The existence of invariant measures

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Introduction

These are the notes accompanying a course on the existence of invariant measures at the Kurt Gödel Research Center for Mathematical Logic at the University of Vienna in Fall 2017. I am grateful to the head of the KGRC, Sy Friedman, for his encouragement, as well as to all of the participants.

Contents

Part I

Basic notions

1. Quasi-invariance

Suppose that X is a standard Borel space and E is a countable Borel equivalence relation on X. We say that a Borel measure μ on X is *E-quasi-invariant* if $\mu(B) > 0 \iff \mu(T(B)) > 0$ for all Borel sets $B \subseteq X$ and Borel automorphisms $T \colon X \to X$ whose graphs are contained in E.

Proposition 1.1. *Suppose that X is a standard Borel space, Γ is a countable group of Borel automorphisms of X, and μ is a Borel measure on X with the property that $\mu(B) > 0 \iff \mu(\gamma B) > 0$ for all Borel sets $B \subseteq X$ and $\gamma \in \Gamma$. Then $\mu(B) > 0 \iff \mu(T(B)) > 0$ for all Borel sets $B \subseteq X$ and Borel functions $T \colon B \to X$ whose graphs are contained in E_Γ^X.*

Proof. Set $B_\gamma = \{x \in B \mid T(x) = \gamma \cdot x\}$ for all $\gamma \in \Gamma$. Then

$$\begin{aligned} \mu(B) > 0 &\iff \exists \gamma \in \Gamma \ \mu(B_\gamma) > 0 \\ &\iff \exists \gamma \in \Gamma \ \mu(\gamma B_\gamma) > 0 \\ &\iff \exists \gamma \in \Gamma \ \mu(T(B_\gamma)) > 0 \\ &\iff \mu(T(B)) > 0, \end{aligned}$$

which completes the proof. ⊠

The following observations often allow one to reduce questions about Borel measures to the E-quasi-invariant case.

Proposition 1.2. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and μ is a Borel measure on X. Then there is an E-quasi-invariant Borel measure ν on X such that $\mu \ll \nu$ and μ and ν agree on every E-invariant Borel set $B \subseteq X$.*

Proof. Fix a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ of positive real numbers whose sum is one, appeal to the Feldman-Moore theorem to obtain a group $\Gamma = \{\gamma_n \mid n \in \mathbb{N}\}$ of Borel automorphisms of X whose induced orbit equivalence relation is E, and define $\nu = \sum_{n \in \mathbb{N}} \epsilon_n (\gamma_n)_* \mu$.

To see that ν is E-quasi-invariant, note that if $B \subseteq X$ is a Borel set and $\gamma \in \Gamma$, then

$$\begin{aligned} \nu(B) > 0 &\iff \exists \delta \in \Gamma \ \mu(\delta B) > 0 \\ &\iff \exists \delta \in \Gamma \ \mu(\delta \gamma B) > 0 \\ &\iff \nu(\gamma B) > 0. \end{aligned}$$

To see that $\mu \ll \nu$, note that if $B \subseteq X$ is Borel and $\mu(B) > 0$, then $((1_\Gamma)_* \mu)(B) > 0$, so $\nu(B) > 0$.

To see that $\mu(B) = \nu(B)$ for all E-invariant Borel sets $B \subseteq X$, note that $B = \gamma^{-1}B$ for all $\gamma \in \Gamma$, so $\nu(B) = \sum_{n \in \mathbb{N}} \epsilon_n \mu(B) = \mu(B)$. ⊠

Proposition 1.3 (Kechris-Miller). *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and μ is a Borel probability measure on X. Then there is a μ-conull Borel set $B \subseteq X$ such that $\mu \upharpoonright B$ is $(E \upharpoonright B)$-quasi-invariant.*

Proof. We can assume that X is a Polish space. Fix a basis $\mathcal{U} = \{U_n \mid n \in \mathbb{N}\}$ for X that is closed under finite unions, as well as a group $\Gamma = \{\gamma_n \mid n \in \mathbb{N}\}$ of Borel automorphisms of X whose induced orbit equivalence relation is E. Let S be the set of pairs $(m, n) \in \mathbb{N} \times \mathbb{N}$ for which there is a Borel set $B_{m,n} \subseteq U_n$ such that $\mu(B_{m,n}) > \mu(U_n)/2$ and $\mu(\gamma_m B_{m,n}) = 0$. Then the set $B = \sim\bigcup_{(m,n) \in S} \gamma_m B_{m,n}$ is μ-conull.

Suppose, towards a contradiction, that $\mu \upharpoonright B$ is not $(E \upharpoonright B)$-quasi-invariant. Then there is a μ-positive Borel set $C \subseteq B$ and a Borel automorphism $T \colon B \to B$ such that $T(C)$ is μ-null and $\mathrm{graph}(T) \subseteq E$. Fix $m \in \mathbb{N}$ for which the set $D = \{x \in C \mid T(x) = \gamma_m \cdot x\}$ is μ-positive. As Borel probability measures on Polish spaces are regular, there exists $n \in \mathbb{N}$ such that $\mu(D \cap U_n) > \mu(U_n)/2$. But then $(m, n) \in S$ and $B_{m,n} \cap D \neq \emptyset$, contradicting the fact that $\gamma_m D \subseteq B$. ⊠

Remark 1.4. Proposition 1.3 trivially implies its strengthening in which the set B is moreover E-complete.

2. Invariance

Suppose that Γ is a group. A function $\rho \colon E \to \Gamma$ is a *cocycle* if $\rho(x, z) = \rho(x, y)\rho(y, z)$ whenever $x \mathrel{E} y \mathrel{E} z$.

One can think of a cocycle $\rho \colon E \to (0, \infty)$ as assigning a notion of relative size to each E-class C, with the *ρ-size* of a point $y \in C$ relative to a point $z \in C$ being $\rho(y, z)$. More generally, the *ρ-size* of a set $Y \subseteq C$ relative to z is given by $|Y|_z^\rho = \sum_{y \in Y} \rho(y, z)$. We say that Y is *ρ-infinite* if this quantity is infinite. As the definition of cocycle ensures that $|Y|_{z'}^\rho = \sum_{y \in Y} \rho(y, z') = \sum_{y \in Y} \rho(y, z)\rho(z, z') = |Y|_z^\rho \, \rho(z, z')$ for all $z' \in C$, it follows that the notion of being ρ-infinite does not depend on the choice of $z \in C$. It also follows that if $Z \subseteq C$ is non-empty, then $|Y|_x^\rho / |Z|_x^\rho$ does not depend on the choice of $x \in C$. We refer to this quantity as the *ρ-size* of Y relative to Z, which we denote by $|Y|_Z^\rho$.

Given a Borel cocycle $\rho \colon E \to (0, \infty)$, we say that a Borel measure μ on X is *ρ-invariant* if

$$\mu(T(B)) = \int_B \rho(T(x), x) \; d\mu(x)$$

for all Borel sets $B \subseteq X$ and Borel automorphisms $T: X \to X$ whose graphs are contained in E. Intuitively, this says that the global notion of size given by μ is compatible with the local notion of size given by ρ. When ρ is constant, we say that μ is *E-invariant*.

Proposition 2.1. *Suppose that X is a standard Borel space, Γ is a countable group of Borel automorphisms of X, $\rho: E_\Gamma^X \to (0,\infty)$ is a Borel cocycle, and μ is a Borel measure on X with the property that $\mu(\gamma B) = \int_B \rho(\gamma \cdot x, x) \; d\mu(x)$ for all Borel sets $B \subseteq X$ and $\gamma \in \Gamma$. Then $\mu(T(B)) = \int_B \rho(T(x), x) \; d\mu(x)$ for all Borel sets $B \subseteq X$ and Borel injections $T: B \to X$ whose graphs are contained in E_Γ^X.*

Proof. Fix an enumeration $(\gamma_n)_{n \in \mathbb{N}}$ of Γ, and recursively define $B_n = \{x \in B \setminus \bigcup_{m<n} B_m \mid T(x) = \gamma_n \cdot x\}$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} \mu(T(B)) &= \textstyle\sum_{n \in \mathbb{N}} \mu(T(B_n)) \\ &= \textstyle\sum_{n \in \mathbb{N}} \mu(\gamma_n B_n) \\ &= \textstyle\sum_{n \in \mathbb{N}} \int_{B_n} \rho(\gamma_n \cdot x, x) \; d\mu(x) \\ &= \textstyle\sum_{n \in \mathbb{N}} \int_{B_n} \rho(T(x), x) \; d\mu(x) \\ &= \textstyle\int_B \rho(T(x), x) \; d\mu(x), \end{aligned}$$

which completes the proof. ⊠

Proposition 2.2. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho: E \to (0,\infty)$ is a Borel cocycle, and μ is a ρ-invariant Borel measure on X. Then*

$$\mu(\phi^{-1}(B)) = \textstyle\int_B |\phi^{-1}(\{x\})|_x^\rho \; d\mu(x)$$

for all Borel functions $\phi: X \to X$ whose graphs are contained in E and Borel sets $B \subseteq X$.

Proof. By the Lusin-Novikov uniformization theorem, there are Borel sets $B_n \subseteq B$ and Borel injections $T_n: B_n \to X$ with the property that $(\mathrm{graph}(T_n))_{n \in \mathbb{N}}$ partitions $\mathrm{graph}(\phi^{-1}) \cap (B \times X)$. Fix Borel extensions $T_n': B \to X$ of T_n whose graphs are contained in E. Then

$$\begin{aligned} \textstyle\int_B |\phi^{-1}(\{x\})|_x^\rho \; d\mu(x) &= \textstyle\int_B \sum_{n \in \mathbb{N}} \chi_{B_n}(x) \rho(T_n'(x), x) \; d\mu(x) \\ &= \textstyle\sum_{n \in \mathbb{N}} \int_{B_n} \rho(T_n(x), x) \; d\mu(x) \\ &= \textstyle\sum_{n \in \mathbb{N}} \mu(T_n(B_n)) \\ &= \mu(\phi^{-1}(B)), \end{aligned}$$

by Proposition 2.1. ⊠

A similar change-of-variables argument yields a general formula for calculating an integral along a Borel transversal of a finite Borel subequivalence relation.

PROPOSITION 2.3. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho\colon E \to (0, \infty)$ is a Borel cocycle, and μ is a ρ-invariant Borel measure. Then*

$$\int \phi \ d\mu = \int_A \sum_{y \in [x]_F} \phi(y)\rho(y, x) \ d\mu(x)$$

for all Borel functions $\phi\colon X \to [0, \infty)$, finite Borel subequivalence relations F of E, and Borel transversals $A \subseteq X$ of F.

PROOF. Fix Borel sets $A_n \subseteq A$, Borel injections $T_n\colon A_n \to X$ with the property that $(\mathrm{graph}(T_n))_{n \in \mathbb{N}}$ partitions $F \cap (A \times X)$, and Borel extensions $T'_n\colon A \to X$ of T_n whose graphs are contained in E. Then

$$\begin{aligned}
\int \phi \ d\mu &= \sum_{n \in \mathbb{N}} \int_{T_n(A_n)} \ \phi \ d\mu \\
&= \sum_{n \in \mathbb{N}} \int_{A_n} \phi \circ T_n \ d((T_n^{-1})_* \mu) \\
&= \sum_{n \in \mathbb{N}} \int_{A_n} (\phi \circ T_n)(x) \rho(T_n(x), x) \ d\mu(x) \\
&= \int_A \sum_{n \in \mathbb{N}} \chi_{A_n}(x)(\phi \circ T'_n)(x) \rho(T'_n(x), x) \ d\mu(x) \\
&= \int_A \sum_{y \in [x]_F} \phi(y)\rho(y, x) \ d\mu(x),
\end{aligned}$$

by Proposition 2.1. ⊠

In particular, this yields the following means of computing measures using finite Borel subequivalence relations.

PROPOSITION 2.4. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho\colon E \to (0, \infty)$ is a Borel cocycle, and μ is a ρ-invariant Borel measure. Then*

$$\mu(B) = \int_A |B \cap [x]_F|^\rho_x \ d\mu(x)$$

for all Borel sets $B \subseteq X$, finite Borel subequivalence relations F of E, and Borel transversals $A \subseteq X$ of F.

PROOF. Simply observe that

$$\begin{aligned}
\mu(B) &= \int \chi_B \ d\mu \\
&= \int_A \sum_{y \in [x]_F} \chi_B(y)\rho(y, x) \ d\mu(x) \\
&= \int_A |B \cap [x]_F|^\rho_x \ d\mu(x),
\end{aligned}$$

by Proposition 2.3. ⊠

Proposition 2.5. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho\colon E \to (0,\infty)$ is a Borel cocycle, and μ is a ρ-invariant Borel measure. Then*

$$\mu(B) = \int |B \cap [x]_F|^{\rho}_{[x]_F} \, d\mu(x)$$

for all Borel $B \subseteq X$ and finite Borel subequivalence relations F of E.

Proof. Fix a Borel transversal $A \subseteq X$ of F, and observe that

$$\begin{aligned}\int |B \cap [x]_F|^{\rho}_{[x]_F} \, d\mu(x) &= \int_A \sum_{y \in [x]_F} |B \cap [y]_F|^{\rho}_{[y]_F} \rho(y,x) \, d\mu(x) \\ &= \int_A |B \cap [x]_F|^{\rho}_{[x]_F} |[x]_F|^{\rho}_x \, d\mu(x) \\ &= \int_A |B \cap [x]_F|^{\rho}_x \, d\mu(x),\end{aligned}$$

by Proposition 2.3, in which case $\mu(B) = \int |B \cap [x]_F|^{\rho}_{[x]_F} \, d\mu(x)$ by Proposition 2.4. ⊠

We close this section by noting the connection between invariance and quasi-invariance.

Proposition 2.6. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and μ is an E-quasi-invariant σ-finite Borel measure on X. Then there is a Borel cocycle $\rho\colon E \to (0,\infty)$ for which μ is ρ-invariant.*

Proof. Fix a countable group Γ of Borel automorphisms whose induced orbit equivalence relation is E. For each $\gamma \in \Gamma$, fix a Borel Radon-Nikodým derivative $\phi_\gamma\colon X \to (0,\infty)$ of $\gamma_*\mu$ with respect to μ.

Lemma 2.7. *Suppose that $\gamma, \delta \in \Gamma$. Then:*

1. *$\gamma \cdot x = \delta \cdot x \implies \phi_{\gamma^{-1}}(x) = \phi_{\delta^{-1}}(x)$ for μ-almost all $x \in X$.*
2. *$\phi_{(\gamma\delta)^{-1}}(x) = \phi_{\gamma^{-1}}(\delta \cdot x)\phi_{\delta^{-1}}(x)$ for μ-almost all $x \in X$.*

Proof. To see (1), note that if $A = \{x \in X \mid \gamma \cdot x = \delta \cdot x\}$, then $(\gamma^{-1})_*\mu \restriction A = (\delta^{-1})_*\mu \restriction A$, so the almost everywhere uniqueness of Radon-Nikodým derivatives yields that $\phi_{\gamma^{-1}}(x) = \phi_{\delta^{-1}}(x)$ for $(\mu \restriction A)$-almost all $x \in A$. To see (2), note that if $B \subseteq X$ is Borel, then

$$\begin{aligned}\int_B \phi_{\gamma^{-1}}(\delta \cdot x)\phi_{\delta^{-1}}(x) \, d\mu(x) &= \int_B \phi_{\gamma^{-1}}(\delta \cdot x) \, d((\delta^{-1})_*\mu)(x) \\ &= \int_{\delta B} \phi_{\gamma^{-1}}(x) \, d\mu(x) \\ &= ((\gamma^{-1})_*\mu)(\delta B) \\ &= \mu(\gamma\delta B) \\ &= (((\gamma\delta)^{-1})_*\mu)(B),\end{aligned}$$

so the almost everywhere uniqueness of Radon-Nikodým derivatives ensures that $\phi_{(\gamma\delta)^{-1}}(x) = \phi_{\gamma^{-1}}(\delta \cdot x)\phi_{\delta^{-1}}(x)$ for μ-almost all $x \in X$. ⊠

As μ is E-quasi-invariant, Lemma 2.7 ensures that the E-invariant Borel set $C \subseteq X$ of $x \in X$ such that $\gamma \cdot y = \delta \cdot y \Longrightarrow \phi_{\gamma^{-1}}(y) = \phi_{\delta^{-1}}(y)$ and $\phi_{(\gamma\delta)^{-1}}(y) = \phi_{\gamma^{-1}}(\delta \cdot y)\phi_{\delta^{-1}}(y)$ for all $\gamma, \delta \in \Gamma$ and $y \in [x]_E$ is μ-conull. The former condition ensures that we obtain a Borel function $\rho\colon E \upharpoonright C \to (0, \infty)$ by setting $\rho(x, y) = \phi_{\gamma^{-1}}(y)$ for all $\gamma \in \Gamma$ and $x, y \in C$ with the property that $x = \gamma \cdot y$. The latter condition implies that if $\gamma, \delta \in \Gamma$ and $x, y, z \in C$ have the property that $x = \gamma \cdot y$ and $y = \delta \cdot z$, then $\rho(x, z) = \phi_{(\gamma\delta)^{-1}}(z) = \phi_{\gamma^{-1}}(\delta \cdot z)\phi_{\delta^{-1}}(z) = \rho(x, y)\rho(y, z)$, thus ρ is a cocycle. As $\mu(\gamma B) = ((\gamma^{-1})_*\mu)(B) = \int_B \phi_{\gamma^{-1}} \, d\mu = \int_B \rho(\gamma \cdot x, x) \, d\mu(x)$ for all Borel sets $B \subseteq C$ and $\gamma \in \Gamma$, Proposition 2.1 ensures that $\mu \upharpoonright C$ is $(\rho \upharpoonright (E \upharpoonright C))$-invariant, thus any extension of ρ to a Borel cocycle on E is as desired. ⊠

Part II

The existence of invariant σ-finite measures

3. Lacunary sets

Given a digraph G on X, we say that a set $Y \subseteq X$ is a G-*clique* if all pairs of distinct points of Y are G-related. Given a Borel cocycle $\rho\colon E \to \Gamma$ and a set $Z \subseteq \Gamma$, let G_Z^ρ denote the digraph on X with respect to which distinct points x and y are related if and only if they are E-equivalent and $\rho(x, y) \in Z$.

Proposition 3.1. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, Γ is a topological group, $\rho\colon E \to \Gamma$ is a Borel cocycle, and $K \subseteq \Gamma$ is compact. If there is an open neighborhood $U \subseteq \Gamma$ of 1_Γ for which there is no infinite G_U^ρ-clique, then the vertical sections of G_K^ρ are finite.*

Proof. Fix a non-empty open set $V \subseteq \Gamma$ for which $V^{-1}V \subseteq U$, as well as a finite sequence $(\gamma_i)_{i<n}$ of elements of Γ with the property that $K \subseteq \bigcup_{i<n} \gamma_i V$. As $(G_K^\rho)_x \subseteq \bigcup_{i<n} (G_{\gamma_i V}^\rho)_x$ for all $x \in X$, we need only show that each $(G_{\gamma_i V}^\rho)_x$ is a G_U^ρ-clique. But if $i < n$ and $y, z \in (G_{\gamma_i V}^\rho)_x$, then $\rho(y, z) = \rho(y, x)\rho(x, z) \in (\gamma_i V)^{-1}\gamma_i V = V^{-1}V \subseteq U$. ⊠

Remark 3.2. As Borel digraphs on standard Borel spaces with finite vertical sections have Borel $\mathbb{N}$-colorings, it follows that if there is an open neighborhood $U \subseteq \Gamma$ of 1_Γ for which there is a Borel $\mathbb{N}$-coloring of G_U^ρ, then there is a Borel $\mathbb{N}$-coloring of G_K^ρ.

Proposition 3.3. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $G \subseteq E$ is a digraph.*

1. *If there is a Borel coloring $c\colon X \to \mathbb{N}$ of G, then there is an E-complete G-independent Borel set $B \subseteq X$.*
2. *If G is of the form G_U^ρ, where Γ is a separable topological group, $\rho\colon E \to \Gamma$ is a Borel cocycle, and $U \subseteq \Gamma$ is a pre-compact open neighborhood of 1_G, then the converse holds.*

Proof. To see (1), set $A_n = c^{-1}(\{n\})$ and $B_n = A_n \setminus \bigcup_{m<n} [A_m]_E$ for all $n \in \mathbb{N}$. As the Lusin-Novikov uniformization theorem ensures that the latter sets are Borel, it follows that their union is an E-complete G_U^ρ-independent Borel set.

To see (2), appeal to the Lusin-Novikov uniformization theorem to obtain Borel sets $B_n \subseteq B$ and Borel functions $\phi_n\colon B_n \to X$ such that $E \cap (B \times X) = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(\phi_n)$. By breaking up the domains of the functions ϕ_n into countably-many Borel sets and re-indexing, we can assume the sets $K_n = \rho(\mathrm{graph}(\phi_n))$ are pre-compact. As Remark 3.2 yields Borel $\mathbb{N}$-colorings of $G_{K_n U K_n^{-1}}^\rho \cap (B \times B)$, and ϕ_n sends $G_{K_n U K_n^{-1}}^\rho$-independent Borel sets to G_U^ρ-independent Borel sets, there is a Borel $\mathbb{N}$-coloring of each $G_U^\rho \cap (\phi_n(B_n) \times \phi_n(B_n))$, and therefore of G_U^ρ. ⊠

Remark 3.4. It follows that if $U \subseteq \Gamma$ is a pre-compact open neighborhood of 1_Γ, then there is a Borel $\mathbb{N}$-coloring of $G^\rho_U \restriction {\sim}B$, where $B = \{x \in X \mid \forall y \in [x]_E \exists^\infty z \in [x]_E\ \rho(y,z) \in U\}$.

We say that a set $Y \subseteq X$ is *ρ-lacunary* if it is G^ρ_U-independent for some open neighborhood $U \subseteq \Gamma$ of 1_Γ.

Proposition 3.5. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, Γ is a locally compact separable group, and $\rho\colon E \to \Gamma$ is a Borel cocycle. Then the following are equivalent:*

1. *The set X is a countable union of ρ-lacunary Borel sets.*
2. *For every pre-compact open neighborhood $U \subseteq \Gamma$ of 1_Γ there is a Borel $\mathbb{N}$-coloring of G^ρ_U.*
3. *There is an open neighborhood $U \subseteq \Gamma$ of 1_Γ for which there is a Borel $\mathbb{N}$-coloring of G^ρ_U.*
4. *There is an E-complete ρ-lacunary Borel set.*

Proof. To see (1) $\implies$ (2), suppose that there are ρ-lacunary Borel sets $B_n \subseteq X$ such that $X = \bigcup_{n\in\mathbb{N}} B_n$, fix open neighborhoods $U_n \subseteq \Gamma$ of 1_Γ such that B_n is $G^\rho_{U_n}$-independent for all $n \in \mathbb{N}$, and appeal to Remark 3.2 to obtain Borel $\mathbb{N}$-colorings of the digraphs $G^\rho_U \cap (B_n \times B_n)$, and therefore of G^ρ_U.

As (2) $\implies$ (3) $\implies$ (1) is trivial, it only remains to note that (3) $\iff$ (4) is a direct consequence of Proposition 3.3. ⊠

When Γ is locally compact and separable, we say that a Borel cocycle $\rho\colon E \to \Gamma$ is *smooth* if it satisfies the equivalent conditions of Proposition 3.5.

4. Smooth cocycles

When $\Gamma = (0,\infty)$, we say that an injection $T\colon X \to X$ is *strictly ρ-increasing* if its graph is contained in E and $\rho(T(x),x) > 1$ for all $x \in X$.

Proposition 4.1. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho\colon E \to (0,\infty)$ is a smooth Borel cocycle. Then there is an E-invariant Borel set $B \subseteq X$ for which $E \restriction {\sim}B$ is smooth and there is a strictly $(\rho \restriction (E \restriction B))$-increasing Borel automorphism.*

Proof. Fix a partition $(B_n)_{n\in\mathbb{N}}$ of X into ρ-lacunary Borel sets, and let $n(x)$ denote the unique natural number for which $x \in B_{n(x)}$. Let $\preceq$ be the partial order on X with respect to which $x \preceq y$ if and only if $x \mathrel{E} y$, $n(x) = n(y)$, and $\rho(x, y) \leq 1$, and let B be the set of $x \in X$ such that for all $n \in \mathbb{N}$, either $B_n \cap [x]_E = \emptyset$ or $\preceq \restriction (B_n \cap [x]_E)$ is isomorphic to the usual ordering of $\mathbb{Z}$. Then the $(\preceq \restriction B)$-successor function is a strictly $(\rho \restriction (E \restriction B))$-increasing Borel automorphism, and the discreteness of $\preceq$ ensures that $E \restriction \sim B$ is smooth. ⊠

The *quotient* of a cocycle $\rho \colon E \to (0, \infty)$ by a finite subequivalence relation F of E is the function $\rho/F \colon E/F \to (0, \infty)$ given by $(\rho/F)([x]_F, [y]_F) = |[x]_F|^{\rho}_{[y]_F}$.

Proposition 4.2. *Suppose that X is a set, E is an equivalence relation on X, F is a finite subequivalence relation of E, Γ is a group, and $\rho \colon E \to \Gamma$ is a cocycle. Then ρ/F is a cocycle.*

Proof. Simply observe that

$$\begin{aligned}(\rho/F)([x]_F, [z]_F) &= |[x]_F|^{\rho}_{w} / |[z]_F|^{\rho}_{w} \\ &= (|[x]_F|^{\rho}_{w} / |[y]_F|^{\rho}_{w})(|[y]_F|^{\rho}_{w} / |[z]_F|^{\rho}_{w}) \\ &= (\rho/F)([x]_F, [y]_F)(\rho/F)([y]_F, [z]_F)\end{aligned}$$

whenever $w \mathrel{E} x \mathrel{E} y \mathrel{E} z$. ⊠

Proposition 4.3. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to (0, \infty)$ is a Borel cocycle, and F is a finite Borel subequivalence relation of E. Then ρ is smooth if and only if ρ/F is smooth.*

Proof. Proposition 4.2 ensures that if $x \mathrel{E} y \mathrel{E} z$, then

$$\begin{aligned}\rho(x, y) &= \rho(x, z)\rho(z, y) \\ &= \rho(x, z)/\rho(y, z) \\ &= |\{x\}|^{\rho}_{\{y\}} \\ &= |\{x\}|^{\rho}_{[x]_F} |[x]_F|^{\rho}_{[y]_F} |[y]_F|^{\rho}_{\{y\}},\end{aligned}$$

so $\rho(x, y)/(\rho/F)([x]_F, [y]_F) = |\{x\}|^{\rho}_{[x]_F} |[y]_F|^{\rho}_{\{y\}}$.

By partitioning X into countably-many F-invariant Borel sets, we can assume that there is a real number $r > 1$ such that $|[x]_F|^{\rho}_{x} < r$ for all $x \in X$. Then $1/r < |\{x\}|^{\rho}_{[x]_F} |[y]_F|^{\rho}_{\{y\}} < r$ for all $x, y \in X$, so $1/r < \rho(x, y)/(\rho/F)([x]_F, [y]_F) < r$ whenever $x \mathrel{E} y$.

One consequence is that if $Y \subseteq X$ and the quotient $[Y]_F/F$ is $G^{\rho/F}_{(1/r, r)}$-dependent, then Y is $G^{\rho}_{(1/r^2, r^2)}$-dependent, so the smoothness of ρ yields that of ρ/F.

Another consequence is that if $Y \subseteq X$ is both F-invariant and $(G^{\rho}_{(1/r, r)} \setminus F)$-dependent, then the quotient Y/F is $G^{\rho/F}_{(1/r^2, r^2)}$-dependent.

As locally finite Borel graphs on standard Borel spaces have Borel $\mathbb{N}$-colorings, the smoothness of ρ/F therefore yields that of ρ. ⊠

We say that a cocycle $\rho\colon E \to (0,\infty)$ is *aperiodic* if every E-class is ρ-infinite.

PROPOSITION 4.4. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho\colon E \to (0,\infty)$ is an aperiodic smooth Borel cocycle. Then there is a finite Borel subequivalence relation F of E for which there is a strictly (ρ/F)-increasing Borel injection.*

PROOF. By Proposition 4.1, we can assume that E is smooth. As the aperiodicity of ρ yields that of E, there is a partition $(B_n)_{n\in\mathbb{N}}$ of X into Borel transversals of E. For each $x \in X$, let $n(x)$ be the unique natural number for which $x \in B_{n(x)}$, set $n_0(x) = 0$, recursively define $n_{i+1}(x)$ to be the least natural number with the property that the ρ-size of $\{y \in [x]_E \mid n(y) \leq n_{i+1}(x)\}$ relative to $\{y \in [x]_E \mid n(y) \leq n_i(x)\}$ is strictly greater than two, and let $i(x)$ be the least natural number for which $n(x) \leq n_{i(x)}(x)$. Let F be the subequivalence relation of E given by $x \mathrel{F} y \iff (x \mathrel{E} y \text{ and } i(x) = i(y))$, and observe that the Borel injection obtained by sending $[x]_F$ to $[y]_F$ if and only if ($x \mathrel{E} y$ and $i(x) = i(y) - 1$) is strictly (ρ/F)-increasing. ⊠

5. A generalization of the $\mathbb{E}_0$ dichotomy

Given an open neighborhood $U \subseteq \Gamma$ of 1_Γ, a *U-Lipschitz embedding* of a cocycle $\sigma\colon E \to \Gamma$ into a cocycle $\rho\colon F \to \Gamma$ is an embedding $\pi\colon X \to Y$ of E into F such that $\rho(\pi(w), \pi(x)) \in U \cdot \sigma(w, x)$ whenever $w \mathrel{E} x$. Let $\wp_0$ denote the constant cocycle on $\mathbb{E}_0$.

THEOREM 5.1 (Glimm-Effros, Shelah-Weiss, Weiss, Jackson-Kechris-Louveau, Miller). *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, Γ is a locally-compact second-countable group, $\rho\colon E \to \Gamma$ is a Borel cocycle, and $U \subseteq \Gamma$ is an open neighborhood of 1_Γ. Then at least one of the following holds:*

1. *The cocycle ρ is smooth.*
2. *There is a continuous U-Lipschitz embedding of $\wp_0$ into ρ.*

Moreover, if U is pre-compact, then exactly one of these holds.

PROOF. To see that conditions (1) and (2) are mutually exclusive when U is pre-compact, note that if ρ is smooth, then there is a sequence $(B_n)_{n\in\mathbb{N}}$ of G^ρ_U-independent Borel sets with the property that $X = \bigcup_{n\in\mathbb{N}} B_n$. But if $\pi\colon 2^\mathbb{N} \to X$ is a Borel U-Lipschitz embedding of $\wp_0$ into ρ, then $(\pi^{-1}(B_n))_{n\in\mathbb{N}}$ is a sequence of Borel partial transversals of $\mathbb{E}_0$ with the property that $2^{\mathbb{N}} = \bigcup_{n\in\mathbb{N}} \pi^{-1}(B_n)$, contradicting the fact that $\mathbb{E}_0$ is not smooth.

It remains to show that if condition (1) fails, then condition (2) holds. Towards this end, fix a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive real numbers converging to zero. Set $U_0 = U$, and fix pre-compact open neighborhoods $U_{n+1} \subseteq \Gamma$ of 1_Γ such that $U_{n+1}^2 U_{n+1}^{-1} \subseteq U_n$ for all $n \in \mathbb{N}$. A simple induction shows that $(\prod_{m\leq n} U_{m+1})U_{n+1}(\prod_{m\leq n} U_{m+1})^{-1} \subseteq U$ for all $n \in \mathbb{N}$. Fix a countable group Δ of Borel automorphisms of X whose orbit equivalence relation is E, and an increasing sequence $(\Delta_n)_{n\in\mathbb{N}}$ of finite sets containing 1_Δ whose union is Δ. By change of topology results, we can assume that Δ acts on X by homeomorphisms, and that for all $\delta \in \Delta$, the function $\rho_\delta \colon X \to \Gamma$ given by $\rho_\delta(x) = \rho(\delta \cdot x, x)$ is continuous. Fix a compatible complete metric on X.

We will construct open sets $V_n \subseteq X$ and group elements $\delta_n \in \Delta$, from which we define $\delta^s = \prod_{n<|s|} \delta_n^{s(n)}$ for all $s \in 2^{<\mathbb{N}}$, so as to ensure that the following conditions hold:

(a) $\forall n \in \mathbb{N}\ \rho \upharpoonright (E \upharpoonright V_n)$ is non-smooth.
(b) $\forall n \in \mathbb{N}\ V_{n+1} \subseteq \rho_{\delta_n}^{-1}(U_{n+1})$.
(c) $\forall n \in \mathbb{N}\ \overline{V_{n+1}} \cup \delta_n \overline{V_{n+1}} \subseteq V_n$.
(d) $\forall n \in \mathbb{N} \forall \delta \in \Delta_n \forall s,t \in 2^n\ \delta\delta^s V_{n+1} \cap \delta^t \delta_n V_{n+1} = \emptyset$.
(e) $\forall n \in \mathbb{N} \forall s \in 2^{n+1}\ \mathrm{diam}(\delta^s V_{n+1}) \leq \epsilon_n$.

We begin by setting $V_0 = X$. Suppose now that $n \in \mathbb{N}$ and we have already found V_n and $(\delta_i)_{i<n}$. For each $\delta \in \Delta$, let $V_{n,\delta}$ be the set of $x \in V_n \cap \delta^{-1} V_n \cap \rho_\delta^{-1}(U_{n+1})$ such that $\forall \delta' \in \Delta_n \forall s,t \in 2^n\ \delta'\delta^s \cdot x \neq \delta^t \delta \cdot x$. As the horizontal sections of $G^\rho_{U_{n+1}} \cap ((V_n \setminus \bigcup_{\delta\in\Delta} V_{n,\delta}) \times (V_n \setminus \bigcup_{\delta\in\Delta} V_{n,\delta}))$ have size at most $4^n|\Delta_n|$, it follows that there is a Borel $\mathbb{N}$-coloring of $G^\rho_{U_{n+1}} \cap ((V_n \setminus \bigcup_{\delta\in\Delta} V_{n,\delta}) \times (V_n \setminus \bigcup_{\delta\in\Delta} V_{n,\delta}))$, so ρ is smooth on $E \upharpoonright (V_n \setminus \bigcup_{\delta\in\Delta} V_{n,\delta})$, thus there exists $\delta_n \in \Delta$ for which $\rho \upharpoonright (E \upharpoonright V_{n,\delta_n})$ is non-smooth. As V_{n,δ_n} is the union of a countable set $\mathcal{V}_{n+1}$ of open sets $V \subseteq X$ satisfying the analogs of conditions (c), (d), and (e) with V in place of V_{n+1}, there exists $V_{n+1} \in \mathcal{V}_{n+1}$ satisfying conditions (a) – (e). This completes the recursive construction.

Note that if $c \in 2^{\mathbb{N}}$, then $\delta^{c\upharpoonright(n+1)}\overline{V_{n+1}} \subseteq \delta^{c\upharpoonright n}(\overline{V_{n+1}} \cup \delta_n \overline{V_{n+1}}) \subseteq \delta^{c\upharpoonright n} V_n$ for all $n \in \mathbb{N}$ by condition (c), and $\mathrm{diam}(\delta^{c\upharpoonright n} V_n) \to 0$ by condition (e), so we obtain a continuous function $\pi \colon 2^{\mathbb{N}} \to X$ by letting $\pi(c)$ be the unique element of $\bigcap_{n\in\mathbb{N}} \delta^{c\upharpoonright n} V_n$, for all $c \in 2^{\mathbb{N}}$.

Observe now that if $c \in 2^{\mathbb{N}}$, $k \in \mathbb{N}$, and $s \in 2^k$, then

$$\begin{aligned}\{\delta^s \cdot \pi((0)^k \frown c)\} &= \delta^s \cdot \textstyle\bigcap_{n \geq k} \delta^{((0)^k \frown c) \upharpoonright n} V_n \\ &= \textstyle\bigcap_{n \geq k} \delta^{(s \frown c) \upharpoonright n} V_n \\ &= \{\pi(s \frown c)\},\end{aligned}$$

in which case $\rho(\pi(s \frown c), \pi((0)^k \frown c))$ can be expressed as

$$\textstyle\prod_{i<k} \rho((\prod_{i \leq j < k} \delta_j^{s(j)}) \cdot \pi((0)^k \frown c), (\prod_{i<j<k} \delta_j^{s(j)}) \cdot \pi((0)^k \frown c)),$$

and is therefore in $\prod_{i<k} U_{i+1}$ by k applications of condition (b), so $\rho(\pi(s \frown c), \pi(t \frown c)) \in (\prod_{i<k} U_{i+1})(\prod_{i<k} U_{i+1})^{-1}$ for all $c \in 2^{\mathbb{N}}$, $k \in \mathbb{N}$, and $s, t \in 2^k$, thus $c \mathbin{\mathbb{E}_0} d \Longrightarrow (\pi(c) \mathrel{E} \pi(d)$ and $\rho(\pi(c), \pi(d)) \in U)$.

But if $c, d \in 2^{\mathbb{N}}$, $n \in \mathbb{N}$, and $c(n) < d(n)$, then $\pi(c) \in \delta^{c \upharpoonright n} V_{n+1}$ and $\pi(d) \in \delta^{d \upharpoonright n} \delta_n V_{n+1}$, so condition (d) yields that $\forall \delta \in \Delta_n \ \delta \cdot \pi(c) \neq \pi(d)$, thus $c \neq d \Longrightarrow \pi(c) \neq \pi(d)$ and $\neg c \mathbin{\mathbb{E}_0} d \Longrightarrow \neg \pi(c) \mathrel{E} \pi(d)$. ⊠

6. Invariant measures and smoothness

We say that a Borel cocycle $\rho \colon E \to \Gamma$ is a *Borel coboundary* if there is a Borel function $\phi \colon X \to \Gamma$ such that $\rho(x,y) = \phi(x)\phi(y)^{-1}$ for all $(x,y) \in E$. When Γ is locally compact, we say that a set $Y \subseteq X$ is *ρ-bounded* if it is $G^{\rho}_{\sim U}$-independent for some pre-compact open neighborhood $U \subseteq \Gamma$ of 1_Γ.

Proposition 6.1. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, Γ is a locally-compact separable group, $\rho \colon E \to \Gamma$ is a Borel cocycle, and $U \subseteq \Gamma$ is an open neighborhood of 1_Γ.*

1. *If ρ is a Borel coboundary, then there is an E-complete $G^{\rho}_{\sim U}$-independent Borel set $B \subseteq X$.*
2. *If $\Gamma = (0, \infty)$ and U is pre-compact, then the converse holds.*

Proof. To see (1), suppose that $\phi \colon X \to \Gamma$ is a Borel function with the property that $\rho(x,y) = \phi(x)\phi(y)^{-1}$ for all $(x,y) \in E$. Fix an enumeration $(\gamma_n)_{n \in \mathbb{N}}$ of a dense subset of Γ, as well as an open set $V \subseteq \Gamma$ for which $VV^{-1} \subseteq U$, and let $n(x)$ be the least natural number for which $\phi([x]_E) \cap V\gamma_{n(x)} \neq \emptyset$. Then the set $B = \{x \in X \mid \phi(x) \in V\gamma_{n(x)}\}$ is E-complete and $G^{\rho}_{\sim U}$-independent.

To see (2), suppose that $B \subseteq X$ is an E-complete ρ-bounded Borel set, define $\phi \colon X \to (0, \infty)$ by $\phi(x) = \sup\{\rho(x,y) \mid y \in B \cap \phi([x]_E)\}$. Given $x \mathrel{E} y$, fix a sequence $(z_n)_{n \in \mathbb{N}}$ of points of $[x]_E$ with the property that $\phi(x) = \lim_{n\to\infty} \rho(x, z_n)$ and $\phi(y) = \lim_{n\to\infty} \rho(y, z_n)$, and note that

$$\begin{aligned}\rho(x,y) &= \lim_{n\to\infty} \rho(x,z_n)\rho(z_n,y)\\ &= \lim_{n\to\infty} \rho(x,z_n)/\lim_{n\to\infty}\rho(y,z_n)\\ &= \phi(x)/\phi(y),\end{aligned}$$

by continuity. ⊠

We say that Borel cocycles $\rho\colon E \to \Gamma$ and $\sigma\colon E \to \Gamma$ are *Borel cohomologous* if there is a Borel function $\phi\colon X \to \Gamma$ with the property that $\rho(x, y) = \phi(x)\sigma(x, y)\phi^{-1}(y)$ whenever $x \mathrel{E} y$.

PROPOSITION 6.2. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\phi\colon X \to (0, \infty)$ is a Borel function witnessing that Borel cocycles $\rho, \sigma\colon E \to (0, \infty)$ are Borel cohomologous. Then for every σ-invariant Borel measure μ, the corresponding Borel measure ν, given by $\nu(B) = \int_B \phi \, d\mu$ for all Borel sets $B \subseteq X$, is ρ-invariant.*

PROOF. Observe that if $B \subseteq X$ is a Borel set and $T\colon X \to X$ is a Borel automorphism whose graph is contained in E, then

$$\begin{aligned}\nu(T(B)) &= \textstyle\int_{T(B)} \phi \, d\mu\\ &= \textstyle\int_B \phi \circ T \, d((T^{-1})_*\mu)\\ &= \textstyle\int_B (\phi \circ T)(x)\sigma(T(x), x) \, d\mu(x)\\ &= \textstyle\int_B \rho(T(x), x)\phi(x) \, d\mu(x)\\ &= \textstyle\int_B \rho(T(x), x) \, d\nu(x),\end{aligned}$$

by σ-invariance. ⊠

PROPOSITION 6.3. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho\colon E \to (0, \infty)$ is a smooth Borel cocycle, and μ is a ρ-invariant σ-finite Borel measure on X. Then there is a μ-conull Borel set on which E is smooth.*

PROOF. By breaking X into countably-many Borel sets, we can assume that μ is finite. By Proposition 4.1, there is an E-invariant Borel set $B \subseteq X$ for which $E \restriction {\sim}B$ is smooth and there is a strictly $(\rho \restriction (E \restriction B))$-increasing Borel automorphism $T\colon B \to B$. But then $\mu(B) = \mu(T(B)) = \int_B \rho(T(x), x) \, d\mu(x)$, thus $\mu(B) = 0$. ⊠

PROPOSITION 6.4. *Suppose that X is a non-empty standard Borel space, E is a smooth Borel equivalence relation on X, and μ is an E-ergodic Borel measure. Then there is a μ-conull E-class.*

PROOF. We can clearly assume, without loss of generality, that μ is non-zero. Fix a Borel reduction $\pi \colon X \to 2^{\mathbb{N}}$ of E to equality, define $d \in 2^{\mathbb{N}}$ by $d(n) = i \iff \{c \in 2^{\mathbb{N}} \mid c(n) = i\}$ is $(\pi_* \mu)$-conull, and observe that $\pi^{-1}(\{d\})$ is a μ-conull E-class. ⊠

THEOREM 6.5 (Glimm-Effros, Shelah-Weiss, Weiss, Miller). *Suppose that X is a non-empty standard Borel space, E is a countable Borel equivalence relation on X, and $\rho \colon E \to (0, \infty)$ is a Borel cocycle. Then the following are equivalent:*

1. *The cocycle ρ is smooth.*
2. *Every ρ-invariant σ-finite Borel measure concentrates on a Borel set on which E is smooth.*
3. *Every E-ergodic ρ-invariant σ-finite Borel measure concentrates on an E-class.*

PROOF. Proposition 6.3 yields (1) $\implies$ (2), while Proposition 6.4 yields (2) $\implies$ (3). To see $\neg$(1) $\implies$ $\neg$(3), fix a pre-compact open neighborhood $U \subseteq (0, \infty)$ of 1, and appeal to Theorem 5.1 to obtain a continuous U-Lipschitz embedding $\pi \colon 2^{\mathbb{N}} \to X$ of ϱ_0 into ρ. Define $\mu_0 = \pi_* \mathbb{\mu}_0$ and $B = \pi(2^{\mathbb{N}})$. The fact that $\mathbb{\mu}_0$ is continuous, $\mathbb{E}_0$-ergodic, and $\mathbb{E}_0$-invariant ensures that $\mu_0 \upharpoonright B$ is continuous, $(E \upharpoonright B)$-ergodic, and $(E \upharpoonright B)$-invariant.

LEMMA 6.6. *There are Borel sets $B_n \subseteq B$ and Borel injections $T_n \colon B_n \to X$, whose graphs are contained in E, with the property that $(T_n(B_n))_{n \in \mathbb{N}}$ partitions $[B]_E$.*

PROOF. Fix a group $\Gamma = \{\gamma_n \mid n \in \mathbb{N}\}$ of Borel automorphisms for which $E = E_\Gamma^X$. For each $x \in [B]_E$, let $n(x)$ be the least natural number such that $\gamma_{n(x)} \cdot x \in B$. Set $A_n = \{x \in [B]_E \mid n(x) = n\}$, $B_n = \gamma_n A_n$, and $T_n = \gamma_n^{-1} \upharpoonright B_n$ for all $n \in \mathbb{N}$. ⊠

Define $\mu = \sum_{n \in \mathbb{N}} (T_n)_* (\mu_0 \upharpoonright B_n)$.

LEMMA 6.7. *The measure μ is E-invariant.*

PROOF. Suppose that $T \colon X \to X$ is a Borel automorphism whose graph is contained in E, and $A \subseteq X$ is Borel. For all $m, n \in \mathbb{N}$, define $A_{m,n} = A \cap T_m(B_m) \cap (T^{-1} \circ T_n)(B_n)$, as well as $A'_{m,n} = T_m^{-1}(A_{m,n})$ and $A''_{m,n} = (T_n^{-1} \circ T)(A_{m,n})$, and observe that $(T_n^{-1} \circ T \circ T_m)(A'_{m,n}) = A''_{m,n}$, so $\mu(A_{m,n}) = \mu_0(A'_{m,n}) = \mu_0(A''_{m,n}) = \mu(T(A_{m,n}))$. It follows that $\mu(A) = \sum_{m,n \in \mathbb{N}} \mu(A_{m,n}) = \sum_{m,n \in \mathbb{N}} \mu(T(A_{m,n})) = \mu(T(A))$. ⊠

As B is ρ-bounded, Proposition 6.1 ensures that $\rho \upharpoonright (E \upharpoonright [B]_E)$ is a Borel coboundary, so Proposition 6.2 implies that μ is equivalent to a ρ-invariant σ-finite Borel measure ν. As $\mu_0 \upharpoonright B$ is continuous and

$(E \upharpoonright B)$-ergodic, it follows that μ is continuous and E-ergodic, thus the same holds of ν. ⊠

Part III

The existence of invariant probability measures

7. Compressibility

We say that a function $\phi \colon X \to X$ whose graph is contained in E is *ρ-increasing* at a finite set $S \subseteq [x]_E$ if $|\phi^{-1}(S)|^\rho_x \leq |S|^\rho_x$, and *strictly ρ-increasing* at a finite set $S \subseteq [x]_E$ if $|\phi^{-1}(S)|^\rho_x < |S|^\rho_x$. A *compression* of ρ over a subequivalence relation F of E is a function $\phi \colon X \to X$, whose graph is contained in E, that is ρ-increasing at every F-class, and for which the set of F-classes at which it is strictly ρ-increasing is (E/F)-complete.

Proposition 7.1. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to (0, \infty)$ is a Borel cocycle, and there is a Borel compression $\phi \colon X \to X$ of ρ over a finite Borel subequivalence relation F of E. Then there is no ρ-invariant Borel probability measure.*

Proof. Proposition 2.2 ensures that $\mu(X) = \int |\phi^{-1}(\{x\})|^\rho_x \, d\mu(x)$. Fix a Borel transversal $A \subseteq X$ of F. Proposition 2.3 then implies that

$$\begin{aligned}
\int |\phi^{-1}(\{x\})|^\rho_x \, d\mu(x) &= \int_A \sum_{y \in [x]_F} |\phi^{-1}(\{y\})|^\rho_y \rho(y,x) \, d\mu(x) \\
&= \int_A \sum_{y \in [x]_F} |\phi^{-1}(\{y\})|^\rho_x \, d\mu(x) \\
&= \int_A |\phi^{-1}([x]_F)|^\rho_x \, d\mu(x),
\end{aligned}$$

so $\mu(X) = \int_A |[x]_F|^\rho_x \, d\mu(x) = \int_A |\phi^{-1}([x]_F)|^\rho_x \, d\mu(x)$ by Proposition 2.4.

As the set $B = \{x \in A \mid |\phi^{-1}([x]_F)|^\rho_x < |[x]_F|^\rho_x\}$ is E-complete, it follows that if $\mu(X) > 0$, then $\mu(B) > 0$. As $|\phi^{-1}([x]_F)|^\rho_x \leq |[x]_F|^\rho_x$ for all $x \in A$, it follows that if $\mu(B) > 0$, then $\mu(X) = \infty$. ⊠

A *compression* of ρ is a compression of ρ over equality.

Proposition 7.2. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to (0, \infty)$ is a Borel cocycle, and F is a finite Borel subequivalence relation of E for which there is a Borel compression $\phi \colon X/F \to X/F$ of ρ/F. Then there is a Borel compression of ρ over F.*

Proof. By the Lusin-Novikov uniformization theorem, there is a Borel uniformization $\psi \colon X \to X$ of $\{(x,y) \in E \mid \phi([x]_F) = [y]_F\}$. But every uniformization of this set is a compression of ρ over F. ⊠

A *compression* of E is a compression of the constant cocycle on E, or equivalently, a Borel injection $\phi \colon X \to X$, whose graph is contained in E, such that $\sim\phi(X)$ is E-complete.

Proposition 7.3. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and there is a Borel* *compression* $\phi\colon X \to X$ *of the constant cocycle on* E *over a finite Borel subequivalence relation* F *of* E*. Then there is a Borel compression of* E*.*

Proof. By the Lusin-Novikov uniformization theorem, there is an injective Borel uniformization $\psi\colon X \to X$ of $\{(x,y) \in E \mid \phi(x) \; F \; y\}$. But every injective uniformization of this set is a compression of E. ⊠

We next consider the connection between injective compressions and smoothness.

Proposition 7.4 (Dougherty-Jackson-Kechris, Miller). *Suppose that* X *is a standard Borel space,* E *is a countable Borel equivalence relation on* X*, and* $\rho\colon E \to (0,\infty)$ *is a Borel cocycle. Then the following are equivalent:*

1. *There is an injective Borel compression of the quotient of* ρ *by a finite Borel subequivalence relation of* E*.*
2. *There is a Borel subequivalence relation of* E *on which* ρ *is aperiodic and smooth.*

Proof. By Proposition 4.4, it is sufficient to show (1) $\implies$ (2). By Proposition 4.3, we can assume that there is an injective Borel compression $\phi\colon X \to X$ of ρ. Set $A = \{x \in X \mid |\phi^{-1}(\{x\})|^\rho_x < 1\}$, and let F be the orbit equivalence relation generated by ϕ. As the sets $A_r = \{x \in X \mid |\phi^{-1}(\{x\})|^\rho_x < r\}$ are $(\rho \restriction F)$-lacunary for all $r < 1$, it follows that $\rho \restriction (F \restriction A)$ is smooth, thus $\rho \restriction (F \restriction [A]_F)$ is aperiodic and smooth. By the Lusin-Novikov uniformization theorem, there is a Borel extension $\psi\colon X \to [A]_F$ of the identity function on $[A]_F$ whose graph is contained in E, in which case the restriction of ρ to the pullback of $F \restriction [A]_F$ through ψ is aperiodic and smooth. ⊠

We will eventually establish Nadkarni's theorem that the existence of a Borel compression of a countable Borel equivalence relation E is equivalent to the inexistence of an E-invariant Borel probability measure. The following observations rule out the most straightforward generalizations to Borel cocycles.

Proposition 7.5. *Suppose that* X *is a standard Borel space and* E *is an aperiodic smooth countable Borel equivalence relation on* X*. Then there is a Borel cocycle* $\rho\colon E \to (0,\infty)$ *that admits neither an invariant Borel probability measure nor a compression.*

Proof. Fix a strictly decreasing sequence $(r_n)_{n\in\mathbb{N}}$ of positive real numbers for which $\sum_{n\in\mathbb{N}} r_n = \infty$. As E is both aperiodic and smooth, there is a partition $(B_n)_{n\in\mathbb{N}}$ of X into Borel transversals of E. For each

$x \in X$, let $n(x)$ denote the unique natural number for which $x \in B_{n(x)}$, and define $\rho \colon E \to (0, \infty)$ by $\rho(x, y) = r_{n(x)}/r_{n(y)}$ for all $(x, y) \in E$.

The fact that $\sum_{n \in \mathbb{N}} r_n = \infty$ ensures that ρ is aperiodic, and the smoothness of E implies that of ρ. Proposition 7.4 therefore yields a Borel compression of the quotient of ρ by a finite Borel subequivalence relation, so Proposition 7.2 ensures that there is a Borel compression of ρ over a finite Borel subequivalence relation, thus Proposition 7.1 implies that there is no ρ-invariant Borel probability measure.

To see that there is no compression of ρ, note that if the graph of a function $\phi \colon X \to X$ is contained in E and $|\phi^{-1}(\{x\})|_x^\rho \leq 1$ for all $x \in X$, then a straightforward induction on $n(x)$, using the fact that $(r_n)_{n \in \mathbb{N}}$ is strictly decreasing, shows that $\phi(x) = x$ for all $x \in X$. ⊠

Proposition 7.6. *Suppose that X is a standard Borel space and E is an aperiodic countable Borel equivalence relation on X for which there is an E-invariant Borel probability measure. Then there is a Borel coboundary $\rho \colon E \to (0, \infty)$ that admits neither an invariant Borel probability measure nor an injective Borel compression of its quotient by a finite Borel subequivalence relation of E.*

Proof. Set $A_0 = B_0 = X$, and given $n \in \mathbb{N}$ and an E-complete Borel set $B_n \subseteq X$ on which E is aperiodic, fix a Borel subequivalence relation F_n of $E \restriction B_n$ whose classes are all of cardinality two (prove that this can be done!), as well as disjoint Borel transversals $A_{n+1}, B_{n+1} \subseteq B_n$ of F_n, and let $\iota_n \colon B_n \to B_n$ be the involution whose graph is F_n. For all $x \in X$, let $n(x)$ be the maximal natural number for which $x \in A_{n(x)}$, and define $\rho \colon E \to (0, \infty)$ by $\rho(x, y) = 2^{n(x)-n(y)}$ for all $(x, y) \in E$.

To see that there is no ρ-invariant Borel probability measure, note that if μ is a ρ-invariant Borel measure, then the observation that $A_{n+1} = \iota_n(B_{n+1}) = \iota_n(A_{n+2}) \sqcup \iota_n(B_{n+2}) = \iota_n(A_{n+2}) \sqcup (\iota_n \circ \iota_{n+1})(A_{n+2})$ yields $\mu(A_{n+1}) = \int_{A_{n+2}} \rho(\iota_n(x), x) + \rho((\iota_n \circ \iota_{n+1})(x), x) \, d\mu(x) = \mu(A_{n+2})$ for all $n \in \mathbb{N}$, thus $\mu(X) \in \{0, \infty\}$.

Suppose, towards a contradiction, that there is an injective Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E. Proposition 7.4 then ensures that there is a Borel subequivalence relation F of E on which ρ is aperiodic and smooth, in which case Proposition 4.1 yields an F-invariant Borel set $A \subseteq X$ such that $F \restriction {\sim}A$ is smooth and there is a strictly $(\rho \restriction (F \restriction A))$-increasing Borel automorphism $T \colon A \to A$. Fix an E-invariant Borel probability measure μ.

As the aperiodicity of $\rho \restriction F$ yields that of F, Proposition 7.4 ensures that there is a Borel compression of the quotient of $F \restriction {\sim}A$ by a finite Borel subequivalence relation, so Proposition 7.2 implies that there is a

Borel compression of $F \restriction {\sim}A$ over a finite Borel subequivalence relation, thus $\mu({\sim}A) = 0$ by Proposition 7.1.

Observe now that the facts that $A_0 = A_1 \sqcup B_1 = A_1 \sqcup \iota_0(A_1)$ and $A_{n+1} = \iota_n(B_{n+1}) = \iota_n(A_{n+2}) \sqcup \iota_n(B_{n+2}) = \iota_n(A_{n+2}) \sqcup (\iota_n \circ \iota_{n+1})(A_{n+2})$ ensure that $\mu(A_n) = 2\mu(A_{n+1})$ for all $n \in \mathbb{N}$, so $\mu(\bigcup_{n\in\mathbb{N}} A_{n+1}) = 1$, whereas $\mu(\bigcup_{n\in\mathbb{N}} A_{n+2}) = 1/2$. But the definition of ρ ensures that $T(A \cap \bigcup_{n\in\mathbb{N}} A_{n+1}) \subseteq \bigcup_{n\in\mathbb{N}} A_{n+2}$, contradicting F-invariance. ⊠

8. The existence of invariant probability measures

Given a finite set $S \subseteq X$ for which $S \times S \subseteq E$, let μ_S^ρ be the Borel probability measure on X given by $\mu_S^\rho(B) = |B \cap S|_S^\rho$.

PROPOSITION 8.1. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho\colon E \to (0,\infty)$ is a Borel cocycle, $\phi\colon X \to [0,\infty)$ is Borel, $\delta > 0$, and $\epsilon > \sup_{(x,y)\in E} \phi(x) - \phi(y)$. Then there exist an E-invariant Borel set $B \subseteq X$ and a finite Borel subequivalence relation F of $E \restriction B$ for which $\rho \restriction (E \restriction {\sim}B)$ is smooth and $\delta\epsilon > \sup_{(x,y)\in E\restriction B} \int \phi \; d\mu^\rho_{[x]_F} - \int \phi \; d\mu^\rho_{[y]_F}$.*

PROOF. By repeatedly applying the corresponding special case of the proposition over the corresponding quotients, we can assume that $\delta > 2/3$. For each $x \in X$, let $\overline{\phi}([x]_E)$ be the average of $\inf \phi([x]_E)$ and $\sup \phi([x]_E)$. Fix a maximal Borel set $\mathcal{S}$ of pairwise disjoint non-empty finite sets $S \subseteq X$ with the property that $S \times S \subseteq E$ and $\epsilon(\delta - 1/2) > |\int \phi \; d\mu_S^\rho - \overline{\phi}([S]_E)|$. Set $C = \{x \in {\sim}\bigcup \mathcal{S} \mid \phi(x) < \overline{\phi}([x]_E)\}$ and $D = \{x \in {\sim}\bigcup \mathcal{S} \mid \phi(x) > \overline{\phi}([x]_E)\}$.

LEMMA 8.2. *Suppose that $(x,y) \in E$. Then there exists a real number $r > 1$ such that x has only finitely-many $G^\rho_{(1/r,r)}$-neighbors in C or y has only finitely-many $G^\rho_{(1/r,r)}$-neighbors in D.*

PROOF. As $\delta > 2/3$, a trivial calculation reveals that $-\epsilon(\delta - 1/2)$ is strictly below the average of $-\epsilon/2$ and $\epsilon(\delta - 1/2)$, or equivalently, that the average of $-\epsilon(\delta - 1/2)$ and $\epsilon/2$ is strictly below $\epsilon(\delta - 1/2)$. It follows that by choosing $m, n \in \mathbb{N}$ for which m/n is sufficiently close to $\rho(y,x)$, we can ensure that the ratios $s = m/(m + n\rho(y,x))$ and $t = n\rho(y,x)/(m + n\rho(y,x))$ are sufficiently close to $1/2$ so as to guarantee that the sums $s(\overline{\phi}([x]_E) - \epsilon/2) + t(\overline{\phi}([x]_E) + \epsilon(\delta - 1/2))$ and $s(\overline{\phi}([x]_E) - \epsilon(\delta - 1/2)) + t(\overline{\phi}([x]_E) + \epsilon/2)$ both lie strictly between $\overline{\phi}([x]_E) - \epsilon(\delta - 1/2)$ and $\overline{\phi}([x]_E) + \epsilon(\delta - 1/2)$. Fix $r > 1$ such that they lie strictly between $(\overline{\phi}([x]_E) - \epsilon(\delta - 1/2))r^2$ and $(\overline{\phi}([x]_E) + \epsilon(\delta - 1/2))/r^2$.

Suppose, towards a contradiction, that there exist sets $S \subseteq C$ and $T \subseteq D$ of $G^\rho_{(1/r,r)}$-neighbors of x and y of cardinalities m and

n. Then $m/r < |S|^\rho_x < mr$ and $n\rho(y,x)/r < |T|^\rho_x < n\rho(y,x)r$, so $(m+n\rho(y,x))/r < |S \cup T|^\rho_x < (m+n\rho(y,x))r$, from which it follows that $s/r^2 < |S|^\rho_x/|S \cup T|^\rho_x < sr^2$ and $t/r^2 < |T|^\rho_x/|S \cup T|^\rho_x < tr^2$. As $\int \phi \ d\mu^\rho_S$ lies between $\overline{\phi}([x]_E) - \epsilon/2$ and $\overline{\phi}([x]_E) - \epsilon(\delta - 1/2)$, and $\int \phi \ d\mu^\rho_T$ lies between $\overline{\phi}([x]_E) + \epsilon(\delta - 1/2)$ and $\overline{\phi}([x]_E) + \epsilon/2$, it follows that $\int \phi \ d\mu^\rho_{S\cup T}$ lies between $(s(\overline{\phi}([x]_E) - \epsilon/2) + t(\overline{\phi}([x]_E) + \epsilon(\delta - 1/2)))/r^2$ and $(s(\overline{\phi}([x]_E) - \epsilon(\delta - 1/2)) + t(\overline{\phi}([x]_E) + \epsilon/2))r^2$, and therefore strictly between $\overline{\phi}([x]_E) - \epsilon(\delta - 1/2)$ and $\overline{\phi}([x]_E) + \epsilon(\delta - 1/2)$, contradicting the maximality of $\mathcal{S}$. ⊠

Letting B be the complement of $[C]_E \cap [D]_E$, it follows from Lemma 8.2 that $\rho \restriction (E \restriction \sim B)$ is smooth. Let F be the equivalence relation on B whose classes are the subsets of B in $\mathcal{S}$. ⊠

PROPOSITION 8.3. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho\colon E \to (0,\infty)$ is a Borel cocycle, $\phi, \psi\colon X \to [0,\infty)$ are Borel, and $r > 1$. Then there exist an E-invariant Borel set $B \subseteq X$, a Borel set $C \subseteq B$, and a finite Borel subequivalence relation F of $E \restriction B$ such that $\rho \restriction (E \restriction \sim B)$ is smooth and $\int_C \phi \ d\mu^\rho_{[x]_F} \le \int_{B\setminus C} \psi \ d\mu^\rho_{[x]_F} \le r \int_C \phi \ d\mu^\rho_{[x]_F}$ for all $x \in B$.*

PROOF. We can assume that $\phi, \psi\colon X \to (0,\infty)$. Fix a maximal Borel set $\mathcal{S}$ of pairwise disjoint non-empty finite sets $S \subseteq X$ such that $S \times S \subseteq E$ and $1 < \int_{S\setminus T} \psi \ d\mu^\rho_S \ / \ \int_T \phi \ d\mu^\rho_S < r$ for some set $T \subseteq S$. Define $D_{U,V} = (\phi^{-1}(U) \cap \psi^{-1}(V)) \setminus \bigcup \mathcal{S}$ for all sets $U, V \subseteq (0,\infty)$.

LEMMA 8.4. *For all $x \in X$, there exists $s > 1$ such that x has only finitely-many $G^\rho_{(1/s,s)}$-neighbors in $D_{(\phi(x)/s,\phi(x)s),(\psi(x)/s,\psi(x)s)}$.*

PROOF. Fix positive natural numbers m and n with the property that $1 < (\psi(x)/\phi(x))(n/m) < r$. Then there exists $s > 1$ sufficiently small that $s^6 < (\psi(x)/\phi(x))(n/m) < r/s^6$. Suppose, towards a contradiction, that there is a set $S \subseteq D_{(\phi(x)/s,\phi(x)s),(\psi(x)/s,\psi(x)s)}$ of $G^\rho_{(1/s,s)}$-neighbors of x of cardinality $k = m + n$, and fix a set $T \subseteq S$ such that $|T| = m$. Then $\phi(x)\mu^\rho_S(T)/s < \int_T \phi \ d\mu^\rho_S < \phi(x)\mu^\rho_S(T)s$ and $(m/k)/s^2 < \mu^\rho_S(T) < (m/k)s^2$, which together yield the inequality that $\phi(x)(m/k)/s^3 < \int_T \phi \ d\mu^\rho_S < \phi(x)(m/k)s^3$. Along similar lines, the facts that $\psi(x)\mu^\rho_S(S \setminus T)/s < \int_{S\setminus T} \psi \ d\mu^\rho_S < \psi(x)\mu^\rho_S(S \setminus T)s$ and $(n/k)/s^2 < \mu^\rho_S(S \setminus T) < (n/k)s^2$ together yield the inequality that $\psi(x)(n/k)/s^3 < \int_{S\setminus T} \psi \ d\mu^\rho_S < \psi(x)(n/k)s^3$, from which it follows that $\int_{S\setminus T} \psi \ d\mu^\rho_S \ / \ \int_T \phi \ d\mu^\rho_S$ lies strictly between $(\psi(x)/\phi(x))(n/m)/s^6$ and $(\psi(x)/\phi(x))(n/m)s^6$, and therefore strictly between 1 and r, contradicting the maximality of $\mathcal{S}$. ⊠

Letting B be the complement of $[\sim \bigcup \mathcal{S}]_E$, it follows from Lemma 8.4 that $\rho \restriction (E \restriction \sim B)$ is smooth. Let F be the Borel equivalence relation on B whose classes are the subsets of B in $\mathcal{S}$, and appeal to the Lusin-Novikov uniformization theorem to obtain a Borel set $C \subseteq B$ such that $1 < \int_{B \setminus C} \psi \, d\mu^\rho_{[x]_F} \,/\, \int_C \phi \, d\mu^\rho_{[x]_F} < r$ for all $x \in B$. ⊠

We are now ready to establish our primary result.

THEOREM 8.5 (Nadkarni, Becker-Kechris, Miller). *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho \colon E \to (0, \infty)$ is a Borel cocycle. Then exactly one of the following holds:*

1. *There is a finite-to-one Borel compression of ρ over a finite Borel subequivalence relation of E.*
2. *There is a ρ-invariant Borel probability measure.*

PROOF. Proposition 7.1 ensures that conditions (1) and (2) are mutually exclusive. To see that one of them holds, fix a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ of positive real numbers converging to zero, as well as a countable group Γ of Borel automorphisms whose induced orbit equivalence relation is E, and define $\rho_\gamma \colon X \to (0, \infty)$ by $\rho_\gamma(x) = \rho(\gamma \cdot x, x)$ for all $\gamma \in \Gamma$.

Fix a Polish topology on $[0, \infty)$, compatible with its underlying Borel structure, with respect to which every interval of the form $[p, q)$, where $p, q \in \mathbb{Q}$ are non-negative, is clopen. Fix a zero-dimensional Polish topology on X, compatible with its underlying Borel structure, with respect to which Γ acts by homeomorphisms and each ρ_γ is continuous. Finally, fix a compatible complete metric on X, as well as a countable algebra $\mathcal{U} \subseteq \mathcal{P}(X)$ forming a basis for X, containing the pullback of every interval of the form $[p, q)$, where $p, q \in \mathbb{Q}$ are non-negative, under each of the functions ρ_γ, and closed under multiplication by elements of Γ, in addition to an increasing sequence $(\mathcal{U}_n)_{n \in \mathbb{N}}$ of finite subsets of $\mathcal{U}$ whose union is $\mathcal{U}$.

We say that a function $\phi \colon X \to [0, \infty)$ is *$\mathcal{U}$-simple* if it is a finite linear combination of characteristic functions of sets in $\mathcal{U}$. Note that for all $\epsilon > 0$, $\gamma \in \Gamma$, and $Y \subseteq X$ on which ρ_γ is bounded, there is such a function with the further property that $|\phi(y) - \rho_\gamma(y)| \leq \epsilon$ for all $y \in Y$.

By recursively applying Propositions 8.1 and 8.3 to functions of the form $[x]_F \mapsto \mu^\rho_{[x]_F}(A)$ and $[x]_F \mapsto \mu^\rho_{[x]_F}(B) - \mu^\rho_{[x]_F}(A)$, and throwing out countably-many E-invariant Borel sets $B \subseteq X$ for which $\rho \restriction (E \restriction B)$ is smooth, we obtain increasing sequences of finite algebras $\mathcal{A}_n \supseteq \mathcal{U}_n$ of Borel subsets of X and finite Borel subequivalence relations F_n of E with the following properties:

(a) $\forall n \in \mathbb{N} \forall A \in \mathcal{A}_n \forall (x, y) \in E \ \mu^\rho_{[x]_{F_{n+1}}}(A) - \mu^\rho_{[y]_{F_{n+1}}}(A) \leq \epsilon_n$.

(b) $\forall n \in \mathbb{N} \forall A, B \in \mathcal{A}_n\ (\forall x \in X\ \mu^\rho_{[x]_{F_n}}(A) \leq \mu^\rho_{[x]_{F_n}}(B) \implies$
$\exists C \in \mathcal{A}_{n+1} \forall x \in X\ 0 \leq \mu^\rho_{[x]_{F_{n+1}}}(B \setminus C) - \mu^\rho_{[x]_{F_{n+1}}}(A) \leq \epsilon_n)$.

Set $\mathcal{A} = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ and $F = \bigcup_{n \in \mathbb{N}} F_n$. Condition (a) ensures that for all $x \in X$, we obtain a finitely-additive probability measure μ_x on $\mathcal{U}$ by setting $\mu_x(U) = \lim_{n \to \infty} \mu^\rho_{[x]_{F_n}}(U)$ for all $U \in \mathcal{U}$.

Lemma 8.6. *Suppose that* $(U_n)_{n \in \mathbb{N}} \in \mathcal{U}^{\mathbb{N}}$ *is a partition of a set in* $\mathcal{U}$ *and* $B = \{x \in X \mid \sum_{n \in \mathbb{N}} \mu_x(U_n) < \mu_x(\bigcup_{n \in \mathbb{N}} U_n)\}$. *Then there is a finite-to-one Borel compression of* $\rho \restriction (E \restriction B)$ *over a finite Borel subequivalence relation of* $E \restriction B$.

Proof. Note first that if $x \in B$, then $\sum_{m \geq n} \mu_x(U_m) \to 0$ and $\mu_x(\bigcup_{m \geq n} U_m) \not\to 0$, so there exist $\delta > 0$ and $n \in \mathbb{N}$ with the property that $\delta + 2 \sum_{m \geq n} \mu_x(U_m) \leq \mu_x(\bigcup_{m \geq n} U_m)$. By partitioning B into countably-many E-invariant Borel sets and passing to terminal segments of $(U_n)_{n \in \mathbb{N}}$ on each set, we can assume that there exists $\delta > 0$ such that $\delta + 2 \sum_{n \in \mathbb{N}} \mu_x(U_n) \leq \mu_x(\bigcup_{n \in \mathbb{N}} U_n)$ for all $x \in X$. Fix a sequence $(\delta_n)_{n \in \mathbb{N}}$ of positive real numbers whose sum is at most δ.

Sublemma 8.7. *There are pairwise disjoint sets* $A_n \subseteq \bigcup_{m > n} U_m$ *in* $\mathcal{A}$ *with the property that for all* $n \in \mathbb{N}$, *there exists* $k \in \mathbb{N}$ *such that* $\forall x \in B\ 0 \leq \mu^\rho_{[x]_{F_k}}(A_n) - \mu^\rho_{[x]_{F_k}}(U_n) \leq \delta_n$.

Proof. Suppose that $n \in \mathbb{N}$ and we have already found $(A_m)_{m<n}$. Note that if $x \in B$, then

$$\begin{aligned} \mu_x(U_n) + \delta &\leq \mu_x(\textstyle\bigcup_{m \in \mathbb{N}} U_m) + \mu_x(U_n) - 2 \sum_{m \in \mathbb{N}} \mu_x(U_m) \\ &\leq \mu_x(\textstyle\bigcup_{m \in \mathbb{N}} U_m) - \mu_x(U_n) - 2 \sum_{m<n} \mu_x(U_m), \end{aligned}$$

in which case

$$\begin{aligned} \mu_x(U_n) + \delta_n &\leq \mu_x\left(\textstyle\bigcup_{m \in \mathbb{N}} U_m\right) - \mu_x(U_n) - \sum_{m<n} 2\mu_x(U_m) + \delta_m \\ &\leq \mu_x\left(\textstyle\bigcup_{m > n} U_m\right) - \sum_{m<n} \mu_x(U_m) + \delta_m, \end{aligned}$$

so if $k \in \mathbb{N}$ is sufficiently large, then

$$\begin{aligned} \mu^\rho_{[x]_{F_k}}(U_n) &\leq \mu^\rho_{[x]_{F_k}}\left(\textstyle\bigcup_{m>n} U_m\right) - \sum_{m<n} \mu^\rho_{[x]_{F_k}}(U_m) + \delta_m \\ &\leq \mu^\rho_{[x]_{F_k}}\left(\textstyle\bigcup_{m>n} U_m\right) - \sum_{m<n} \mu^\rho_{[x]_{F_k}}(A_m) \\ &\leq \mu^\rho_{[x]_{F_k}}\left(\textstyle\bigcup_{m>n} U_m\right) - \mu^\rho_{[x]_{F_k}}\left(\textstyle\bigcup_{m<n} A_m\right) \\ &\leq \mu^\rho_{[x]_{F_k}}(\textstyle\bigcup_{m>n} U_m \setminus \bigcup_{m<n} A_m), \end{aligned}$$

by condition (a). It then follows from condition (b) that there exists $A_n \subseteq \bigcup_{m>n} U_m \setminus \bigcup_{m<n} A_m$ in $\mathcal{A}$ with $0 \leq \mu^\rho_{[x]_{F_k}}(A_n) - \mu^\rho_{[x]_{F_k}}(U_n) \leq \delta_n$ for all $x \in B$, for sufficiently large $k \in \mathbb{N}$. ⊠

Fix natural numbers k_n such that $\mu^\rho_{[x]_{F_{k_n}}}(U_n) \leq \mu^\rho_{[x]_{F_{k_n}}}(A_n)$ for all $n \in \mathbb{N}$ and $x \in B$, as well as Borel functions $\phi_n \colon B \cap U_n \to A_n$ whose graphs are contained in F_{k_n} for all $n \in \mathbb{N}$. Then the union of $\bigcup_{n \in \mathbb{N}} \phi_n$ and the identity function on $B \setminus \bigcup_{n \in \mathbb{N}} U_n$ is a finite-to-one Borel compression of $\rho \restriction (E \restriction B)$ over the union of $\bigcup_{n \in \mathbb{N}} F_{k_n} \restriction (A_n \cap B)$ and equality on B. ⊠

Lemma 8.6 ensures that, after throwing out countably-many E-invariant Borel sets $B \subseteq X$ for which there is a finite-to-one Borel compression of $\rho \restriction (E \restriction B)$ over a finite Borel subequivalence relation of $E \restriction B$, we can assume that for all $\delta > 0$ and $U \in \mathcal{U}$, there is a partition $(U_n)_{n \in \mathbb{N}}$ of U into sets in $\mathcal{U}$ of diameter at most δ such that $\mu_x(U) = \sum_{n \in \mathbb{N}} \mu_x(U_n)$ for all $x \in X$. It follows that each μ_x is a measure on $\mathcal{U}$, and therefore has a unique extension to a Borel probability measure $\overline{\mu}_x$ on X.

Lemma 8.8. *Suppose that $\gamma \in \Gamma$, $U \in \mathcal{U}$, ρ_γ is bounded on U, and $B = \{x \in X \mid \overline{\mu}_x(\gamma U) \neq \int_U \rho_\gamma \ d\overline{\mu}_x\}$. Then there is a finite-to-one Borel compression of $\rho \restriction (E \restriction B)$ over a finite Borel subequivalence relation of $E \restriction B$.*

Proof. By the symmetry of our argument, it is enough to establish the analogous lemma for the set $B = \{x \in X \mid \overline{\mu}_x(\gamma U) < \int_U \rho_\gamma \ d\overline{\mu}_x\}$. By breaking up B into countably-many E-invariant Borel sets, we can assume that $B = \{x \in X \mid \delta + \overline{\mu}_x(\gamma U) < \int_U \rho_\gamma \ d\overline{\mu}_x\}$ for some $\delta > 0$.

Sublemma 8.9. *For all $\epsilon > 0$, there exists $n \in \mathbb{N}$ with the property that $| \int_U \rho_\gamma \ d\overline{\mu}_x - \int_U \rho_\gamma \ d\mu^\rho_{[x]_{F_n}} | \leq \epsilon$ for all $x \in X$.*

Proof. Fix a $\mathcal{U}$-simple function $\phi \colon X \to [0, \infty)$ with the property that $|\phi(x) - \rho_\gamma(x)| \leq \epsilon/3$ for all $x \in U$. By condition (a), there exists $n \in \mathbb{N}$ such that $| \int_U \phi \ d\overline{\mu}_x - \int_U \phi \ d\mu^\rho_{[x]_{F_n}} | \leq \epsilon/3$ for all $x \in X$. Then

$$\begin{aligned}\left| \int_U \rho_\gamma \ d\overline{\mu}_x - \int_U \rho_\gamma \ d\mu^\rho_{[x]_{F_n}} \right| &\leq \left| \int_U \rho_\gamma \ d\overline{\mu}_x - \int_U \phi \ d\overline{\mu}_x \right| + \\ &\quad \left| \int_U \phi \ d\overline{\mu}_x - \int_U \phi \ d\mu^\rho_{[x]_{F_n}} \right| + \\ &\quad \left| \int_U \phi \ d\mu^\rho_{[x]_{F_n}} - \int_U \rho_\gamma \ d\mu^\rho_{[x]_{F_n}} \right| \\ &\leq \epsilon,\end{aligned}$$

for all $x \in X$. ⊠

Condition (a) and Sublemma 8.9 ensure that there exists $n \in \mathbb{N}$ such that $\mu^\rho_{[x]_{F_n}}(\gamma U) < \int_U \rho_\gamma \ d\mu^\rho_{[x]_{F_n}}$ for all $x \in B$. As the former quantity is $|\gamma U \cap [x]_{F_n}|^\rho_x / |[x]_{F_n}|^\rho_x$ and the latter is $|\gamma U \cap \gamma [x]_{F_n}|^\rho_x / |[x]_{F_n}|^\rho_x$, it follows that $|\gamma U \cap [x]_{F_n}|^\rho_x < |\gamma U \cap \gamma[x]_{F_n}|^\rho_x$ for all $x \in B$, so any function from $B \cap \gamma U$ to $B \cap \gamma U$, sending $\gamma U \cap [x]_{F_n}$ to $\gamma U \cap \gamma[x]_{F_n}$ for all $x \in B \cap \gamma U$, is a compression of $\rho \restriction (E \restriction (B \cap \gamma U))$ over the equivalence relation $(\gamma \times \gamma)(F_n) \restriction (B \cap \gamma U)$. The Lusin-Novikov uniformization theorem yields a Borel such function, and every such function trivially extends to a finite-to-one Borel compression of $\rho \restriction (E \restriction B)$ over a finite Borel subequivalence relation of $E \restriction B$. ⊠

Lemma 8.8 ensures that, after throwing out countably-many E-invariant Borel sets $B \subseteq X$ for which there is a finite-to-one Borel compression of $\rho \restriction (E \restriction B)$ over a finite Borel subequivalence relation of $E \restriction B$, we can assume that $\overline{\mu}_x(\gamma U) = \int_U \rho_\gamma \, d\overline{\mu}_x$ for all $\gamma \in \Gamma$, $U \in \mathcal{U}$ on which ρ_γ is bounded, and $x \in X$. As our choice of topologies ensures that every open set $U \subseteq X$ is a disjoint union of sets in $\mathcal{U}$ on which ρ_γ is bounded, we obtain the same conclusion even when $U \subseteq X$ is an arbitrary open set. As every Borel probability measure on a Polish space is regular, we obtain the same conclusion even when $U \subseteq X$ is an arbitrary Borel set. Proposition 2.1 therefore ensures that each $\overline{\mu}_x$ is ρ-invariant. ⊠

9. Coboundaries and invariant measures

Suppose that $R \subseteq X \times X$ is a Borel set whose vertical sections are countable and $\rho \colon R \to \Gamma$ is Borel. We say that a Borel measure μ on X is *ρ-invariant* if $\mu(T(B)) = \int_B \rho(T(x), x) \, d\mu(x)$ for all Borel sets $B \subseteq X$ and Borel injections $T \colon B \to X$ whose graphs are contained in R^{-1}. Proposition 2.1 ensures that this agrees with the usual notion when R is an equivalence relation and ρ is a cocycle.

The *composition* of sets $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is given by $R \circ S = \{(x, z) \in X \times Z \mid \exists y \in Y \; x \; R \; y \; S \; z\}$. The Lusin-Novikov uniformization theorem ensures that the class of Borel sets whose vertical sections are countable is closed under composition.

Proposition 9.1. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $R, S \subseteq E$ are Borel, and $\rho \colon E \to (0, \infty)$ is a Borel cocycle. Then every $(\rho \restriction (R \cup S))$-invariant Borel measure μ is $(\rho \restriction (R \circ S))$-invariant.*

Proof. Note first that if $B \subseteq X$ is a Borel set, $T_S \colon B \to X$ is a Borel injection whose graph is contained in S^{-1}, and $T_R \colon T_S(B) \to X$

is a Borel injection whose graph is contained in R^{-1}, then

$$\begin{aligned}\mu((T_R \circ T_S)(B)) &= \textstyle\int_{T_S(B)} \rho(T_R(x), x) \ d\mu(x) \\ &= \textstyle\int_B \rho((T_R \circ T_S)(x), T_S(x)) \ d((T_S^{-1})_*\mu)(x) \\ &= \textstyle\int_B \rho((T_R \circ T_S)(x), T_S(x))\rho(T_S(x), x) \ d\mu(x) \\ &= \textstyle\int_B \rho((T_R \circ T_S)(x), x) \ d\mu(x).\end{aligned}$$

But the Lusin-Novikov uniformization theorem ensures that every Borel injection whose graph is contained in $(R \circ S)^{-1}$ can be decomposed into countably-many Borel injections of the form $T_R \circ T_S$ as above. ⊠

We say that a set $Y \subseteq X$ has *ρ-density* at least ϵ if there is a finite Borel subequivalence relation F of E such that $\mu^\rho_{[x]_F}(Y) \geq \epsilon$ for all $x \in X$. We say that a Borel set $B \subseteq X$ has *positive ρ-density* if there exists $\epsilon > 0$ for which B has ρ-density at least ϵ.

Proposition 9.2. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to (0, \infty)$ is a Borel cocycle, and $B \subseteq X$ is a Borel set with positive ρ-density. Then every $(\rho \restriction (E \restriction B))$-invariant finite Borel measure μ extends to a ρ-invariant finite Borel measure.*

Proof. Fix $\epsilon > 0$ for which B has ρ-density at least ϵ, as well as a finite Borel subequivalence relation F of E such that $\mu^\rho_{[x]_F}(B) \geq \epsilon$ for all $x \in X$, and let $\overline{\mu}$ be the Borel measure on X given by

$$\overline{\mu}(A) = \textstyle\int |A \cap [x]_F|^\rho_{B \cap [x]_F} \ d\mu(x)$$

for all Borel sets $A \subseteq X$.

As $\overline{\mu}(X) \leq \mu(B)/\epsilon$, it follows that $\overline{\mu}$ is finite, and Proposition 2.5 ensures that $\mu = \overline{\mu} \restriction B$.

Lemma 9.3. *Suppose that $\phi \colon X \to [0, \infty)$ is a Borel function. Then $\int \phi \ d\overline{\mu} = \int \sum_{y \in [x]_F} \phi(y) |\{y\}|^\rho_{B \cap [x]_F} \ d\mu(x)$.*

Proof. It is sufficient to check the special case that ϕ is the characteristic function of a Borel set, which is a direct consequence of the definition of $\overline{\mu}$. ⊠

Lemma 9.4. *The measure $\overline{\mu}$ is $(\rho \restriction F)$-invariant.*

PROOF. Simply observe that if $A \subseteq X$ is a Borel set and $T\colon X \to X$ is a Borel automorphism whose graph is contained in F, then

$$\begin{aligned} \int_A \rho(T(x), x)\ d\overline{\mu}(x) &= \int \sum_{y \in A \cap [x]_F} \rho(T(y), y) |\{y\}|^\rho_{B \cap [x]_F}\ d\mu(x) \\ &= \int \sum_{y \in A \cap [x]_F} |\{T(y)\}|^\rho_{B \cap [x]_F}\ d\mu(x) \\ &= \int |T(A \cap [x]_F)|^\rho_{B \cap [x]_F}\ d\mu(x) \\ &= \int |T(A) \cap [x]_F|^\rho_{B \cap [x]_F}\ d\mu(x) \\ &= \overline{\mu}(T(A)), \end{aligned}$$

by Lemma 9.3. ⊠

As $E = F \circ (E \upharpoonright B) \circ F$, two applications of Proposition 9.1 ensure that $\overline{\mu}$ is ρ-invariant. ⊠

The primary argument of this section will hinge on the following approximation lemma.

PROPOSITION 9.5. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho\colon E \to (0, \infty)$ is a Borel cocycle. Then for all Borel sets $A \subseteq X$ and positive real numbers $r < 1$, there exist an E-invariant Borel set $B \subseteq X$, a Borel set $C \subseteq B$, and a finite Borel subequivalence relation F of $E \upharpoonright C$ such that $\rho \upharpoonright (E \upharpoonright {\sim}B)$ is smooth, $r < |A \cap [x]_F|^\rho_{[x]_F \setminus A} < 1$ for all $x \in C$, and $A \cap [x]_E \subseteq C$ or $[x]_E \setminus A \subseteq C$ for all $x \in B$.*

PROOF. Fix a maximal Borel set $\mathcal{S}$ of pairwise disjoint non-empty finite sets $S \subseteq X$ for which $S \times S \subseteq E$ and $r < |A \cap S|^\rho_{S \setminus A} < 1$. Set $D = A \setminus \bigcup \mathcal{S}$ and $D' = ({\sim}A) \setminus \bigcup \mathcal{S}$.

LEMMA 9.6. *Suppose that $(x, x') \in E$. Then there exists a real number $s > 1$ with the property that x has only finitely-many $G^\rho_{(1/s,s)}$-neighbors in D or x' has only finitely-many $G^\rho_{(1/s,s)}$-neighbors in D'.*

PROOF. Fix $n, n' \in \mathbb{N}$ such that $(n/n')\rho(x, x')$ lies strictly between r and 1, and fix $s > 1$ sufficiently small that $(n/n')\rho(x, x')$ lies strictly between rs^2 and $1/s^2$. Suppose, towards a contradiction, that there are sets $S \subseteq D$ and $S' \subseteq D'$ of $G^\rho_{(1/s,s)}$-neighbors of x and x' of cardinalities n and n'. Then $n/s < |S|^\rho_x < ns$ and $n'\rho(x', x)/s < |S'|^\rho_x < n'\rho(x', x)s$, so the ρ-size of S relative to S' lies strictly between $(n/n')\rho(x, x')/s^2$ and $(n/n')\rho(x, x')s^2$. As these bounds lie strictly between r and 1, this contradicts the maximality of $\mathcal{S}$. ⊠

Letting B be the complement of $[D]_E \cap [D']_E$, it follows from Lemma 9.6 that $\rho \upharpoonright (E \upharpoonright {\sim}B)$ is smooth. Set $C = B \cap \bigcup \mathcal{S}$, and let F be the equivalence relation on C whose classes are the subsets of C in $\mathcal{S}$. ⊠

We say that a Borel set $B \subseteq X$ has *σ-positive ρ-density* if X is the union of countably-many E-invariant Borel sets $A_n \subseteq X$ for which $A_n \cap B$ has positive $(\rho \restriction (E \restriction A_n))$-density.

THEOREM 9.7. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho\colon E \to (0, \infty)$ is a Borel cocycle, and $A \subseteq X$ is an E-complete Borel set. Then X is the union of an E-invariant Borel set $B \subseteq X$ for which $\rho \restriction (E \restriction B)$ is smooth, an E-invariant Borel set $C \subseteq X$ for which $A \cap C$ has σ-positive $(\rho \restriction (E \restriction C))$-density, and an E-invariant Borel set $D \subseteq X$ for which there is a finite-to-one Borel compression of the quotient of $\rho \restriction (E \restriction D)$ by a finite Borel subequivalence relation of $E \restriction D$.*

PROOF. Fix a positive real number $r < 1$. We will show that, after throwing out countably-many E-invariant Borel sets $B \subseteq X$ for which $\rho \restriction (E \restriction B)$ is smooth, as well as countably-many E-invariant Borel sets $C \subseteq X$ for which $A \cap C$ has positive $(\rho \restriction (E \restriction C))$-density, there are increasing sequences of finite Borel subequivalence relations F_n of E and E-complete F_n-invariant Borel sets $A_n \subseteq X$ with the property that $r < |A_n \cap [x]_{F_{n+1}}|^{\rho}_{(A_{n+1} \setminus A_n) \cap [x]_{F_{n+1}}} < 1$ for all $n \in \mathbb{N}$ and $x \in A_n$.

We begin by setting $A_0 = A$ and letting F_0 be equality. Suppose now that $n \in \mathbb{N}$ and we have already found A_n and F_n. By applying Proposition 9.5 to A_n/F_n, and throwing out an E-invariant Borel set $B \subseteq X$ for which $\rho \restriction (E \restriction B)$ is smooth, we obtain a finite Borel subequivalence relation $F_{n+1} \supseteq F_n$ of E and an F_{n+1}-invariant Borel set $A_{n+1} \subseteq X$ such that $r < |A_n \cap [x]_{F_{n+1}}|^{\rho}_{[x]_{F_{n+1}} \setminus A_n} < 1$ for all $x \in A_{n+1}$, and $A_n \cap [x]_E \subseteq A_{n+1}$ or $[x]_E \setminus A_n \subseteq A_{n+1}$ for all $x \in X$. By throwing out an E-invariant Borel set $C \subseteq X$ for which $A \cap C$ has positive $(\rho \restriction (E \restriction C))$-density, we can assume that $A_n \subseteq A_{n+1}$, completing the recursive construction.

Set $B_n = A_n \setminus \bigcup_{m<n} A_m$ and define $\phi_n\colon B_n/F_n \to B_{n+1}/F_{n+1}$ by setting $\phi_n(B_n \cap [x]_{F_n}) = B_{n+1} \cap [x]_{F_{n+1}}$ for all $n \in \mathbb{N}$ and $x \in B_n$. Then the union of $\bigcup_{n\in\mathbb{N}} \phi_n$ and the identity function on $\sim\bigcup_{n\in\mathbb{N}} A_n$ is a Borel compression of the quotient of ρ by the union of $\bigcup_{n\in\mathbb{N}} F_n \restriction B_n$ and equality. ⊠

As a corollary, we can now establish the converse of Proposition 7.2 for Borel coboundaries.

THEOREM 9.8. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho\colon E \to (0, \infty)$ is a Borel coboundary, and there is a Borel compression of ρ over a finite Borel subequivalence relation of E. Then there is a Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E.*

Proof. By Proposition 6.1, there is a pre-compact open neighborhood $U \subseteq (0, \infty)$ of 1 for which there is an E-complete Borel set $A \subseteq X$ such that $\rho(E \restriction A) \subseteq U$. By Theorem 9.7, after throwing out E-invariant Borel sets $B \subseteq X$ and $D \subseteq X$ for which $\rho \restriction (E \restriction B)$ is smooth and there is a finite-to-one Borel compression of the quotient of $\rho \restriction (E \restriction D)$ by a finite Borel subequivalence relation of $E \restriction D$, we can assume that A has σ-positive ρ-density.

Note that there is no $(\rho \restriction (E \restriction A))$-invariant Borel probability measure μ, since otherwise, by passing to an $(E \restriction A)$-invariant μ-positive Borel set, we could assume that A has positive ρ-density, in which case Proposition 9.2 would yield a ρ-invariant Borel probability measure, contradicting Proposition 7.1. Proposition 6.2 therefore ensures that there is no $(E \restriction A)$-invariant Borel probability measure, so the special cases of Proposition 7.4 and Theorem 8.5 for constant cocycles yield an aperiodic smooth Borel subequivalence relation F of $E \restriction A$.

It follows that $\rho \restriction F$ is smooth, and the fact that $\rho \restriction (E \restriction A)$ is bounded ensures that $\rho \restriction F$ is also aperiodic. Fix a Borel extension $\phi \colon X \to A$ of the identity function on A whose graph is contained in E, and observe that ρ is aperiodic and smooth on the pullback of F through ϕ, in which case Proposition 4.4 yields an injective Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E. ⊠

10. Uniform ergodic decomposition

Recall that a *decomposition* of a Borel probability measure μ on X is a Borel function $\phi \colon X \to P(X)$ such that $\phi^{-1}(\{\phi(x)\})$ is $\phi(x)$-conull for all $x \in X$ and $\mu(B) = \int \phi(x)(B) \ d\mu(x)$ for all Borel sets $B \subseteq X$. A *decomposition* of a set $P \subseteq P(X)$ is a function $\phi \colon X \to P(X)$ that is a decomposition of every $\mu \in P$.

Theorem 10.1 (Ditzen). *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho \colon E \to (0, \infty)$ is a Borel cocycle for which there is a ρ-invariant Borel probability measure. Then there is a hyperfinite Borel subequivalence relation F of E for which there is an E-invariant Borel decomposition of the family of all ρ-invariant Borel probability measures into F-ergodic ρ-invariant Borel probability measures.*

Proof. By the proof of Theorem 8.5, we can assume that X is a Polish space for which there exist a countable algebra $\mathcal{U} \subseteq \mathcal{P}(X)$ of open sets forming a basis for X, an increasing sequence $(F_n)_{n \in \mathbb{N}}$ of finite Borel subequivalence relations of E, as well as an E-invariant function $\phi \colon X \to P(X)$ with the property that $\phi(x)$ is ρ-invariant for all $x \in X$ and $\forall U \in \mathcal{U}$ $\mu^{\rho}_{[x]_{F_n}}(U) \to \phi(x)(U)$ μ-almost everywhere for all ρ-invariant Borel probability measures μ. Define $F = \bigcup_{n \in \mathbb{N}} F_n$.

Lemma 10.2. *Suppose that $A \subseteq X$ is an F-invariant Borel set, $B \subseteq X$ is Borel, and μ is a ρ-invariant Borel probability measure. Then $\mu(A \cap B) = \int_A \phi(x)(B)\ d\mu(x)$.*

Proof. Observe first that if $U \in \mathcal{U}$, then Proposition 2.5 ensures that $\mu(A \cap U) = \int_A \mu^{\rho}_{[x]_{F_n}}(U)\ d\mu(x)$ for all $n \in \mathbb{N}$, from which it follows that $\mu(A \cap U) = \lim_{n \to \infty} \int_A \mu^{\rho}_{[x]_{F_n}}(U)\ d\mu(x) = \int_A \phi(x)(U)\ d\mu(x)$. The fact that every Borel probability measure on a Polish space is regular therefore implies that $\mu(A \cap B) = \int_A \phi(x)(B)\ d\mu(x)$. ⊠

Recall that the ergodic decomposition theorem for a single Borel probability measure μ on X can be established by first producing a Borel function $\phi\colon X \to \mathcal{P}(X)$ satisfying the conclusion of Lemma 10.2 for μ, and then noting that every such function has the property that $\phi^{-1}(\{\phi(x)\})$ is $\phi(x)$-conull and $\phi(x)$ is F-ergodic for μ-almost all $x \in X$. We can therefore assume that the latter conclusion holds for every ρ-invariant Borel probability measure μ.

Lemma 10.3. *Suppose that μ is an E-ergodic ρ-invariant Borel probability measure. Then $\phi^{-1}(\{\mu\})$ is μ-conull.*

Proof. As the E-ergodicity of μ ensures that ϕ is constant on a μ-conull set, Lemma 10.2 implies that $\forall U \in \mathcal{U}$ $\mu(U) = \phi(x)(U)$ for μ-almost all $x \in X$. As every Borel probability measure on a Polish space is regular, it follows that $\mu = \phi(x)$ for all such x. ⊠

It now follows that if μ is a ρ-invariant Borel probability measure, then μ is E-ergodic $\implies \phi^{-1}(\{\mu\})$ is μ-conull $\implies \mu$ is F-ergodic, thus the set $B = \{x \in X \mid \phi(x) \text{ is } F\text{-ergodic}\}$ is Borel. Setting $A = {\sim}B$, we therefore obtain the desired decomposition by redefining $\phi \upharpoonright A$ to be any $(E \upharpoonright A)$-invariant Borel function sending each point of A to an F-ergodic ρ-invariant Borel probability measure. ⊠

11. Generic compressibility

We say that a binary relation R on X is *aperiodic* if its vertical sections are all infinite, and *countable* if its vertical sections are all countable. We say that a set $Y \subseteq X$ is *R-complete* if it intersects every vertical section of R, and *R-invariant* if $R_y \subseteq Y$ for all $y \in Y$.

Theorem 11.1. *Suppose that X is a Polish space, R is an aperiodic countable Borel binary relation on X, and S is an aperiodic transitive Borel subrelation of R. Then there is a comeager R-invariant Borel set*

$C \subseteq X$ for which there is a Borel injection $T \colon C \to C$, whose graph is contained in S, such that $\bigcap_{n \in \mathbb{N}} T^n(C) = \emptyset$.

Proof. Fix Borel sets $A_n \subseteq X$ and Borel injections $T_n \colon A_n \to X$ such that $R = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(T_n)$, and set $A'_n = \{x \in A_n \mid x \, S \, T_n(x)\}$ for all $n \in \mathbb{N}$. Fix a decreasing sequence $(B_n)_{n \in \mathbb{N}}$ of S-complete Borel sets whose intersection is empty.

We recursively define Borel sets $D_s \subseteq \sim B_{|s|}$ for all $s \in \mathbb{N}^{<\mathbb{N}}$, beginning with $D_\emptyset = \emptyset$. Given $s \in 2^{<\mathbb{N}}$ for which we have found $(D_t)_{t \sqsubseteq s}$, set $D_{s \frown (n)} = A'_n \cap T_n^{-1}(B_{|s|+1} \setminus B_{|s|+2}) \setminus (B_{|s|+1} \cup \bigcup_{t \sqsubseteq s} D_t)$ for all $n \in \mathbb{N}$. Now define $D = \{(b, x) \in \mathbb{N}^\mathbb{N} \times X \mid x \in \bigcup_{n \in \mathbb{N}} D_{b \restriction n}\}$.

Lemma 11.2. *Every horizontal section of D is dense.*

Proof. Suppose that $x \in X$. To see that D^x is dense, note that if $s \in \mathbb{N}^{<\mathbb{N}}$, then there exist $i \in \mathbb{N}$ for which $x \notin B_{|s|+i}$, $y \in B_{|s|+i+1}$ for which $x \, S \, y$, and $n \in \mathbb{N}$ for which $T_n(x) = y$. Let j be the unique natural number for which $y \in B_{|s|+i+j+1} \setminus B_{|s|+i+j+2}$, and observe that $x \in \bigcup_{u \sqsubseteq s \frown t \frown (n)} D_u$, thus $\mathcal{N}_{s \frown t \frown (n)} \subseteq D^x$, for all $t \in \mathbb{N}^{i+j}$. ⊠

As the horizontal sections of D are open, Lemma 11.2 ensures that $\forall x \in X \forall^* b \in \mathbb{N}^\mathbb{N} \; b \in \bigcap_{n \in \mathbb{N}} D^{T_n(x)}$, in which case the Kuratowski-Ulam theorem implies that $\forall^* b \in \mathbb{N}^\mathbb{N} \forall^* x \in X \; b \in \bigcap_{n \in \mathbb{N}} D^{T_n(x)}$. Fix $b \in \mathbb{N}^\mathbb{N}$ for which the set $C = \{x \in X \mid b \in \bigcap_{n \in \mathbb{N}} D^{T_n(x)}\}$ is comeager, and observe that the function $T = \bigcup_{n \in \mathbb{N}} T_{b(n)} \restriction (C \cap D_{b \restriction (n+1)})$ is as desired. ⊠

Theorem 11.3 (Kechris-Miller). *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho \colon E \to (0, \infty)$ is a Borel cocycle. Then there are E-invariant Borel sets $B \subseteq C \subseteq X$ such that C is comeager, $E \restriction (C \setminus B)$ is smooth, and there is an injective Borel compression of $\rho \restriction (E \restriction B)$.*

Proof. If the set $A = \{x \in X \mid \forall y \in [x]_E \exists^\infty z \in [x]_E \; \rho(y, z) \leq 1\}$ is countable, then E is smooth, and there is nothing to prove. Otherwise, there is an E-invariant infinite meager Borel set $M \subseteq A$. Fix an aperiodic countable Borel equivalence relation F on X such that $A \setminus M$ is an F-invariant set on which E and F agree, and fix a Borel cocycle $\sigma \colon F \to (0, \infty)$, agreeing with ρ on $E \restriction (A \setminus M)$, for which the transitive binary relation $S = \{(x, y) \in F \mid \sigma(x, y) \leq 1\}$ is aperiodic. By Theorem 11.1, there is a comeager F-invariant Borel set $D \subseteq X$ for which there is an injective Borel compression of $\sigma \restriction (F \restriction D)$. Then the sets $B = (A \setminus M) \cap D$ and $C = (\sim A) \cup B$ are as desired. ⊠

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