb{N}^{<\mathbb{N}}$ for which $t \sqsubseteq \pi(s)$. Then $s_m \wedge s_n = s$ for all distinct $m, n \in \mathbb{N}$, thus $s_n \to s \frown (\infty)$. ⊠

A set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is $\sqsubseteq$-*dense* if $\forall s \in \mathbb{N}^{<\mathbb{N}} \exists t \in T \; s \sqsubseteq t$. More generally, a set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is $\sqsubseteq$-*dense* below $r \in \mathbb{N}^{<\mathbb{N}}$ if $\forall s \in \mathbb{N}^{<\mathbb{N}} \exists t \in T \; r \frown s \sqsubseteq t$.

Proposition 5.7. *Suppose that $T \subseteq \mathbb{N}^{<\mathbb{N}}$. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $\pi(\mathbb{N}^{<\mathbb{N}}) \subseteq T$ or $\pi(\mathbb{N}^{<\mathbb{N}}) \subseteq {\sim}T$.*

Proof. Fix $S \in \{T, {\sim}T\}$ which is $\sqsubseteq$-dense below some $s \in \mathbb{N}^{<\mathbb{N}}$, and recursively construct a function $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_s \cap S$ with the property that $\pi(t) \frown (i) \sqsubseteq \pi(t \frown (i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$. ⊠

Proposition 5.8. *Suppose that $C \subseteq \mathbb{N}^{\mathbb{N}}$ is a non-meager set with the Baire property. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ with the property that $\overline{\pi}(\mathbb{N}^{\mathbb{N}}) \subseteq C$.*

Proof. Fix $s \in \mathbb{N}^{<\mathbb{N}}$ for which C is comeager in $\mathcal{N}_s \cap \mathbb{N}^{\mathbb{N}}$, as well as dense open sets $U_n \subseteq \mathcal{N}_s \cap \mathbb{N}^{\mathbb{N}}$ with the property that $\bigcap_{n \in \mathbb{N}} U_n \subseteq C$, set $T_n = \{t \in \mathbb{N}^{<\mathbb{N}} \mid \mathcal{N}_t \cap \mathbb{N}^{\mathbb{N}} \subseteq U_n\}$ for all $n \in \mathbb{N}$, and recursively construct a function $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_s \cap \mathbb{N}^{<\mathbb{N}}$ such that $\pi(\mathbb{N}^n) \subseteq T_n$ for all $n \in \mathbb{N}$ and $\pi(t) \frown (i) \sqsubseteq \pi(t \frown (i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$. ⊠

6. The Jayne-Rogers theorem

It is trivial to see that if a function is σ-continuous with closed witnesses, then it is G_δ-measurable. Here we establish the Jayne-Rogers theorem that the converse holds (see [JR82]) by weakening G_δ-measurability to σ-continuity with closed witnesses in condition (1) of Theorem 4.2. However, we first establish several preliminary facts concerning $\wedge$-embeddings.

Proposition 6.1. *Suppose that X is a second countable topological space and $\phi \colon \mathbb{N}^{\mathbb{N}} \to X$ is Baire measurable. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ for which $\phi \circ \overline{\pi}$ is continuous.*

Proof. Fix a comeager set $C \subseteq \mathbb{N}^{\mathbb{N}}$ on which ϕ is continuous, and appeal to Proposition 5.8 to obtain a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ with the property that $\overline{\pi}(\mathbb{N}^{\mathbb{N}}) \subseteq C$. ⊠

Proposition 6.2. *Suppose that X is a metric space and $\phi \colon \mathbb{N}^{\mathbb{N}} \to X$ is continuous. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ with the property that* $\operatorname{diam} \phi(\mathcal{N}_{\pi(t)}) \to 0$.

Proof. Fix a sequence $(\epsilon_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$ of positive real numbers converging to zero, note that the continuity of ϕ ensures that for all $t \in \mathbb{N}^{<\mathbb{N}}$ the set $T_t = \{s \in \mathbb{N}^{<\mathbb{N}} \mid \operatorname{diam} \phi(\mathcal{N}_s) < \epsilon_t\}$ is $\sqsubseteq$-dense, and recursively construct a function $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $\pi(t) \in T_t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$ and $\pi(t) \frown (i) \sqsubseteq \pi(t \frown (i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$. ⊠

Given a countable set I and a topological space X, we say that a sequence $(X_i)_{i \in I}$ of subsets of X is *discrete* if for all $x \in X$ there is an open neighborhood U of x such that all but finitely many $i \in I$ have the property that $U \cap X_i = \emptyset$.

Proposition 6.3. *Suppose that X is a metric space, $\phi \colon \mathbb{N}^\mathbb{N} \to X$, and $\operatorname{diam} \phi(\mathcal{N}_t) \to 0$. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $(\phi(\mathcal{N}_{\pi(t \frown (i))}))_{i \in \mathbb{N}}$ is convergent or discrete for all $t \in \mathbb{N}^{<\mathbb{N}}$.*

Proof. For each $t \in \mathbb{N}^{<\mathbb{N}}$, the fact that $\operatorname{diam} \phi(\mathcal{N}_t) \to 0$ ensures that there is an injection $\iota_t \colon \mathbb{N} \to \mathbb{N}$ for which $(\phi(\mathcal{N}_{t \frown (\iota_t(i))}))_{i \in \mathbb{N}}$ is convergent or discrete. Define $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ by choosing $\pi(\emptyset) \in \mathbb{N}^{<\mathbb{N}}$ arbitrarily and setting $\pi(t \frown (i)) = \pi(t) \frown (\iota_{\pi(t)}(i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$. ⊠

Proposition 6.4. *Suppose that X is a metric space and $\phi \colon \mathbb{N}^\mathbb{N} \to X$ is continuous. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $\phi \circ \overline{\pi}$ is constant or injective.*

Proof. If there exists $s \in \mathbb{N}^{<\mathbb{N}}$ for which $\phi \restriction \mathcal{N}_s$ is constant, then the function $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ given by $\pi(t) = s \frown t$ has the property that $\phi \circ \overline{\pi}$ is constant. A straightforward induction therefore allows us to assume that $\phi(\mathcal{N}_t)$ is infinite for all $t \in \mathbb{N}^{<\mathbb{N}}$.

Lemma 6.5. *For all $t \in \mathbb{N}^{<\mathbb{N}}$, there is a function $\iota_t \colon \mathbb{N} \to \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$ such that $(\iota_t(i)(0))_{i \in \mathbb{N}}$ is injective and the sets of the form $\phi(\mathcal{N}_{t \frown \iota_t(i)})$ for $i \in \mathbb{N}$ are pairwise disjoint.*

Proof. Fix extensions $b_i \in \mathbb{N}^\mathbb{N}$ of $t \frown (i)$ with the property that $\phi(b_i) \notin \{\phi(b_j) \mid j < i\}$ for all $i \in \mathbb{N}$. Fix a subsequence $(a_i)_{i \in \mathbb{N}}$ of $(b_i)_{i \in \mathbb{N}}$ for which $\{\phi(a_i) \mid i \in \mathbb{N}\}$ is discrete. For each $i \in \mathbb{N}$, fix $\epsilon_i > 0$ such that $\phi(a_j) \notin \mathcal{B}(\phi(a_i), \epsilon_i)$ for all $j \in \mathbb{N} \setminus \{i\}$, and fix $\iota_t(i) \in \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$ such that $t \frown \iota_t(i) \sqsubseteq a_i$ and $\phi(\mathcal{N}_{t \frown \iota_t(i)}) \subseteq \mathcal{B}(\phi(a_i), \epsilon_i/2)$. ⊠

Define $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ by choosing $\pi(\emptyset) \in \mathbb{N}^{<\mathbb{N}}$ arbitrarily and setting $\pi(t \frown (i)) = \pi(t) \frown \iota_{\pi(t)}(i)$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$. ⊠

Proposition 6.6. *Suppose that X is a metric space and $\phi \colon \mathbb{N}^\mathbb{N} \to X$ is a continuous function that does not have constant value $x \in X$. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $x \notin \overline{(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N})}$.*

Proof. Fix $s \in \mathbb{N}^{<\mathbb{N}}$ such that $x \notin \overline{\phi(\mathcal{N}_s)}$, and define $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ by $\pi(t) = s \frown t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$. ⊠

Proposition 6.7. *Suppose that X is a metric space and $\phi \colon \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \to X$. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ for which $\phi \circ \overline{\pi}$ is constant or injective.*

Proof. If for no finite set $F \subseteq \phi(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$ and $t \in \mathbb{N}^{<\mathbb{N}}$ is it the case that $\phi(\mathcal{N}_t) \subseteq F$, then fix an enumeration $(t_n)_{n \in \mathbb{N}}$ of $\mathbb{N}^{<\mathbb{N}}$ such that $t_m \sqsubseteq t_n \implies m \leq n$ for all $m, n \in \mathbb{N}$, and recursively construct $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $\phi(\pi(t_n) \frown (\infty)) \notin \{\phi(\pi(t_m) \frown (\infty)) \mid m < n\}$ and $\pi(t_n') \frown (n) \sqsubseteq \pi(t_n)$ for all $n > 0$, where t_n' is the maximal proper initial segment of t_n.

Otherwise, there exists $x \in \phi(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$ with the property that the set $S = \{s \in \mathbb{N}^{<\mathbb{N}} \mid \phi(s \frown (\infty)) = x\}$ is $\sqsubseteq$-dense below some $t \in \mathbb{N}^{<\mathbb{N}}$, in which case we can recursively construct a function $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_t \cap S$ with the property that $\pi(u) \frown (i) \sqsubseteq \pi(u \frown (i))$ for all $i \in \mathbb{N}$ and $u \in \mathbb{N}^{<\mathbb{N}}$. ⊠

Proposition 6.8. *Suppose that X is a metric space, $\phi \colon \mathbb{N}^{\mathbb{N}}_* \to X$ is a function for which $\phi \upharpoonright \mathbb{N}^{\mathbb{N}}$ is continuous and injective, $\phi \upharpoonright \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$ is injective, and $(\phi(\mathcal{N}_{t \frown (i)} \cap \mathbb{N}^{\mathbb{N}}))_{i \in \mathbb{N}}$ has only finitely-many limit points for all $t \in \mathbb{N}^{<\mathbb{N}}$, and $(t_n)_{n \in \mathbb{N}}$ is an enumeration of $\mathbb{N}^{<\mathbb{N}}$ with the property that $t_m \sqsubseteq t_n \implies m \leq n$ for all $m, n \in \mathbb{N}$. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that for no natural numbers $m < n$ is it the case that $(\phi \circ \overline{\pi})(t_m \frown (\infty))$ is in the closure of $(\phi \circ \overline{\pi})(\mathcal{N}_{t_n} \cap \mathbb{N}^{\mathbb{N}})$ or $(\phi \circ \overline{\pi})(t_n \frown (\infty))$ is a limit point of $((\phi \circ \overline{\pi})(\mathcal{N}_{t_m \frown (i)} \cap \mathbb{N}^{\mathbb{N}}))_{i \in \mathbb{N}}$.*

Proof. Note that if $u \in \mathbb{N}^{<\mathbb{N}}$ and $F \subseteq \mathbb{N}^{<\mathbb{N}}$ is finite, then the continuity and injectivity of $\phi \upharpoonright \mathbb{N}^{\mathbb{N}}$ yield an extension $v \in \mathbb{N}^{<\mathbb{N}}$ such that for no $t \in F$ is $\phi(t \frown (\infty))$ in the closure of $\phi(\mathcal{N}_v \cap \mathbb{N}^{\mathbb{N}})$, and the injectivity of $\phi \upharpoonright \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$ and the fact that $(\phi(\mathcal{N}_{t \frown (i)} \cap \mathbb{N}^{\mathbb{N}}))_{i \in \mathbb{N}}$ has only finitely-many limit points for all $t \in F$ yield an extension $w \in \mathbb{N}^{<\mathbb{N}}$ of v such that for no $t \in F$ is $\phi(w \frown (\infty))$ a limit point of $(\phi(\mathcal{N}_{t \frown (i)} \cap \mathbb{N}^{\mathbb{N}}))_{i \in \mathbb{N}}$. We can therefore recursively define a function $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ with the property that for no natural numbers $m < n$ is it the case that $\phi(\pi(t_m) \frown (\infty))$ is in the closure of $\phi(\mathcal{N}_{\pi(t_n)} \cap \mathbb{N}^{\mathbb{N}})$ or $\phi(\pi(t_n) \frown (\infty))$ is a limit point of $(\phi(\mathcal{N}_{\pi(t_m) \frown (i)} \cap \mathbb{N}^{\mathbb{N}}))_{i \in \mathbb{N}}$, and $\phi(t_n') \frown (n) \sqsubseteq \phi(t_n)$ for all $n \in \mathbb{N}$, where t_n' is the maximal proper initial segment of t_n. ⊠

The following fact simultaneously implies the Jayne-Rogers theorem and strengthens Theorem 4.3.

Theorem 6.9 (Jayne-Rogers)**.** *Suppose that X is an analytic metric space, Y is a separable metric space, and $\phi \colon X \to Y$ is Borel. Then exactly one of the following holds:*

1. *The function ϕ is σ-continuous with closed witnesses.*
2. *There is a continuous function $\psi \colon \mathbb{N}^{\mathbb{N}}_* \to X$ with the property that $\overline{(\phi \circ \psi)(\mathbb{N}^{\mathbb{N}})} \cap (\phi \circ \psi)(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}) = \emptyset$.*

Proof. By Theorem 4.3, we can assume that $X = \mathbb{N}^{\mathbb{N}}_*$ and for no $t \in \mathbb{N}^{<\mathbb{N}}$ is $\phi(t \frown (\infty))$ a limit point of $(\phi(\mathcal{N}_{t \frown (i)} \cap \mathbb{N}^{\mathbb{N}}))_{i \in \mathbb{N}}$. By Proposition 6.1, we can assume that $\phi \restriction \mathbb{N}^{\mathbb{N}}$ is continuous.

By Proposition 6.4, we can assume that $\phi \restriction \mathbb{N}^{\mathbb{N}}$ is constant or injective. In the former case, the fact that for no $t \in \mathbb{N}^{<\mathbb{N}}$ is $\phi(t \frown (\infty))$ a limit point of $(\phi(\mathcal{N}_{t \frown (i)} \cap \mathbb{N}^{\mathbb{N}}))_{i \in \mathbb{N}}$ ensures that the closure of $\phi(\mathbb{N}^{\mathbb{N}})$ is disjoint from $\phi(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$. So we can assume that we are in the latter.

By Proposition 6.7, we can assume that $\phi \restriction \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$ is constant or injective. In the former case, Proposition 6.6 allows us to further assume that the closure of $\phi(\mathbb{N}^{\mathbb{N}})$ is disjoint from $\phi(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$. So we can assume that we are in the latter.

By Propositions 6.2 and 6.3, we can assume that $(\phi(\mathcal{N}_{t \frown (i)}))_{i \in \mathbb{N}}$ is convergent or discrete for all $t \in \mathbb{N}^{<\mathbb{N}}$. Fix an enumeration $(t_n)_{n \in \mathbb{N}}$ of $\mathbb{N}^{<\mathbb{N}}$ such that $t_m \sqsubseteq t_n \implies m \leq n$ for all $m, n \in \mathbb{N}$. By Proposition 6.8, we can assume that for no natural numbers $m < n$ is it the case that $\phi(t_m \frown (\infty))$ is in the closure of $\phi(\mathcal{N}_{t_n} \cap \mathbb{N}^{\mathbb{N}})$ or $\phi(t_n \frown (\infty))$ is a limit point of $(\phi(\mathcal{N}_{t_m \frown (i)} \cap \mathbb{N}^{\mathbb{N}}))_{i \in \mathbb{N}}$.

Suppose, towards a contradiction, that there is a sequence $(b_k)_{k \in \mathbb{N}}$ of elements of $\mathbb{N}^{\mathbb{N}}$ for which there exists $n \in \mathbb{N}$ with the property that $\phi(b_k) \to \phi(t_n \frown (\infty))$. Note that if $m < n$, then there are only finitely-many $i \in \mathbb{N}$ for which some b_k extends $t_m \frown (i)$, since $\phi(t_n \frown (\infty))$ is not a limit point of $(\phi(\mathcal{N}_{t_m \frown (i)}))_{i \in \mathbb{N}}$. A straightforward induction therefore yields $m \geq n$ such that infinitely many of the sequences b_k extend t_m. As $\phi(t_n \frown (\infty))$ is not a limit point of $(\phi(\mathcal{N}_{t_n \frown (i)} \cap \mathbb{N}^{\mathbb{N}}))_{i \in \mathbb{N}}$, we can assume that $m > n$. But then $\phi(t_n \frown (\infty))$ is in the closure of $\phi(\mathcal{N}_{t_m} \cap \mathbb{N}^{\mathbb{N}})$, a contradiction. ⊠

7. Baire measurable functions on $\mathbb{N}^{\mathbb{N}}$

Here we provide a basis for the class of Baire measurable functions from $\mathbb{N}^{\mathbb{N}}$ to separable metric spaces. We begin with a strengthening of Proposition 6.4.

Proposition 7.1. *Suppose that X is a metric space and $\phi \colon \mathbb{N}^{\mathbb{N}} \to X$ is a continuous injection. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $\forall i \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}}$ $\overline{\phi(\mathcal{N}_{\pi(t \frown (i))})} \cap \overline{\bigcup_{j \in \mathbb{N} \setminus \{i\}} \phi(\mathcal{N}_{\pi(t \frown (j))})} = \emptyset$.*

Proof. The main point is the following observation.

Lemma 7.2. *For all $t \in \mathbb{N}^{<\mathbb{N}}$, there is a function $\iota_t \colon \mathbb{N} \to \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$ such that $(\iota_t(i)(0))_{i \in \mathbb{N}}$ is injective and the closures of $\phi(\mathcal{N}_{t \frown \iota_t(i)})$ and $\bigcup_{j \in \mathbb{N} \setminus \{i\}} \phi(\mathcal{N}_{t \frown \iota_t(j)})$ are disjoint for all $i \in \mathbb{N}$.*

Proof. Fix extensions $b_i \in \mathbb{N}^\mathbb{N}$ of $t \frown (i)$ with the property that $\phi(b_i) \notin \{\phi(b_j) \mid j < i\}$ for all $i \in \mathbb{N}$. Fix a subsequence $(a_i)_{i \in \mathbb{N}}$ of $(b_i)_{i \in \mathbb{N}}$ for which $\{\phi(a_i) \mid i \in \mathbb{N}\}$ is discrete. For each $i \in \mathbb{N}$, fix $\epsilon_i > 0$ such that $\phi(a_j) \notin \mathcal{B}(\phi(a_i), \epsilon_i)$ for all $j \in \mathbb{N} \setminus \{i\}$, and fix $\iota_t(i) \in \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$ such that $t \frown \iota_t(i) \sqsubseteq a_i$ and $\phi(\mathcal{N}_{t \frown \iota_t(i)}) \subseteq \mathcal{B}(\phi(a_i), \epsilon_i/3)$.

Suppose, towards a contradiction, that there exists $i \in \mathbb{N}$ for which some $x \in X$ is in the closures of $\phi(\mathcal{N}_{t \frown \iota_t(i)})$ and $\bigcup_{j \in \mathbb{N} \setminus \{i\}} \phi(\mathcal{N}_{t \frown \iota_t(j)})$. Then there exist $j \in \mathbb{N} \setminus \{i\}$ and $y \in \phi(\mathcal{N}_{t \frown \iota_t(j)})$ with the property that $d_X(x, y) \leq \max\{\epsilon_i, \epsilon_j\}/3$, in which case

$$\begin{aligned} d_X(\phi(a_i), \phi(a_j)) &\leq d_X(\phi(a_i), x) + d_X(x, y) + d_X(y, \phi(a_j)) \\ &< \epsilon_i/3 + \max\{\epsilon_i, \epsilon_j\}/3 + \epsilon_j/3 \\ &\leq \max\{\epsilon_i, \epsilon_j\}, \end{aligned}$$

so $\phi(a_i) \in \mathcal{B}(\phi(a_j), \epsilon_j)$ or $\phi(a_j) \in \mathcal{B}(\phi(a_i), \epsilon_i)$, a contradiction. ⊠

Define $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ by choosing $\pi(\emptyset) \in \mathbb{N}^{<\mathbb{N}}$ arbitrarily and setting $\pi(t \frown (i)) = \pi(t) \frown \iota_{\pi(t)}(i)$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$. ⊠

The following fact strengthens the well-known fact that Baire measurable functions between Polish spaces are either constant or injective on a perfect set.

Theorem 7.3. *Suppose that X is a separable metric space and $\phi \colon \mathbb{N}^\mathbb{N} \to X$ is Baire measurable. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $\phi \circ \overline{\pi}$ is constant or extends to a closed continuous embedding on $\mathbb{N}^\mathbb{N}$ or $\mathbb{N}^\mathbb{N}_*$.*

Proof. By Remark 5.4, we are free to replace ϕ by its composition with the extension of any $\wedge$-embedding. For example, by Proposition 6.1, we can assume that ϕ is continuous.

By Proposition 6.4, we can assume that ϕ is constant or injective. As we are finished in the latter case, we can assume that we are in the former. Propositions 5.7, 6.2, 6.3, and 7.1 therefore yield a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $(\phi(\mathcal{N}_{\pi(t \frown (i))}))_{i \in \mathbb{N}}$ is convergent for all $t \in \mathbb{N}^{<\mathbb{N}}$ or discrete for all $t \in \mathbb{N}^{<\mathbb{N}}$, and

$$\forall i \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}} \ \overline{\phi(\mathcal{N}_{\pi(t \frown (i))})} \cap \overline{\bigcup_{j \in \mathbb{N} \setminus \{i\}} \phi(\mathcal{N}_{\pi(t \frown (j))})} = \emptyset.$$

As $\overline{\pi}(\mathcal{N}_t) \subseteq \mathcal{N}_{\pi(t)}$ for all $t \in \mathbb{N}^{<\mathbb{N}}$, it follows that

$$\forall i \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}} \ \overline{(\phi \circ \overline{\pi})(\mathcal{N}_{t \frown (i)})} \cap \overline{\bigcup_{j \in \mathbb{N} \setminus \{i\}} (\phi \circ \overline{\pi})(\mathcal{N}_{t \frown (j)})} = \emptyset.$$

So by replacing ϕ with $\phi \circ \overline{\pi}$, we can assume that $\operatorname{diam} \phi(\mathcal{N}_t) \to 0$, $(\phi(\mathcal{N}_{t \frown (i)}))_{i \in \mathbb{N}}$ is convergent for all $t \in \mathbb{N}^{<\mathbb{N}}$ or $(\phi(\mathcal{N}_{t \frown (i)}))_{i \in \mathbb{N}}$ is discrete for all $t \in \mathbb{N}^{<\mathbb{N}}$, and

$$(\dagger) \qquad \forall i \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}} \ \overline{\phi(\mathcal{N}_{t \frown (i)})} \cap \overline{\bigcup_{j \in \mathbb{N} \setminus \{i\}} \phi(\mathcal{N}_{t \frown (j)})} = \emptyset.$$

We next check that if $(\phi(\mathcal{N}_{t \frown (i)}))_{i \in \mathbb{N}}$ is discrete for all $t \in \mathbb{N}^{<\mathbb{N}}$, then ϕ is a closed continuous embedding. It is sufficient to show that every sequence $(b_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{N}^{\mathbb{N}}$ for which $(\phi(b_n))_{n \in \mathbb{N}}$ converges to some $x \in X$ is itself convergent. But a straightforward recursive argument yields $b \in \mathbb{N}^{\mathbb{N}}$ such that x is in the closure of $\phi(\mathcal{N}_{b \restriction i})$ for all $i \in \mathbb{N}$, so $(\dagger)$ ensures that x is not in the closure of $\bigcup_{j \in \mathbb{N} \setminus \{b(i)\}} \phi(\mathcal{N}_{b \restriction i \frown (j)})$ for all $i \in \mathbb{N}$, thus $(b_n \restriction i)_{n \in \mathbb{N}}$ is eventually constant with value $b \restriction i$ for all $i \in \mathbb{N}$, hence $b_n \to b$.

It remains to check that if $(\phi(\mathcal{N}_{t \frown (i)}))_{i \in \mathbb{N}}$ is convergent for all $t \in \mathbb{N}^{<\mathbb{N}}$, then the extension of ϕ to $\mathbb{N}^{\mathbb{N}}_*$ given by $\overline{\phi}(t \frown (\infty)) = \lim_{i \to \infty} \phi(\mathcal{N}_{t \frown (i)})$ for all $t \in \mathbb{N}^{<\mathbb{N}}$ is a closed continuous embedding. To see that $\overline{\phi}$ is injective, note that if $c, d \in \mathbb{N}^{\mathbb{N}}_*$ are distinct, then there is a least $i \in \mathbb{N}$ with $c(i) \neq d(i)$. By reversing the roles of c and d if necessary, we can assume that $c(i) \neq \infty$. Set $t = c \restriction i = d \restriction i$, and appeal to $(\dagger)$ to see that $\overline{\phi}(c)$ is in the closure of $\phi(\mathcal{N}_{t \frown (c(i))})$ but $\overline{\phi}(d)$ is not, so $\overline{\phi}(c) \neq \overline{\phi}(d)$. To see that $\overline{\phi}$ is continuous, suppose that $c \in \mathbb{N}^{\mathbb{N}}_*$ and U is an open neighborhood of $\overline{\phi}(c)$, and fix an open neighborhood V of $\overline{\phi}(c)$ whose closure is contained in U. If $c \in \mathbb{N}^{\mathbb{N}}$, then there exists $i \in \mathbb{N}$ for which $\phi(\mathcal{N}_{c \restriction i}) \subseteq V$, thus $\mathcal{N}_{c \restriction i}$ is an open neighborhood of c whose image under $\overline{\phi}$ is contained in U. Otherwise, there exists $t \in \mathbb{N}^{<\mathbb{N}}$ for which $c = t \frown (\infty)$, as well as $i \in \mathbb{N}$ for which $\phi(\mathcal{N}_t \setminus \bigcup_{j<i} \mathcal{N}_{t \frown (j)}) \subseteq V$. Then $\mathcal{N}_t \setminus \bigcup_{j<i} \mathcal{N}_{t \frown (j)}$ is an open neighborhood of c whose image under $\overline{\phi}$ is contained in U. ⊠

A *closed continuous embedding* of a function $\phi \colon X \to Y$ into a function $\phi' \colon X' \to Y'$ is a pair of closed continuous embeddings $\pi_X \colon X \to X'$ and $\pi_Y \colon \overline{\phi(X)} \to \overline{\phi'(X')}$ such that $\pi_Y \circ \phi = \phi' \circ \pi_X$.

For each topological space X, let c_X denote the unique function from X to the trivial topological space $\{\infty\}$. Given topological spaces $X \subseteq Y$, define $\iota_{X,Y} \colon X \to Y$ by $\iota_{X,Y}(x) = x$ for all $x \in X$.

Proposition 7.4. *Suppose that X is a separable metric space, $\phi \colon \mathbb{N}^{\mathbb{N}} \to X$ is Baire measurable, $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ is a $\wedge$-embedding, and $\phi \circ \overline{\pi}$ is constant or extends to a closed continuous embedding on $\mathbb{N}^{\mathbb{N}}$ or*

$\mathbb{N}^{\mathbb{N}}_*$. *Then there exist* $\phi_0 \in \{c_{\mathbb{N}^{\mathbb{N}}}\} \cup \{\iota_{\mathbb{N}^{\mathbb{N}},Z} \mid Z \in \{\mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}_*\}\}$ *and* $\psi \colon \overline{\phi_0(\mathbb{N}^{\mathbb{N}})} \to \overline{\phi(\mathbb{N}^{\mathbb{N}})}$ *with the property that* $(\overline{\pi} \upharpoonright \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, \psi)$ *is a closed continuous embedding of* ϕ_0 *into* ϕ.

Proof. If $\phi \circ \overline{\pi}$ is constant, then set $\phi_0 = c_{\mathbb{N}^{\mathbb{N}}}$ and let ψ be the unique function from $c_{\mathbb{N}^{\mathbb{N}}}(\mathbb{N}^{\mathbb{N}})$ to $(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}})$. If $\phi \circ \overline{\pi}$ extends to a closed continuous embedding ψ on $Z \in \{\mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}_*\}$, then set $\phi_0 = \iota_{\mathbb{N}^{\mathbb{N}},Z}$. ⊠

8. The Solecki dichotomy

Here we strengthen [Sol98, Theorem 3.1] by providing a basis for the class of non-σ-continuous-with-closed-witnesses Baire-class-one functions between analytic metric spaces.

Proposition 8.1. *Suppose that* X *is a metric space and* $\phi \colon \mathbb{N}^{\mathbb{N}}_* \to X$ *has the property that* $\phi \upharpoonright \mathbb{N}^{\mathbb{N}}$ *is continuous. Then there is a* $\wedge$*-embedding* $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ *such that either* $\overline{(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}})} \cap \overline{(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})} = \emptyset$ *or* $\phi \circ \overline{\pi}$ *is continuous at every point of* $\mathbb{N}^{\mathbb{N}}$.

Proof. We can assume that there is no $s \in \mathbb{N}^{<\mathbb{N}}$ with the property that $\inf\{d_X(\phi(s \frown b), \phi(s \frown t \frown (\infty))) \mid b \in \mathbb{N}^{\mathbb{N}} \text{ and } t \in \mathbb{N}^{<\mathbb{N}}\} > 0$, since otherwise the $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ given by $\pi(t) = s \frown t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$ has the property that $\overline{(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}})} \cap \overline{(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})} = \emptyset$.

Lemma 8.2. *Suppose that* $\epsilon > 0$ *and* $s \in \mathbb{N}^{<\mathbb{N}}$. *Then there exists* $t \in \mathbb{N}^{<\mathbb{N}}$ *with* $d_X(\phi(s \frown t \frown b), \phi(s \frown t \frown (\infty))) < \epsilon$ *for all* $b \in \mathbb{N}^{\mathbb{N}}$.

Proof. Fix $\delta < \epsilon$ and $u \in \mathbb{N}^{<\mathbb{N}}$ with $\operatorname{diam} \phi(\mathcal{N}_{s \frown u} \cap \mathbb{N}^{\mathbb{N}}) < \delta$, and $b \in \mathbb{N}^{\mathbb{N}}$ and $v \in \mathbb{N}^{<\mathbb{N}}$ with $d_X(\phi(s \frown u \frown b), \phi(s \frown u \frown v \frown (\infty))) < \epsilon - \delta$, and set $t = u \frown v$. ⊠

Fix a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ of positive real numbers converging to zero, and recursively construct a function $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ with the property that $d_X(\phi(\pi(t) \frown b), \phi(\pi(t) \frown (\infty))) < \epsilon_{|t|}$ for all $b \in \mathbb{N}^{\mathbb{N}}$ and $t \in \mathbb{N}^{<\mathbb{N}}$ and $\pi(t) \frown (i) \sqsubseteq \pi(t \frown (i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$. ⊠

We say that a metric space is *ϵ-discrete* if all distinct points have distance at least ϵ from one another. The following fact is a straightforward generalization of Proposition 6.7.

Proposition 8.3. *Suppose that* X *is a metric space,* $\phi \colon \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \to X$, $\epsilon > 0$, *and* $t \in \mathbb{N}^{<\mathbb{N}}$. *Then there is a* $\wedge$*-embedding* $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_t \cap \mathbb{N}^{<\mathbb{N}}$ *with the property that* $\phi \circ \overline{\pi}$ *is an injection into an* ϵ*-discrete set or* $(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$ *is contained in the* ϵ*-ball around a point of* $\phi(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$.

Proof. If for no finite set $F \subseteq \phi(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$ and extension u of t is it the case that $\phi(\mathcal{N}_u) \subseteq \mathcal{B}(F, \epsilon)$, then fix an enumeration $(t_n)_{n \in \mathbb{N}}$ of $\mathbb{N}^{<\mathbb{N}}$

with the property that $t_m \sqsubseteq t_n \implies m \leq n$ for all $m, n \in \mathbb{N}$, and recursively construct $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_t \cap \mathbb{N}^{<\mathbb{N}}$ such that $\phi(\pi(t_n) \frown (\infty)) \notin \mathcal{B}(\{\phi(\pi(t_m) \frown (\infty)) \mid m < n\}, \epsilon)$ and $\pi(t'_n) \frown (n) \sqsubseteq \pi(t_n)$ for all $n > 0$, where t'_n is the maximal proper initial segment of t_n.

Otherwise, there exists $x \in \phi(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$ with the property that the set $S = \{s \in \mathbb{N}^{<\mathbb{N}} \mid \phi(s \frown (\infty)) \in \mathcal{B}(x, \epsilon)\}$ is $\sqsubseteq$-dense below some extension u of t, in which case we can recursively construct a function $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_u \cap S$ with the property that $\pi(v) \frown (i) \sqsubseteq \pi(v \frown (i))$ for all $i \in \mathbb{N}$ and $v \in \mathbb{N}^{<\mathbb{N}}$. ⊠

Proposition 8.4. *Suppose that X is a metric space and $\phi \colon \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \to X$. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $\phi \circ \overline{\pi}$ is an injection into an ϵ-discrete set for some $\epsilon > 0$ or* $\operatorname{diam}(\phi \circ \overline{\pi})(\mathcal{N}_t) \to 0$.

Proof. Suppose that for no $\epsilon > 0$ is there a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $\phi \circ \overline{\pi}$ is an injection into an ϵ-discrete set, fix a sequence $(\epsilon_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$ of positive real numbers converging to zero, and recursively apply Proposition 8.3 to the functions $\phi_t = \phi \circ \prod_{n < |t|} \overline{\pi_{t \upharpoonright n}}$ to obtain $\wedge$-embeddings $\pi_t \colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_t \cap \mathbb{N}^{<\mathbb{N}}$ such that $(\phi \circ \prod_{n \leq |t|} \overline{\pi_{t \upharpoonright n}})(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$ is contained in an ϵ_t-ball for all $t \in \mathbb{N}^{<\mathbb{N}}$. Let π be the $\wedge$-embedding obtained from applying Proposition 5.2 to $(\pi_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$, and observe that $\operatorname{diam}(\phi \circ \overline{\pi})(\mathcal{N}_t) \to 0$. ⊠

Define $p \colon \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ by setting $p(t \frown (\infty)) = t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$.

Theorem 8.5. *Suppose that X is an analytic metric space, Y is a separable metric space, and $\phi \colon X \to Y$ is a Baire-class-one function that is not σ-continuous with closed witnesses. Then there exists $\phi_0 \in \{c_{\mathbb{N}^{\mathbb{N}}}\} \cup \{\iota_{\mathbb{N}^{\mathbb{N}}, Z} \mid Z \in \{\mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}_*\}\}$ for which there is a closed continuous embedding of $\phi_0 \cup p$ into ϕ.*

Proof. By Theorem 6.9, there is a continuous function $\psi \colon \mathbb{N}^{\mathbb{N}}_* \to X$ such that $\overline{(\phi \circ \psi)(\mathbb{N}^{\mathbb{N}})} \cap (\phi \circ \psi)(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}) = \emptyset$. As $(\psi, \mathrm{id}_{\overline{(\phi \circ \psi)(\mathbb{N}^{\mathbb{N}}_*)}})$ is a closed continuous embedding of $\phi \circ \psi$ into ϕ, by replacing the latter with the former, we can assume that $X = \mathbb{N}^{\mathbb{N}}_*$ and $\overline{\phi(\mathbb{N}^{\mathbb{N}})} \cap \phi(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}) = \emptyset$.

By Proposition 6.1, there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ for which $(\phi \circ \overline{\pi}) \upharpoonright \mathbb{N}^{\mathbb{N}}$ is continuous. By composing π with the $\wedge$-embedding given by Proposition 8.1, we can assume that $\overline{(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}})} \cap \overline{(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})} = \emptyset$ or $\phi \circ \overline{\pi}$ is continuous at every point of $\mathbb{N}^{\mathbb{N}}$. As the former possibility would imply that $\phi \circ \overline{\pi}$ is not Baire class one, it follows that the latter holds. By Proposition 8.4, we can assume that either there exists $\epsilon > 0$ for which $(\phi \circ \overline{\pi}) \upharpoonright \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$ is an injection into an ϵ-discrete set, or $\operatorname{diam}(\phi \circ \overline{\pi})(\mathcal{N}_t \cap (\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})) \to 0$. As the former possibility contradicts the facts that $(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}}) \cap (\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}) = \emptyset$ and

$(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N}) \subseteq \overline{(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})}$, it follows that the latter holds. By Proposition 6.7, we can assume that $(\phi \circ \overline{\pi}) \upharpoonright \mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}$ is either constant or injective.

Lemma 8.6. *Suppose that $(s_n)_{n \in \mathbb{N}}$ is an injective sequence of elements of $\mathbb{N}^{<\mathbb{N}}$ for which $((\phi \circ \overline{\pi})(s_n))_{n \in \mathbb{N}}$ converges, and $b_n \in \mathbb{N}^\mathbb{N}$ is an extension of s_n for all $n \in \mathbb{N}$. Then $\lim_{n \to \infty} (\phi \circ \overline{\pi})(b_n) = \lim_{n \to \infty} (\phi \circ \overline{\pi})(s_n)$.*

Proof. Simply note that $(\phi \circ \overline{\pi})(b_n) \in \overline{(\phi \circ \overline{\pi})(\mathcal{N}_{s_n} \cap (\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}))}$ for all $n \in \mathbb{N}$ and $\operatorname{diam} (\phi \circ \overline{\pi})(\mathcal{N}_{s_n} \cap (\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})) \to 0$. ⊠

As $\overline{(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N})} \cap (\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}) = \emptyset$, it follows from Lemma 8.6 that $(\phi \circ \overline{\pi}) \upharpoonright \mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}$ is not constant, and is therefore injective. It similarly follows that $(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})$ is discrete. By Theorem 7.3, we can assume that $(\phi \circ \overline{\pi}) \upharpoonright \mathbb{N}^\mathbb{N}$ is constant or extends to a closed continuous embedding on $\mathbb{N}^\mathbb{N}$ or $\mathbb{N}^\mathbb{N}_*$.

We will now complete the proof by showing that there exist $\phi_0 \in \{c_{\mathbb{N}^\mathbb{N}}\} \cup \{\iota_{\mathbb{N}^\mathbb{N}, Z} \mid Z \in \{\mathbb{N}^\mathbb{N}, \mathbb{N}^\mathbb{N}_*\}\}$ and $\psi \colon \overline{\phi_0(\mathbb{N}^\mathbb{N}_*)} \cup \mathbb{N}^{<\mathbb{N}} \to \overline{\phi(X)}$ for which $(\overline{\pi} \upharpoonright \mathbb{N}^\mathbb{N}_* \to \mathbb{N}^\mathbb{N}_*, \psi)$ is a closed continuous embedding of $\phi_0 \cup p$ into ϕ.

If $(\phi \circ \overline{\pi}) \upharpoonright \mathbb{N}^\mathbb{N}$ is constant with value $x \in X$, then set $\phi_0 = c_{\mathbb{N}^\mathbb{N}}$, and observe that the extension ψ of $\phi \circ \overline{\pi} \circ p^{-1}$ to the *one-point compactification* $\mathbb{N}^{<\mathbb{N}}_* = \mathbb{N}^{<\mathbb{N}} \cup \{\infty\}$ of $\mathbb{N}^{<\mathbb{N}}$ given by $\psi(\infty) = x$ is a continuous injection, and is therefore a closed continuous embedding by the compactness of $\mathbb{N}^{<\mathbb{N}}_*$.

If $(\phi \circ \overline{\pi}) \upharpoonright \mathbb{N}^\mathbb{N}$ is a closed continuous embedding, then set $\phi_0 = \iota_{\mathbb{N}^\mathbb{N}, \mathbb{N}^\mathbb{N}}$, and observe that the function $\psi = \phi \circ \overline{\pi} \circ (\phi_0 \cup p)^{-1}$ is a continuous injection. To see that it is closed, it is enough to show that every injective sequence $(a_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{N}^{\leq \mathbb{N}}$ for which $(\psi(a_n))_{n \in \mathbb{N}}$ converges to some point $x \in X$ has a subsequence converging to a point of $\mathbb{N}^{\leq \mathbb{N}}$. As $\mathbb{N}^{\leq \mathbb{N}}_*$ is compact, by passing to a subsequence, we can assume that $(a_n)_{n \in \mathbb{N}}$ converges to a point of $\mathbb{N}^{\leq \mathbb{N}}_*$. As every point of $\mathbb{N}^{<\mathbb{N}}$ is isolated, it therefore converges to a point of $\mathbb{N}^\mathbb{N}_*$. And if there exists $t \in \mathbb{N}^{<\mathbb{N}}$ for which $a_n \to t \frown (\infty)$, then fix extensions $b_n \in \mathbb{N}^\mathbb{N}$ of a_n for all $n \in \mathbb{N}$, and observe that $b_n \to t \frown (\infty)$ and $\psi(b_n) \to x$ by Lemma 8.6, contradicting the fact that $(\phi \circ \overline{\pi}) \upharpoonright \mathbb{N}^\mathbb{N}$ is closed.

If $(\phi \circ \overline{\pi}) \upharpoonright \mathbb{N}^\mathbb{N}$ extends to a closed continuous embedding $\overline{\phi \circ \overline{\pi}}$ on $\mathbb{N}^\mathbb{N}_*$, then set $\phi_0 = \iota_{\mathbb{N}^\mathbb{N}, \mathbb{N}^\mathbb{N}_*}$, and observe that the function $\psi = \overline{\phi \circ \overline{\pi}} \circ (\phi_0 \cup p)^{-1}$ is injective. To see that it is continuous, suppose that $(t_n)_{n \in \mathbb{N}}$ is an injective sequence of elements of $\mathbb{N}^{<\mathbb{N}}$ converging to $t \frown (\infty)$ for some $t \in \mathbb{N}^{<\mathbb{N}}$, fix extensions $b_n \in \mathbb{N}^\mathbb{N}$ of t_n for all $n \in \mathbb{N}$, and observe that the continuity of $\overline{\phi \circ \overline{\pi}} \upharpoonright \mathbb{N}^\mathbb{N}_*$ ensures that $\psi(b_n) \to \psi(t \frown (\infty))$, thus

Lemma 8.6 implies that $\psi(t_n) \to \psi(t \frown (\infty))$. As $\mathbb{N}_*^{\leq\mathbb{N}}$ is compact, it follows that ψ is a closed continuous embedding. ⊠

9. Functions on $\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$

Here we provide a basis for the class of all functions from $\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$ to analytic metric spaces.

Proposition 9.1. *Suppose that X is a topological space, $\phi\colon \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \to X$ is injective, and $x \in X$. Then there is a $\wedge$-embedding $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $x \notin (\phi \circ \overline{\pi})(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})$.*

Proof. Fix $s \in \mathbb{N}^{<\mathbb{N}}$ such that $x \notin \phi(\mathcal{N}_s)$, and define $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ by $\pi(t) = s \frown t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$. ⊠

Proposition 9.2. *Suppose that X is a metric space and $\phi\colon \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \to X$. Then there is a $\wedge$-embedding $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ with the property that $((\phi \circ \overline{\pi})(t \frown (i, \infty)))_{i\in\mathbb{N}}$ is convergent or $\{(\phi \circ \overline{\pi})(t \frown (i, \infty)) \mid i \in \mathbb{N}\}$ is closed and discrete for all $t \in \mathbb{N}^{<\mathbb{N}}$.*

Proof. For each $t \in \mathbb{N}^{<\mathbb{N}}$, there is an injection $\iota_t\colon \mathbb{N} \to \mathbb{N}$ for which $(\phi(t \frown (\iota_t(i), \infty)))_{i\in\mathbb{N}}$ is convergent or $\{\phi(t \frown (\iota_t(i), \infty)) \mid i \in \mathbb{N}\}$ is closed and discrete. Define $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ by choosing $\pi(\emptyset) \in \mathbb{N}^{<\mathbb{N}}$ arbitrarily and setting $\pi(t \frown (i)) = \pi(t) \frown (\iota_{\pi(t)}(i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$, and note that $(\phi \circ \overline{\pi})(t \frown (i, \infty)) = \phi(\pi(t \frown (i)) \frown (\infty)) = \phi(\pi(t) \frown (\iota_{\pi(t)}(i), \infty))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{\mathbb{N}}$. ⊠

Proposition 9.3. *Suppose that X is a metric space, $\phi\colon \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \to X$, $F \subseteq X$ is finite, and $t \in \mathbb{N}^{<\mathbb{N}}$. Then there is a $\wedge$-embedding $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_t \cap \mathbb{N}^{<\mathbb{N}}$ such that either $((\phi \circ \overline{\pi})(u \frown (\infty)))_{u\in\mathbb{N}^{<\mathbb{N}}}$ converges to an element of F or the closure of $(\phi \circ \overline{\pi})(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})$ is disjoint from F.*

Proof. If the set $S_\epsilon = \{s \in \mathbb{N}^{<\mathbb{N}} \mid \phi(s \frown (\infty)) \in \mathcal{B}(F, \epsilon)\}$ is $\sqsubseteq$-dense below t for all $\epsilon > 0$, then there exist an extension u of t as well as $x \in F$ such that the set $S_{\epsilon,x} = \{s \in \mathbb{N}^{<\mathbb{N}} \mid \phi(s \frown (\infty)) \in \mathcal{B}(x, \epsilon)\}$ is $\sqsubseteq$-dense below u for all $\epsilon > 0$. Fix a sequence $(\epsilon_v)_{v\in\mathbb{N}^{<\mathbb{N}}}$ of positive real numbers converging to zero, and recursively construct a function $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_u \cap \mathbb{N}^{<\mathbb{N}}$ such that $\pi(v) \in S_{\epsilon_v,x}$ for all $v \in \mathbb{N}^{<\mathbb{N}}$ and $\pi(v) \frown (i) \sqsubseteq \pi(v \frown (i))$ for all $i \in \mathbb{N}$ and $v \in \mathbb{N}^{<\mathbb{N}}$, and observe that $(\phi \circ \overline{\pi})(v \frown (\infty)) \to x$.

Otherwise, fix $\epsilon > 0$ and an extension u of t with the property that $\mathcal{N}_u \cap S_\epsilon = \emptyset$, define $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_u \cap \mathbb{N}^{<\mathbb{N}}$ by $\pi(v) = u \frown v$, and note that the closure of $(\phi \circ \overline{\pi})(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})$ is disjoint from F. ⊠

For the rest of this section, it will be convenient to fix an enumeration $(t_n)_{n\in\mathbb{N}}$ of $\mathbb{N}^{<\mathbb{N}}$ such that $t_m \sqsubseteq t_n \implies m \leq n$ for all $m, n \in \mathbb{N}$.

Proposition 9.4. *Suppose that X is a metric space and $\phi \colon \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \to X$. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ with the property that $((\phi \circ \overline{\pi})(t \frown (\infty)))_{t \in \mathbb{N}^{<\mathbb{N}}}$ converges or for no natural numbers $m < n$ is $(\phi \circ \overline{\pi})(t_m \frown (\infty))$ or a limit point of $\{(\phi \circ \overline{\pi})(t_m \frown (i, \infty)) \mid i \in \mathbb{N}\}$ in the closure of $(\phi \circ \overline{\pi})(\mathcal{N}_{t_n})$.*

Proof. Suppose that for no $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ is the sequence $((\phi \circ \overline{\pi})(t \frown (\infty)))_{t \in \mathbb{N}^{<\mathbb{N}}}$ convergent. By Proposition 9.2, we can assume that $(\phi(t \frown (i, \infty)))_{i \in \mathbb{N}}$ is convergent or $\{\phi(t \frown (i, \infty)) \mid i \in \mathbb{N}\}$ is closed and discrete for all $t \in \mathbb{N}^{<\mathbb{N}}$. By recursively applying Lemma 9.3 to the functions $\phi_t = \phi \circ \prod_{n<|t|} \overline{\pi_{t \upharpoonright n}}$, we obtain $\wedge$-embeddings $\pi_t \colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_t \cap \mathbb{N}^{<\mathbb{N}}$ such that for no $m < n$ is $(\phi \circ \prod_{k \leq |t_m|} \overline{\pi_{t_m \upharpoonright k}})(t_m \frown (\infty))$ or a limit point of $\{(\phi \circ \prod_{k \leq |t_m|} \overline{\pi_{t_m \upharpoonright k}})(t_m \frown (i, \infty)) \mid i \in \mathbb{N}\}$ in the closure of $(\phi \circ \prod_{k \leq |t_n|} \overline{\pi_{t_n \upharpoonright k}})(\mathcal{N}_{t_n})$. Let π be the $\wedge$-embedding obtained from applying Proposition 5.2 to $(\pi_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$, and observe that for no natural numbers $m < n$ is it the case that $(\phi \circ \overline{\pi})(t_m \frown (\infty))$ or a limit point of $\{(\phi \circ \overline{\pi})(t_m \frown (i, \infty)) \mid i \in \mathbb{N}\}$ in the closure of $(\phi \circ \overline{\pi})(\mathcal{N}_{t_n})$. ⊠

We next provide an analog of Theorem 7.3 for functions on $\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$.

Theorem 9.5. *Suppose that X is an analytic metric space and $\phi \colon \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \to X$. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $\phi \circ \overline{\pi}$ is constant, $\phi \circ \overline{\pi}$ extends to a closed continuous embedding on $\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}_*$, or $\phi \circ \overline{\pi} \circ p^{-1}$ extends to a closed continuous embedding on $\mathbb{N}^{<\mathbb{N}}$, $\mathbb{N}^{<\mathbb{N}}_*$, $\mathbb{N}^{<\mathbb{N}} \cup (\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$, $\mathbb{N}^{\leq\mathbb{N}}$, or $\mathbb{N}^{\leq\mathbb{N}}_*$.*

Proof. Set $\psi = \phi \circ p^{-1}$. By Proposition 6.7, we can assume that ψ is either constant or injective. As we are done in the former case, we can assume that we are in the latter.

By Proposition 8.4, we can ensure that $\psi(\mathbb{N}^{<\mathbb{N}})$ is closed and discrete or $\operatorname{diam} \psi(\mathcal{N}_t) \to 0$. As ψ is a closed continuous embedding in the former case, we can assume that we are in the latter.

Let $\overline{\psi}$ be the extension of ψ to a partial function on $\mathbb{N}^{\leq\mathbb{N}}_*$ given by $\overline{\psi}(b) = \lim_{i \to \infty} \psi(b \upharpoonright i)$ and $\overline{\psi}(t \frown (\infty)) = \lim_{i \to \infty} \psi(t \frown (i))$ for all $b \in \mathbb{N}^{\mathbb{N}}$ and $t \in \mathbb{N}^{<\mathbb{N}}$. By Proposition 9.2, we can assume that $\{\psi(t \frown (i)) \mid i \in \mathbb{N}\}$ has a limit point $\implies t \frown (\infty) \in \operatorname{dom}(\overline{\psi})$ for all $t \in \mathbb{N}^{<\mathbb{N}}$.

As each $t \in \mathbb{N}^{<\mathbb{N}}$ is isolated, $\operatorname{diam} \psi(\mathcal{N}_{b \upharpoonright i}) \to 0$ for all $b \in \mathbb{N}^{\mathbb{N}}$, and $\operatorname{diam} \psi(\mathcal{N}_{t \frown (i)}) \to 0$ for all $t \in \mathbb{N}^{<\mathbb{N}}$, it follows that $\overline{\psi}$ is continuous. To see that $\overline{\psi}$ is closed, it is sufficient show that every injective sequence $(c_n)_{n \in \mathbb{N}}$ of points in the domain of $\overline{\psi}$ for which $(\overline{\psi}(c_n))_{n \in \mathbb{N}}$ is convergent has a subsequence converging to a point in the domain of $\overline{\psi}$. By passing to a subsequence, we can assume that the sequence converges to a point of $\mathbb{N}_*^{\leq \mathbb{N}}$. As each point of $\mathbb{N}^{<\mathbb{N}}$ is isolated, it does not converge to a point of $\mathbb{N}^{<\mathbb{N}}$, so the facts that $\operatorname{diam} \psi(\mathcal{N}_t) \to 0$ and $\{\psi(t \frown (i)) \mid i \in \mathbb{N}\}$ has a limit point $\implies t \frown (\infty) \in \operatorname{dom}(\overline{\psi})$ for all $t \in \mathbb{N}^{<\mathbb{N}}$ ensure that it converges to a point of the domain of $\overline{\psi}$.

By Proposition 5.7, we can assume that one of the following holds:

1. $\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \subseteq \operatorname{dom}(\overline{\psi})$ and $\forall t \in \mathbb{N}^{<\mathbb{N}}\ \overline{\psi}(t) = \overline{\psi}(t \frown (\infty))$.
2. $\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \subseteq \operatorname{dom}(\overline{\psi})$ and $\forall t \in \mathbb{N}^{<\mathbb{N}}\ \overline{\psi}(t) \neq \overline{\psi}(t \frown (\infty))$.
3. $(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}) \cap \operatorname{dom}(\overline{\psi}) = \emptyset$.

As the domain of $\overline{\psi}$ is analytic, so too is its intersection with $\mathbb{N}^{\mathbb{N}}$. It follows that the latter intersection has the Baire property, so Proposition 5.8 allows us to assume that one of the following holds:

(a) The domain of $\overline{\psi}$ is disjoint from $\mathbb{N}^{\mathbb{N}}$.
(b) The domain of $\overline{\psi}$ contains $\mathbb{N}^{\mathbb{N}}$.

In the special case that condition (b) holds, Theorem 7.3 allows us to assume that $\overline{\psi} \upharpoonright \mathbb{N}^{\mathbb{N}}$ is either constant or injective.

Proposition 9.4 allows us to assume that $(\psi(t))_{t \in \mathbb{N}^{<\mathbb{N}}}$ converges to some $x \in X$ or for no natural numbers $m < n$ is $\psi(t_m)$ or $\overline{\psi}(t_m \frown (\infty))$ in the closure of $\psi(\mathcal{N}_{t_n})$. In the former case, Proposition 9.1 allows us to assume that $\psi(\mathbb{N}^{<\mathbb{N}})$ is discrete, so the extension of ψ to $\mathbb{N}_*^{<\mathbb{N}}$ sending ∞ to x is a closed continuous embedding, thus we can assume that we are in the latter.

Lemma 9.6. *Suppose that $c, d \in \operatorname{dom}(\overline{\psi})$ are distinct but $\overline{\psi}(c) = \overline{\psi}(d)$. Then there exists $t \in \mathbb{N}^{<\mathbb{N}}$ such that $\{c, d\} = \{t, t \frown (\infty)\}$.*

Proof. To see that $\overline{\psi} \upharpoonright \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$ is injective, observe that if $m < n$, both $t_m \frown (\infty)$ and $t_n \frown (\infty)$ are in the domain of $\overline{\psi}$, and moreover $\overline{\psi}(t_m \frown (\infty)) = \overline{\psi}(t_n \frown (\infty))$, then $\overline{\psi}(t_m \frown (\infty))$ is in the closure of $\psi(\mathcal{N}_{t_n})$, a contradiction.

To see that $\overline{\psi} \upharpoonright \mathbb{N}^{\mathbb{N}}$ is injective when $\mathbb{N}^{\mathbb{N}}$ is contained in the domain of $\overline{\psi}$, note that otherwise it is constant, and let x be this constant value. Then for each $t \in \mathbb{N}^{<\mathbb{N}}$, there is a sequence $(u_i)_{i \in \mathbb{N}}$ of elements of $\mathbb{N}^{<\mathbb{N}}$ such that $\psi(t \frown (i) \frown (u_i)) \to x$, so the fact that $\operatorname{diam} \psi(\mathcal{N}_{t \frown (i)}) \to 0$ ensures that $\overline{\psi}(t \frown (\infty)) = x$, contradicting the fact that $\overline{\psi} \upharpoonright \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$ is injective.

To see that $\overline{\psi}(\mathbb{N}^{\mathbb{N}}) \cap \psi(\mathbb{N}^{<\mathbb{N}}) = \emptyset$, note that if $b \in \operatorname{dom}(\overline{\psi}) \cap \mathbb{N}^{\mathbb{N}}$, $t \in \mathbb{N}^{<\mathbb{N}}$, and $\overline{\psi}(b) = \psi(t)$, then there exists $m < n$ with $t_m = t$ and $t_n \sqsubset b$, so $\psi(t_m)$ is in the closure of $\psi(\mathcal{N}_{t_n})$, a contradiction.

To see that $\overline{\psi}(\mathbb{N}^{\mathbb{N}}) \cap \overline{\psi}(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}) = \emptyset$, note that if $b \in \operatorname{dom}(\overline{\psi}) \cap \mathbb{N}^{\mathbb{N}}$, $t \in \mathbb{N}^{<\mathbb{N}}$, $t \frown (\infty) \in \operatorname{dom}(\overline{\psi})$, and $\overline{\psi}(b) = \overline{\psi}(t \frown (\infty))$, then there exists

$m < n$ with $t_m = t$ and $t_n \sqsubset b$, in which case $\overline{\psi}(t_m \frown (\infty))$ is in the closure of $\psi(\mathcal{N}_{t_n})$, a contradiction.

Observe finally that if $s, t \in \mathbb{N}^{<\mathbb{N}}$ are distinct, $t \frown (\infty) \in \mathrm{dom}(\overline{\psi})$, and $\psi(s) = \overline{\psi}(t \frown (\infty))$, then there exist $m \neq n$ such that $t_m = s$ and $t_n = t$. Then $\psi(t_m)$ is in the closure of $\psi(\mathcal{N}_{t_n})$ and $\overline{\psi}(t_n \frown (\infty))$ is in $\psi(\mathcal{N}_{t_m})$, a contradiction. ⊠

If (1a) or (1b) holds, then $\overline{\psi} \upharpoonright \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$ or $\overline{\psi} \upharpoonright \mathbb{N}^{\mathbb{N}}_*$ is an extension of ϕ to a closed continuous embedding. If (2a), (2b), (3a), or (3b) holds, then $\overline{\psi}$ is an extension of ψ to a closed continuous embedding on $\mathbb{N}^{<\mathbb{N}} \cup (\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$, $\mathbb{N}^{\leq\mathbb{N}}_*$, $\mathbb{N}^{<\mathbb{N}}$, or $\mathbb{N}^{\leq\mathbb{N}}$. ⊠

Proposition 9.7. *Suppose that X is an analytic metric space, $\phi \colon \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \to X$, $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ is a $\wedge$-embedding, and either $\phi \circ \overline{\pi}$ is constant, $\phi \circ \overline{\pi}$ extends to a closed continuous embedding on $\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}_*$, or $\phi \circ \overline{\pi} \circ p^{-1}$ extends to a closed continuous embedding on $\mathbb{N}^{<\mathbb{N}}$, $\mathbb{N}^{<\mathbb{N}}_*$, $\mathbb{N}^{<\mathbb{N}} \cup (\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$, $\mathbb{N}^{\leq\mathbb{N}}$, or $\mathbb{N}^{\leq\mathbb{N}}_*$. Then there exist $\phi_0 \in \{c_{\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}}\} \cup \{\iota_{\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}, Z} \mid Z \in \{\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}_*\}\} \cup \{\iota_{\mathbb{N}^{<\mathbb{N}}, Z} \circ p \mid Z \in \{\mathbb{N}^{<\mathbb{N}}, \mathbb{N}^{<\mathbb{N}}_*, \mathbb{N}^{<\mathbb{N}} \cup (\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}), \mathbb{N}^{\leq\mathbb{N}}, \mathbb{N}^{\leq\mathbb{N}}_*\}\}$ and $\psi \colon \overline{\phi_0(\mathbb{N}^{\mathbb{N}})} \to \overline{\phi(\mathbb{N}^{\mathbb{N}})}$ with the property that $(\overline{\pi} \upharpoonright \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}, \psi)$ is a closed continuous embedding of ϕ_0 into ϕ.*

Proof. If $\phi \circ \overline{\pi}$ is constant, then set $\phi_0 = c_{\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}}$ and let ψ be the unique function from $c_{\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}}(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$ to $(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}})$. If $\phi \circ \overline{\pi}$ extends to a closed continuous embedding ψ on $Z \in \{\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}_*\}$, then set $\phi_0 = \iota_{\mathbb{N}^{\mathbb{N}}, Z}$. And if $\phi \circ \overline{\pi} \circ p^{-1}$ extends to a closed continuous embedding ψ on $Z \in \{\mathbb{N}^{<\mathbb{N}}, \mathbb{N}^{<\mathbb{N}}_*, \mathbb{N}^{<\mathbb{N}} \cup (\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}), \mathbb{N}^{\leq\mathbb{N}}, \mathbb{N}^{\leq\mathbb{N}}_*\}$, then set $\phi_0 = \iota_{\mathbb{N}^{<\mathbb{N}}, Z} \circ p$. ⊠

10. Borel functions that are not Baire class one

Here we provide bases for the classes of non-Baire-class-one Borel functions and non-σ-continuous-with-closed-witnesses Borel functions between analytic metric spaces.

Proposition 10.1. *Suppose that X is a metric space and $\phi \colon \mathbb{N}^{\mathbb{N}}_* \to X$ has the property that $\phi \upharpoonright \mathbb{N}^{\mathbb{N}}$ is continuous and $\phi(\mathbb{N}^{\mathbb{N}}) \nsubseteq \overline{\phi(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})}$. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ with the property that $\overline{(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}})} \cap \overline{(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})} = \emptyset$.*

Proof. Fix $b \in \mathbb{N}^{\mathbb{N}}$ for which $\phi(b)$ is not in the closure of $\phi(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$. Then there is an open neighborhood U of $\phi(b)$ disjoint from $\phi(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$, as well as an open neighborhood V of $\phi(b)$ whose closure is contained in U, in which case the continuity of $\phi \upharpoonright \mathbb{N}^{\mathbb{N}}$ yields a proper initial segment s of b for which $\phi(\mathcal{N}_s \cap \mathbb{N}^{\mathbb{N}}) \subseteq V$. Then the $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ given by $\pi(t) = s \frown t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$ is as desired. ⊠

Given $\phi_{\mathbb{N}^\mathbb{N}} \colon \mathbb{N}^\mathbb{N} \to X$ and $\phi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}} \colon \mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N} \to Y$, let $\phi_{\mathbb{N}^\mathbb{N}} \sqcup \phi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}}$ denote the corresponding function from $\mathbb{N}^\mathbb{N}_*$ to the disjoint union $X \sqcup Y$.

Theorem 10.2. *Suppose that X and Y are analytic metric spaces and $\phi \colon X \to Y$ is a Borel function that is not Baire class one. Then there exist $\phi_{\mathbb{N}^\mathbb{N}} \in \{c_{\mathbb{N}^\mathbb{N}}\} \cup \{\iota_{\mathbb{N}^\mathbb{N},Z} \mid Z \in \{\mathbb{N}^\mathbb{N}, \mathbb{N}^\mathbb{N}_*\}\}$ and $\phi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}} \in \{c_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}}\} \cup \{\iota_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N},Z} \mid Z \in \{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}, \mathbb{N}^\mathbb{N}_*\}\} \cup \{\iota_{\mathbb{N}^{<\mathbb{N}},Z} \circ p \mid Z \in \{\mathbb{N}^{<\mathbb{N}}, \mathbb{N}^{<\mathbb{N}}_*, \mathbb{N}^{<\mathbb{N}} \cup (\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}), \mathbb{N}^{\le\mathbb{N}}, \mathbb{N}^{\le\mathbb{N}}_*\}\}$ for which there is a closed continuous embedding of $\phi_{\mathbb{N}^\mathbb{N}} \sqcup \phi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}}$ into ϕ.*

Proof. Hurewicz's dichotomy theorem for F_σ sets yields a closed continuous embedding $\psi \colon \mathbb{N}^\mathbb{N}_* \to X$ with $(\phi \circ \psi)(\mathbb{N}^\mathbb{N}) \cap \overline{(\phi \circ \psi)(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})} = \emptyset$ (see, for example, [CM, Theorem 11]). As $(\psi, \mathrm{id}_{\overline{(\phi \circ \psi)(\mathbb{N}^\mathbb{N}_*)}})$ is a closed continuous embedding of $\phi \circ \psi$ into ϕ, by replacing the latter with the former, we can assume that $X = \mathbb{N}^\mathbb{N}_*$ and $\phi(\mathbb{N}^\mathbb{N}) \cap \overline{\phi(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})} = \emptyset$.

By Proposition 6.1, there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ for which $(\phi \circ \overline{\pi}) \restriction \mathbb{N}^\mathbb{N}$ is continuous. By composing π with the $\wedge$-embedding given by Proposition 10.1, we can assume that $\overline{(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N})} \cap \overline{(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})} = \emptyset$. By composing π with the $\wedge$-embedding given by Theorem 7.3, we can assume that $\phi \circ \overline{\pi}$ is constant or extends to a closed continuous embedding on $\mathbb{N}^\mathbb{N}$ or $\mathbb{N}^\mathbb{N}_*$. And by composing π with the $\wedge$-embedding given by Theorem 9.5, we can assume that $\phi \circ \overline{\pi}$ is constant, $\phi \circ \overline{\pi}$ extends to a closed continuous embedding on $\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}$ or $\mathbb{N}^\mathbb{N}_*$, or $\phi \circ \overline{\pi} \circ p^{-1}$ extends to a closed continuous embedding on $\mathbb{N}^{<\mathbb{N}}$, $\mathbb{N}^{<\mathbb{N}}_*$, $\mathbb{N}^{<\mathbb{N}} \cup (\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})$, $\mathbb{N}^{\le\mathbb{N}}$, or $\mathbb{N}^{\le\mathbb{N}}_*$.

By Proposition 7.4, there exist $\phi_{\mathbb{N}^\mathbb{N}} \in \{c_{\mathbb{N}^\mathbb{N}}\} \cup \{\iota_{\mathbb{N}^\mathbb{N},Z} \mid Z \in \{\mathbb{N}^\mathbb{N}, \mathbb{N}^\mathbb{N}_*\}\}$ and $\psi_{\mathbb{N}^\mathbb{N}} \colon \overline{\phi_{\mathbb{N}^\mathbb{N}}(\mathbb{N}^\mathbb{N})} \to \overline{\phi(\mathbb{N}^\mathbb{N})}$ for which $(\overline{\pi} \restriction \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}, \psi_{\mathbb{N}^\mathbb{N}})$ is a closed continuous embedding of $\phi_{\mathbb{N}^\mathbb{N}}$ into $\phi \restriction \mathbb{N}^\mathbb{N}$. By Proposition 9.7, there exist $\phi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}} \in \{c_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}}\} \cup \{\iota_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N},Z} \mid Z \in \{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}, \mathbb{N}^\mathbb{N}_*\}\} \cup \{\iota_{\mathbb{N}^{<\mathbb{N}},Z} \circ p \mid Z \in \{\mathbb{N}^{<\mathbb{N}}, \mathbb{N}^{<\mathbb{N}}_*, \mathbb{N}^{<\mathbb{N}} \cup (\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}), \mathbb{N}^{\le\mathbb{N}}, \mathbb{N}^{\le\mathbb{N}}_*\}\}$ and $\psi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}} \colon \overline{\phi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}}(\mathbb{N}^\mathbb{N})} \to \overline{\phi(\mathbb{N}^\mathbb{N})}$ for which $(\overline{\pi} \restriction \mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}, \psi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}})$ is a closed continuous embedding of $\phi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}}$ into $\phi \restriction \mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}$. Then $(\overline{\pi} \restriction \mathbb{N}^\mathbb{N}_* \to \mathbb{N}^\mathbb{N}_*, \psi_{\mathbb{N}^\mathbb{N}} \sqcup \psi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}})$ is a closed continuous embedding of $\phi_{\mathbb{N}^\mathbb{N}} \sqcup \phi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}}$ into ϕ. ⊠

Theorems 8.5 and 10.2 together provide the promised twenty-seven element basis under closed continuous embeddability for the class of non-σ-continuous-with-closed-witnesses Borel functions between analytic metric spaces.

11. $\mathbf{\Delta}^0_2$-measurable $\mathbb{N}$-colorings

Proceeding exactly as in §2, one can endow the set $(2 \times \mathbb{N})^\mathbb{N}_* = (2 \times \mathbb{N})^\mathbb{N} \cup \{t \frown (i, \infty) \mid i < 2 \text{ and } t \in (2 \times \mathbb{N})^{<\mathbb{N}}\}$ with the smallest topology making the sets $\{t\}$ and $\mathcal{N}_t = \{c \in (2 \times \mathbb{N})^{\mathbb{N}}_* \mid t \sqsubseteq c\}$ clopen for all $t \in (2 \times \mathbb{N})^{<\mathbb{N}}$. Let $\mathbb{G}_0(\mathbf{\Delta}^0_2)$ be the digraph on $(2 \times \mathbb{N})^{\mathbb{N}}_*$ given by

$$\mathbb{G}_0(\mathbf{\Delta}^0_2) = \{(t \frown (0, \infty), t \frown (1, \infty)) \mid t \in (2 \times \mathbb{N})^{<\mathbb{N}}\}.$$

Theorem. *Suppose that X is an analytic metric space and G is a digraph on X. Then exactly one of the following holds:*

1. *There is a $\mathbf{\Delta}^0_2$-measurable $\mathbb{N}$-coloring of G.*
2. *There is a continuous homomorphism from $\mathbb{G}_0(\mathbf{\Delta}^0_2)$ to G.*

Proof. To see that conditions (1) and (2) are mutually exclusive, suppose that $(C_i)_{i \in \mathbb{N}}$ is a sequence of closed G-independent sets covering X and $\phi \colon (2 \times \mathbb{N})^{\mathbb{N}}_* \to X$ is a continuous homomorphism from $\mathbb{G}_0(\mathbf{\Delta}^0_2)$ to G, recursively construct $b \in (2 \times \mathbb{N})^{\mathbb{N}}$ such that for no $i \in \mathbb{N}$ is there an extension of $b \upharpoonright (i+1)$ in $\phi^{-1}(C_i)$, and note that $b \notin \bigcup_{i \in \mathbb{N}} \phi^{-1}(C_i)$, a contradiction.

Fix a bijection $f \colon 2 \times \mathbb{N} \to \mathbb{N}$, and define $N_i = f(\{i\} \times \mathbb{N})$ for all $i < 2$. Let H be the $\mathbb{N}$-dimensional dihypergraph on X consisting of all sequences $(x_n)_{n \in \mathbb{N}}$ for which there exists $(\overline{x}_0, \overline{x}_1) \in G$ such that $\overline{x}_i = \lim_{n \to \infty} x_{f(i,n)}$ for all $i < 2$. Note that if a set $Y \subseteq X$ is H-independent, then $\overline{Y}$ is G-independent. So if there is an $\mathbb{N}$-coloring of H, then there is a $\mathbf{\Delta}^0_2$-measurable $\mathbb{N}$-coloring of G.

Lemma 11.1. *The $\mathbb{N}$-dimensional dihypergraph H is box open.*

Proof. Suppose that $(x_n)_{n \in \mathbb{N}} \in H$, fix a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ of positive real numbers converging to zero, and note that $\prod_{n \in \mathbb{N}} \mathcal{B}(x_n, \epsilon_n)$ is a box-open subset of $X^{\mathbb{N}}$ contained in H. ⊠

By Theorem 1.1 and Lemma 11.1, we can assume that there is a continuous homomorphism $\phi \colon \mathbb{N}^{\mathbb{N}} \to X$ from $\mathbb{H}$ to H.

Lemma 11.2. *Suppose that $t \in \mathbb{N}^{<\mathbb{N}}$. Then there exists $(x_{0,t}, x_{1,t}) \in G$ with the property that $x_{i,t} = \lim_{n \to \infty} \phi(\mathcal{N}_{t \frown (f(i,n))})$ for all $i < 2$.*

Proof. As ϕ is a homomorphism from $\mathbb{H}$ to H, it follows that if $(x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \phi(\mathcal{N}_{t \frown (n)})$, then there exists $(x_{0,t}, x_{1,t}) \in G$ such that $x_{f(i,n)} \to x_{i,t}$ for all $i < 2$. If it is not the case that $\phi(\mathcal{N}_{t \frown (f(i,n))}) \to x_{i,t}$ for some $i < 2$, then there is an open neighborhood U of $x_{i,t}$ for which there is an infinite set $N \subseteq \mathbb{N}$ such that for all $n \in N$, there exists $y_{f(i,n)} \in \phi(\mathcal{N}_{t \frown (f(i,n))}) \setminus U$. By shrinking N if necessary, we can assume that it is also co-infinite, in which case $(x_{f(i,n)})_{n \in \sim N} \cup (y_{f(i,n)})_{n \in N}$ does not converge, a contradiction. ⊠

Define $\psi \colon (2 \times \mathbb{N})^{\mathbb{N}} \to X$ by $\psi = \phi \circ f^{\mathbb{N}}$, and let $\overline{\psi}$ be the extension of ψ to $(2 \times \mathbb{N})^{\mathbb{N}}_*$ given by $\psi(t \frown (i, \infty)) = x_{i, f^{|t|}(t)}$. Clearly $\overline{\psi}$ is a homomorphism from $\mathbb{G}_0(\mathbf{\Delta}^0_2)$ to G.

Lemma 11.3. *Suppose that* $t \in (2 \times \mathbb{N})^{<\mathbb{N}}$*. Then* $\overline{\psi}(\mathcal{N}_t) \subseteq \overline{\psi(\mathcal{N}_t)}$*.*

Proof. Note that $\overline{\psi}(\mathcal{N}_t \setminus \mathbb{N}^{\mathbb{N}}) \subseteq \bigcup_{s \sqsupseteq t} \overline{\psi(\mathcal{N}_s)} \subseteq \overline{\psi(\mathcal{N}_t)}$. ⊠

To see that $\overline{\psi}$ is continuous, suppose that $c \in (2 \times \mathbb{N})^{\mathbb{N}}_*$ and U is an open neighborhood of $\overline{\psi}(c)$, and fix an open neighborhood V of $\overline{\psi}(c)$ whose closure is contained in U. If $c \in (2 \times \mathbb{N})^{\mathbb{N}}$, then there exists $i \in \mathbb{N}$ for which $\psi(\mathcal{N}_{c \upharpoonright i}) \subseteq V$, thus $\mathcal{N}_{c \upharpoonright i}$ is an open neighborhood of c whose image under $\overline{\psi}$ is contained in U. Otherwise, there exist $i < 2$ and $t \in (2 \times \mathbb{N})^{<\mathbb{N}}$ for which $c = t \frown (i, \infty)$, as well as $j \in \mathbb{N}$ for which $\psi(\mathcal{N}_t \setminus \bigcup_{k<j} \mathcal{N}_{t \frown (i,k)}) \subseteq V$. Then $\mathcal{N}_t \setminus \bigcup_{k<j} \mathcal{N}_{t \frown (i,k)}$ is an open neighborhood of c whose image under $\overline{\psi}$ is contained in U. ⊠

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