Measure theory and countable Borel equivalence relations

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Introduction

These are the notes accompanying an introductory course to measure theory, with a view towards interactions with descriptive set theory, at the Kurt Gödel Research Center for Mathematical Logic at the University of Vienna in Fall 2016. I am grateful to the head of the KGRC, Sy Friedman, for his encouragement and many useful suggestions, as well as to all of the participants.

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Part I

Measures on families of sets

1. Extensions

A function $\mu\colon \mathcal{U} \subseteq \mathcal{P}(X) \to [0,\infty]$ is said to be *monotone* if $U \subseteq V \implies \mu(U) \leq \mu(V)$ for all sets $U, V \in \mathcal{U}$, *finitely subadditive* if $\mu(\bigcup_{n\leq N} U_n) \leq \sum_{n\leq N} \mu(U_n)$ for all finite sequences $(U_n)_{n\leq N}$ of sets in $\mathcal{U}$ whose union is in $\mathcal{U}$, *σ-subadditive* if $\mu(\bigcup_{n\in\mathbb{N}} U_n) \leq \sum_{n\in\mathbb{N}} \mu(U_n)$ for all sequences $(U_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{U}$ whose union is in $\mathcal{U}$, *finitely additive* if $\mu(\bigcup_{n\leq N} U_n) = \sum_{n\leq N} \mu(U_n)$ for all finite sequences $(U_n)_{n\leq N}$ of pairwise disjoint sets in $\mathcal{U}$ whose union is in $\mathcal{U}$, and *σ-additive* if $\mu(\bigcup_{n\in\mathbb{N}} U_n) = \sum_{n\in\mathbb{N}} \mu(U_n)$ for all sequences $(U_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{U}$ whose union is in $\mathcal{U}$.

Proposition 1.1. *Suppose that $\mathcal{U} \subseteq \mathcal{P}(X)$ is closed under differences and finite intersections and $\mu\colon \mathcal{U} \to [0,\infty]$ is finitely additive. Then μ is monotone and finitely subadditive, and μ is σ-additive if and only if μ is σ-subadditive.*

Proof. To see that μ is monotone, note that if $U \subseteq V$ are in $\mathcal{U}$, then $\nu(V) = \nu(U) + \nu(V \setminus U) \geq \nu(U)$. To see that μ is finitely subadditive, note that if $(U_n)_{n\leq N}$ is a finite sequence of sets in $\mathcal{U}$ whose union is in $\mathcal{U}$, then $\mu(\bigcup_{n\leq N} U_n) = \sum_{n\leq N} \mu(\bigcap_{m<n} U_n \setminus U_m) \leq \sum_{n\leq N} \mu(U_n)$. The same idea can be used to show that if μ is σ-additive, then it is σ-subadditive. To establish the converse, note that if $(U_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{U}$ whose union U is in $\mathcal{U}$, then $\mu(U) = \mu(\bigcap_{n\leq N} U \setminus U_n) + \sum_{n\leq N} \mu(U_n)$ for all $N \in \mathbb{N}$, from which it follows that $\mu(U) \geq \sum_{n\in\mathbb{N}} \mu(U_n)$, thus $\mu(U) = \sum_{n\in\mathbb{N}} \mu(U_n)$. ⊠

An *outer measure* on a set X is a monotone σ-subadditive function $\mu\colon \mathcal{P}(X) \to [0,\infty]$ for which $\mu(\emptyset) = 0$. In what follows, we adopt the convention that the infimum of the empty set is ∞.

Proposition 1.2. *Suppose that $\mathcal{U} \subseteq \mathcal{P}(X)$ is closed under finite intersections and $\mu\colon \mathcal{U} \to [0,\infty]$ is a monotone σ-subadditive function for which $\mu(\emptyset) = 0$. Then the function $\mu^*\colon \mathcal{P}(X) \to [0,\infty]$, given by*

$$\mu^*(Y) = \inf\{\textstyle\sum_{n\in\mathbb{N}} \mu(U_n) \mid (U_n)_{n\in\mathbb{N}} \in \mathcal{U}^{\mathbb{N}} \text{ and } Y \subseteq \bigcup_{n\in\mathbb{N}} U_n\},$$

is an extension of μ to an outer measure.

Proof. It is clear that μ^* is monotone. To see that μ^* is σ-subadditive, suppose that $(X_n)_{n\in\mathbb{N}}$ is a sequence of subsets of X for which $\sum_{n\in\mathbb{N}} \mu^*(X_n) < \infty$, and given $\epsilon > 0$, fix positive real numbers ϵ_n for which $\sum_{n\in\mathbb{N}} \epsilon_n \leq \epsilon$, as well as sequences $(U_{m,n})_{m\in\mathbb{N}} \in \mathcal{U}^{\mathbb{N}}$ with the property that $X_n \subseteq \bigcup_{m\in\mathbb{N}} U_{m,n}$ and $\sum_{n\in\mathbb{N}} \mu(U_{m,n}) \leq \epsilon_n + \mu^*(X_n)$ for all $n \in \mathbb{N}$, and observe that $\mu^*(\bigcup_{n\in\mathbb{N}} X_n) \leq \sum_{m,n\in\mathbb{N}} \mu(U_{m,n}) \leq \epsilon + \sum_{n\in\mathbb{N}} \mu^*(X_n)$, from which it follows that $\mu^*(\bigcup_{n\in\mathbb{N}} X_n) \leq \sum_{n\in\mathbb{N}} \mu^*(X_n)$.

To see that μ^* is an extension of μ, observe that if $U \in \mathcal{U}$, $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}^{\mathbb{N}}$, and $U \subseteq \bigcup_{n\in\mathbb{N}} U_n$, then $\mu(U) \leq \sum_{n\in\mathbb{N}} \mu(U \cap U_n) \leq \sum_{n\in\mathbb{N}} \mu(U_n)$, so $\mu(U) \leq \mu^*(U) \leq \mu(U)$. ⊠

A set $B \subseteq X$ is *Carathéodory measurable* with respect to an outer measure μ on X if $\mu(Y) = \mu(Y \setminus B) + \mu(Y \cap B)$ for all $Y \subseteq X$.

Proposition 1.3. *Suppose that μ is an outer measure on a set X. Then the corresponding family of Carathéodory measurable sets is a σ-algebra on which the restriction of μ is σ-additive.*

Proof. It is clear that the family of Carathéodory measurable sets is closed under complements.

Lemma 1.4. *Suppose that $(B_n)_{n<N}$ is a finite sequence of Carathéodory measurable sets and $Y \subseteq X$. Then $\mu(Y) = \sum_{s\in 2^N} \mu(Y \cap B_s)$, where $B_s = \bigcap_{n\in\mathrm{supp}(s)} B_n \setminus \bigcup_{n\in N\setminus\mathrm{supp}(s)} B_n$.*

Proof. By a straightforward induction. ⊠

To obtain closure under finite unions, note that if $A, B \subseteq X$ are Carathéodory measurable and $Y \subseteq X$, then Lemma 1.4 ensures that $\mu(Y) = \mu(Y \setminus (A \cup B)) + \mu(Y \cap (A \setminus B)) + \mu(Y \cap (B \setminus A)) + \mu(Y \cap (A \cap B)) \leq \mu(Y \setminus (A \cup B)) + \mu(Y \cap (A \cup B))$, so $\mu(Y) = \mu(Y \setminus (A \cup B)) + \mu(Y \cap (A \cup B))$, thus $A \cup B$ is Carathéodory measurable.

To obtain closure under countable disjoint unions, note that if $(B_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint Carathéodory measurable sets and $Y \subseteq X$, then one more application of Lemma 1.4 ensures that $\mu(Y) = \mu(Y \setminus \bigcup_{n\leq N} B_n) + \sum_{n\leq N} \mu(Y \cap B_n)$ for all $N \in \mathbb{N}$, so $\mu(Y) \geq \mu(Y \setminus \bigcup_{n\in\mathbb{N}} B_n) + \sum_{n\in\mathbb{N}} \mu(Y \cap B_n)$, from which it follows that $\mu(Y) \geq \mu(Y \setminus \bigcup_{n\in\mathbb{N}} B_n) + \mu(Y \cap \bigcup_{n\in\mathbb{N}} B_n)$, and therefore that $\mu(Y) = \mu(Y \setminus \bigcup_{n\in\mathbb{N}} B_n) + \mu(Y \cap \bigcup_{n\in\mathbb{N}} B_n)$, hence $\bigcup_{n\in\mathbb{N}} B_n$ is Carathéodory measurable.

By Proposition 1.1, it only remains to show that the restriction of μ to the set of Carathéodory measurable sets is finitely additive. But if $B \subseteq X$ is Carathéodory measurable and $A \subseteq X \setminus B$, then $\mu(A \cup B) = \mu((A \cup B) \setminus B) + \mu((A \cup B) \cap B) = \mu(A) + \mu(B)$. ⊠

A *finitely-additive measure* on $\mathcal{U} \subseteq \mathcal{P}(X)$ is a finitely-additive function $\mu \colon \mathcal{U} \to [0, \infty]$ for which $\mu(\emptyset) = 0$, and a *measure* on $\mathcal{U} \subseteq \mathcal{P}(X)$ is a σ-additive function $\mu \colon \mathcal{U} \to [0, \infty]$ for which $\mu(\emptyset) = 0$. A finitely-additive measure μ on $\mathcal{U} \subseteq \mathcal{P}(X)$ is *finite* if $\mu(X) < \infty$, and *σ-finite* if X is the union of countably-many sets $U \in \mathcal{U}$ for which $\mu(U) < \infty$.

Theorem 1.5 (Carathéodory). *Suppose that $\mathcal{U} \subseteq \mathcal{P}(X)$ is closed under differences and finite intersections. Then every measure on $\mathcal{U}$*

has an extension to a measure on the σ-algebra generated by $\mathcal{U}$. Moreover, every σ-finite measure on $\mathcal{U}$ has a unique such extension.

PROOF. Propositions 1.2 and 1.3 ensure that in order to obtain the desired extension, it is sufficient to check that every set $U \in \mathcal{U}$ is Carathéodory measurable with respect to μ^*. Towards this end, suppose that $Y \subseteq X$. If $\mu^*(Y) = \infty$, then $\mu^*(Y \setminus U) = \infty$ or $\mu^*(Y \cap U) = \infty$, thus $\mu^*(Y) = \mu^*(Y \setminus U) + \mu^*(Y \cap U)$. Otherwise, given $\epsilon > 0$, fix $(U_n)_{n \in \mathbb{N}} \in \mathcal{U}^{\mathbb{N}}$ with $Y \subseteq \bigcup_{n \in \mathbb{N}} U_n$ and $\mu^*(Y) + \epsilon \geq \sum_{n \in \mathbb{N}} \mu(U_n)$. As the latter quantity can be expressed as $\sum_{n \in \mathbb{N}} \mu(U_n \setminus U) + \sum_{n \in \mathbb{N}} \mu(U_n \cap U)$, and is therefore bounded below by $\mu^*(Y \setminus U) + \mu^*(Y \cap U)$, it follows that $\mu^*(Y) \geq \mu^*(Y \setminus U) + \mu^*(Y \cap U)$, thus $\mu^*(Y) = \mu^*(Y \setminus U) + \mu^*(Y \cap U)$, hence U is Carathéodory measurable.

Observe now that if ν is an extension of μ to a measure on the σ-algebra generated by $\mathcal{U}$ and B is in this σ-algebra, then Proposition 1.1 ensures that $\nu \leq \mu^*$. To see that $\nu \geq \mu^*$ when μ is σ-finite, it is sufficient to show that if $\mu(U) < \infty$, then $\nu(B) \geq \mu^*(B)$ for every set $B \subseteq U$ in the σ-algebra generated by $\mathcal{U}$. But this can be seen by noting that $\nu(B) = \nu(U) - \nu(U \setminus B) \geq \mu^*(U) - \mu^*(U \setminus B) = \mu^*(B)$. ⊠

REMARK 1.6. Suppose that C is a countably-infinite set and D is an uncountably-infinite set disjoint from C, let $\mathcal{U}$ denote the algebra of subsets of $C \cup D$ generated by singletons, and let μ denote the measure on $\mathcal{U}$ given by $\mu(B) = |B \cap C|$. Then for each $r \in [0, \infty]$, there is a unique extension of μ to a measure ν on the σ-algebra generated by $\mathcal{U}$ with the property that $\nu(D) = r$.

One typically applies Theorem 1.5 in conjunction with a simpler extension theorem.

PROPOSITION 1.7. *Suppose that $\mathcal{U} \subseteq \mathcal{P}(X)$ is closed under finite intersections. Then every measure on $\mathcal{U}$ has a unique extension to a measure on the closure of $\mathcal{U}$ under countable disjoint unions.*

PROOF. Suppose that μ is a measure on $\mathcal{U}$, and note that if ν is an extension of μ to a measure on the closure of $\mathcal{U}$ under countable disjoint unions, then $\nu(\bigcup_{n \in \mathbb{N}} U_n) = \sum_{n \in \mathbb{N}} \mu(U_n)$ for all sequences $(U_n)_{n \in \mathbb{N}}$ of pairwise disjoint sets in $\mathcal{U}$. To see that this constraint yields a well-defined extension of μ, suppose that $(U_m)_{m \in \mathbb{N}}$ and $(V_n)_{n \in \mathbb{N}}$ are sequences of pairwise disjoint sets in $\mathcal{U}$ whose unions coincide, and observe that $\sum_{m \in \mathbb{N}} \mu(U_m) = \sum_{m,n \in \mathbb{N}} \mu(U_m \cap V_n) = \sum_{n \in \mathbb{N}} \mu(V_n)$. To see that the resulting extension ν is a measure, suppose that $(U_n)_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint sets in the closure of $\mathcal{U}$ under countable disjoint unions, fix sequences $(U_{m,n})_{n \in \mathbb{N}}$ of pairwise disjoint sets in $\mathcal{U}$

with the property that $U_n = \bigcup_{m\in\mathbb{N}} U_{m,n}$ for all $n \in \mathbb{N}$, and observe that $\nu(\bigcup_{n\in\mathbb{N}} U_n) = \nu(\bigcup_{m,n\in\mathbb{N}} U_{m,n}) = \sum_{m,n\in\mathbb{N}} \mu(U_{m,n}) = \sum_{n\in\mathbb{N}} \nu(U_n)$. ⊠

Remark 1.8. Suppose that $\mathcal{U} \subseteq \mathcal{P}(X)$. Given finite sequences $(U_m)_{m\leq M}$ and $(V_n)_{n\leq N}$ of pairwise disjoint sets in $\mathcal{U}$, the fact that $\bigcup_{m\leq M} U_m \cap \bigcup_{n\leq N} V_n = \bigcup_{m\leq M,n\leq N} U_m \cap V_n$ ensures that if $\mathcal{U}$ is closed under finite intersections, then the closure of $\mathcal{U}$ under finite disjoint unions is also closed under finite intersections. Similarly, the fact that $\bigcup_{m\leq M} U_m \setminus \bigcup_{n\leq N} V_n = \bigcup_{m\leq M} \bigcap_{n\leq N} U_m \setminus V_n$ ensures that if $\mathcal{U}$ is closed under finite intersections and differences of sets in $\mathcal{U}$ are in the closure of $\mathcal{U}$ under finite disjoint unions, then the closure of $\mathcal{U}$ under finite disjoint unions is closed under differences. So by combining Theorem 1.5 with Proposition 1.7, we obtain the generalization of Theorem 1.5 in which we merely require that differences of sets in $\mathcal{U}$ are in the closure of $\mathcal{U}$ under finite disjoint unions, rather than in $\mathcal{U}$ itself.

Suppose that μ is a measure on a σ-algebra $\mathcal{B} \subseteq \mathcal{P}(X)$. A set $Y \subseteq X$ is *μ-null* if there exists $B \in \mathcal{B}$ with $Y \subseteq B$ and $\mu(B) = 0$, and *μ-measurable* if there exists $B \in \mathcal{B}$ such that $Y \subseteq B$ and $B \setminus Y$ is μ-null, or equivalently, if there exists $B \in \mathcal{B}$ such that $B \bigtriangleup Y$ is μ-null (since $B \bigtriangleup Y \subseteq A \implies (Y \subseteq A \cup B$ and $(A \cup B) \setminus Y \subseteq A \setminus Y)$).

Proposition 1.9. *Suppose that $\mathcal{B} \subseteq \mathcal{P}(X)$ is a σ-algebra and μ is a measure on $\mathcal{B}$. Then the family of μ-measurable sets is a σ-algebra on which there is a unique measure extending μ.*

Proof. To see that the family $\mathcal{C}$ of μ-measurable subsets of X is closed under complements, suppose that $C \in \mathcal{C}$, fix $B \in \mathcal{B}$ for which $B \bigtriangleup C$ is μ-null, and observe that $(X \setminus B) \bigtriangleup (X \setminus C) = B \bigtriangleup C$, and is therefore also μ-null. To see that $\mathcal{C}$ is closed under countable unions, given a sequence $(C_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{C}$, fix $B_n \in \mathcal{B}$ such that $C_n \subseteq B_n$ and $B_n \setminus C_n$ is μ-null for all $n \in \mathbb{N}$, and note that $\bigcup_{n\in\mathbb{N}} C_n \subseteq \bigcup_{n\in\mathbb{N}} B_n$ and $\bigcup_{n\in\mathbb{N}} B_n \setminus \bigcup_{n\in\mathbb{N}} C_n \subseteq \bigcup_{n\in\mathbb{N}} B_n \setminus C_n$, and is therefore also μ-null.

To see that there is a unique measure on $\mathcal{C}$ extending μ, note that if ν is any such extension, then $\nu(C) = 0$ whenever $C \in \mathcal{C}$ is μ-null, so $\nu(B) = \nu(C)$ whenever $B \subseteq C$ are in $\mathcal{C}$ and $C \setminus B$ is μ-null, thus $\nu(B) = \nu(B \cap C) = \nu(C)$ whenever $B, C \in \mathcal{C}$ and $B \bigtriangleup C$ is μ-null. In particular, it follows that $\nu(C) = \mu(B)$ whenever $B \in \mathcal{B}$, $C \in \mathcal{C}$, and $B \bigtriangleup C$ is μ-null. To see that this constraint yields a well-defined extension of μ, note that if $A, B \in \mathcal{B}$, $C \in \mathcal{C}$, and $A \bigtriangleup C$ and $B \bigtriangleup C$ are μ-null, then the fact that $A \bigtriangleup B \subseteq (A \bigtriangleup C) \cup (B \bigtriangleup C)$ ensures that $\mu(A \bigtriangleup B) = 0$, thus $\mu(A) = \mu(B)$. To see that the resulting extension ν is a measure, suppose that $(C_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{C}$, appeal to the closure of $\mathcal{C}$ under complements to obtain sets

$B_n \in \mathcal{B}$ for which $B_n \subseteq C_n$ and $C_n \setminus B_n$ is μ-null, and observe that $\nu(\bigcup_{n\in\mathbb{N}} C_n) = \mu(\bigcup_{n\in\mathbb{N}} B_n) = \sum_{n\in\mathbb{N}} \mu(B_n) = \sum_{n\in\mathbb{N}} \nu(C_n)$. ⊠

The *completion* of μ is the measure given by Proposition 1.9. We will identify measures with their completions.

2. Integration

A *Borel space* is a set equipped with a σ-algebra of *Borel sets*. A function $\phi\colon X \to Y$ between such spaces is *Borel* if pre-images of Borel sets are Borel.

Proposition 2.1. *Suppose that X is a Borel space and $\phi\colon X \to [0,\infty]$ is Borel. Then there are sequences $(B_n)_{n\in\mathbb{N}}$ of Borel subsets of X and $(r_n)_{n\in\mathbb{N}}$ of positive real numbers such that $\phi = \sum_{n\in\mathbb{N}} r_n \chi_{B_n}$.*

Proof. Fix Borel functions $\phi_n\colon [0,\infty] \to (0,\infty)$ with the property that $\phi_n([0,\infty])$ is countable for all $n \in \mathbb{N}$, and $r = \sum_{n\in\mathbb{N}} \phi_n(r)$ for all $r \in [0,\infty]$. This can be achieved, for example, by setting $\phi_n(\infty) = 1$ for all $n \in \mathbb{N}$ and $\phi_0(r) = \max\{k \in \mathbb{N} \mid k \leq r\}$ for all $r \in [0,\infty)$, and recursively defining $R_n = \{r \in [0,\infty) \mid r \geq 1/2^n + \sum_{m<n} \phi_m(r)\}$ and $\phi_n(r) = (1/2^n)\chi_{R_n}(r)$, for all $n > 0$ and $r \in [0,\infty)$. Now define $B_{n,r} = (\phi_n \circ \phi)^{-1}(\{r\})$ for all $n \in \mathbb{N}$ and $r \in \phi_n([0,\infty])$, and observe that $\phi = \sum_{n\in\mathbb{N}, r\in\phi_n([0,\infty])} r\chi_{B_{n,r}}$. ⊠

We say that a function $s\colon X \to [0,\infty)$ is *simple* if $s(X)$ is finite. Note that a Borel function $s\colon X \to [0,\infty)$ is simple if and only if there exists $N \in \mathbb{N}$ for which there are sequences $(B_n)_{n<N}$ of Borel subsets of X and $(r_n)_{n<N}$ of positive real numbers such that $s = \sum_{n<N} r_n\chi_{B_n}$. A *Borel measure* on a Borel space is a measure on the corresponding family of Borel sets.

Proposition 2.2. *Suppose that X is a Borel space, μ is a Borel measure on X, $(A_m)_{m<M}$ and $(B_n)_{n<N}$ are finite sequences of μ-measurable subsets of X, and $(r_m)_{m<M}$ and $(s_n)_{n<N}$ are finite sequences of reals numbers with the property that $\sum_{m<M} r_m\chi_{A_m} \leq \sum_{n<N} s_n\chi_{B_n}$. Then $\sum_{m<M} r_m\mu(A_m) \leq \sum_{n<N} s_n\mu(B_n)$.*

Proof. By appending the Borel set $A_M = X \setminus \bigcup_{m<M} A_m$ and the real number $r_M = 0$ onto the sequences $(A_m)_{m<M}$ and $(r_m)_{m<M}$, we can assume that $X = \bigcup_{m<M} A_m$, and similarly, that $X = \bigcup_{n<N} B_n$.

For each $t \in 2^M$, set $A_t = \bigcap_{m\in\mathrm{supp}(t)} A_m \setminus \bigcup_{m\in M\setminus\mathrm{supp}(t)} A_m$ and $r_t = \sum_{m\in\mathrm{supp}(t)} r_m$. A straightforward inductive argument shows that by replacing $(A_m)_{m<M}$ and $(r_m)_{m<M}$ with $(A_t)_{t\in 2^M}$ and $(r_m)_{m<M}$ with $(r_t)_{t\in 2^M}$, we can assume that the sets of the form A_m are pairwise disjoint, and similarly, that so too are the sets of the form B_n.

It then follows that $\sum_{m<M} r_m \mu(A_m) = \sum_{m<M,n<N} r_m \mu(A_m \cap B_n)$ and $\sum_{n<N} s_n \mu(B_n) = \sum_{m<M,n<N} s_n \mu(A_m \cap B_n)$, so we can assume that $(A_m)_{m<M} = (B_n)_{n<N}$. But this implies that $r_n \leq s_n$ for all $n < N$, thus $\sum_{n<N} r_n \mu(B_n) \leq \sum_{n<N} s_n \mu(B_n)$. ⊠

We say that a function $\phi\colon X \to Y$ is *μ-measurable* if pre-images of Borel sets are μ-measurable. Proposition 2.2 allows us to define the *integral* of a μ-measurable simple function $s\colon X \to [0,\infty]$ with respect to μ by setting $\int s \; d\mu = \sum_{n<N} r_n \mu(B_n)$, for all finite sequences $(B_n)_{n<N}$ of μ-measurable subsets of X and $(r_n)_{n<N}$ of positive real numbers such that $s = \sum_{n<N} r_n \chi_{B_n}$. Moreover, it allows us to extend this notion to all μ-measurable functions $\phi\colon X \to [0,\infty]$ by setting $\int \phi \; d\mu = \sup\{\int s \; d\mu \mid s \leq \phi \text{ is } \mu\text{-measurable and simple}\}$. We also use $\int \phi(x) \; d\mu(x)$ to denote $\int \phi \; d\mu$, and $\int_B \phi \; d\mu$ to denote $\int \phi \chi_B \; d\mu$.

PROPOSITION 2.3. *Suppose that X is a Borel space, μ is a Borel measure on X, $\phi\colon X \to [0,\infty]$ is μ-measurable, $(B_n)_{n\in\mathbb{N}}$ is a sequence of μ-measurable subsets of X, and $(r_n)_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers with $\phi = \sum_{n\in\mathbb{N}} r_n \chi_{B_n}$. Then $\int \phi \; d\mu = \sum_{n\in\mathbb{N}} r_n \mu(B_n)$.*

PROOF. As $\int \phi \; d\mu \geq \int \sum_{n\leq N} r_n \chi_{B_n} \; d\mu = \sum_{n\leq N} r_n \mu(B_n)$ for all $N \in \mathbb{N}$, it follows that $\int \phi \; d\mu \geq \sum_{n\in\mathbb{N}} r_n \mu(B_n)$, so we need only show that $\int \phi \; d\mu \leq \sum_{n\in\mathbb{N}} r_n \mu(B_n)$. As this holds trivially when $\sum_{n\in\mathbb{N}} r_n \mu(B_n) = \infty$, we can assume that $\sum_{n\in\mathbb{N}} r_n \mu(B_n) < \infty$.

LEMMA 2.4. *Suppose that $\epsilon > 0$. Then $\mu(\phi^{-1}((\epsilon,\infty])) < \infty$.*

PROOF. If $\mu(\phi^{-1}((\epsilon,\infty])) = \infty$, then there exists $N \in \mathbb{N}$ for which the set $B = \{x \in X \mid \sum_{n\leq N} r_n \chi_{B_n}(x) \geq \epsilon\}$ has μ-measure strictly greater than $\sum_{n\in\mathbb{N}} r_n \mu(B_n)/\epsilon$. As $\epsilon \chi_B \leq \sum_{n\leq N} r_n \chi_{B_n}$, Proposition 2.2 yields that $\sum_{n\in\mathbb{N}} r_n \mu(B_n) < \sum_{n\leq N} r_n \mu(B_n)$, a contradiction. ⊠

Suppose now that $s \leq \phi$ is a μ-measurable simple function. Lemma 2.4 then ensures that $\mu(\mathrm{supp}(s)) < \infty$, so for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ for which the set $B = \{x \in X \mid s(x) > \epsilon + \sum_{n\leq N} r_n \chi_{B_n}\}$ has μ-measure at most ϵ. Then $s \leq \max(s)\chi_B + \epsilon\chi_{\mathrm{supp}(s)} + \sum_{n\leq N} r_n \chi_{B_n}$, so $\int s \; d\mu \leq \max(s)\epsilon + \epsilon\mu(\mathrm{supp}(s)) + \sum_{n\in\mathbb{N}} r_n \mu(B_n)$ by Proposition 2.2, thus $\int s \; d\mu \leq \sum_{n\in\mathbb{N}} r_n \mu(B_n)$, hence $\int \phi \; d\mu \leq \sum_{n\in\mathbb{N}} r_n \mu(B_n)$. ⊠

PROPOSITION 2.5. *Suppose that X is a Borel space, μ is a Borel measure on X, and $\phi_n\colon X \to [0,\infty]$ is μ-measurable for all $n \in \mathbb{N}$. Then $\sum_{n\in\mathbb{N}} \phi_n$ is μ-measurable and $\int \sum_{n\in\mathbb{N}} \phi_n \; d\mu = \sum_{n\in\mathbb{N}} \int \phi_n \; d\mu$.*

PROOF. To see that $\sum_{n\in\mathbb{N}} \phi_n$ is μ-measurable, simply note that the function $\phi\colon [0,\infty]^{\mathbb{N}} \to [0,\infty]$ given by $\phi((r_n)_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} r_n$ is Borel.

By Proposition 2.1, there are μ-measurable sets $B_{m,n} \subseteq X$ and real numbers $r_{m,n} \geq 0$ with the property that $\phi_n = \sum_{m \in \mathbb{N}} r_{m,n} \chi_{B_{m,n}}$ for all $n \in \mathbb{N}$. Then $\int \sum_{n \in \mathbb{N}} \phi_n \, d\mu = \sum_{m,n \in \mathbb{N}} r_{m,n} \mu(B_{m,n}) = \sum_{n \in \mathbb{N}} \int \phi_n \, d\mu$ by Proposition 2.3. ⊠

Proposition 2.6. *Suppose that X is a Borel space, μ is a Borel measure on X, $\phi \colon X \to [0, \infty]$ and $\psi \colon X \to [0, \infty]$ are μ-measurable, and $\nu(B) = \int_B \psi \, d\mu$ for all Borel sets $B \subseteq X$. Then ν is a Borel measure on X, ϕ is ν-measurable, and $\int \phi \, d\nu = \int \phi\psi \, d\mu$.*

Proof. Proposition 2.5 directly implies that ν is a measure. As every μ-null set is ν-null, it follows that every μ-measurable set is ν-measurable, thus ϕ is ν-measurable.

By Proposition 2.1, there are μ-measurable sets $B_n \subseteq X$ and real numbers $r_n \geq 0$ with $\phi = \sum_{n \in \mathbb{N}} r_n \chi_{B_n}$. Proposition 2.5 then ensures that $\int \phi \, d\nu = \sum_{n \in \mathbb{N}} r_n \nu(B_n) = \sum_{n \in \mathbb{N}} \int r_n \chi_{B_n} \psi \, d\mu = \int \phi\psi \, d\mu$. ⊠

The *push-forward* of a Borel measure μ on a Borel space X through a Borel function $\psi \colon X \to Y$ is the Borel measure $\psi_* \mu$ on the Borel space Y given by $(\psi_* \mu)(B) = \mu(\psi^{-1}(B))$ for all Borel sets $B \subseteq Y$.

Proposition 2.7. *Suppose that X and Y are Borel spaces, μ is a Borel measure on X, $\psi \colon X \to Y$ is Borel, and $\phi \colon Y \to [0, \infty]$ is $(\psi_* \mu)$-measurable. Then $\phi \circ \psi$ is μ-measurable and $\int \phi \circ \psi \, d\mu = \int \phi \, d(\psi_* \mu)$.*

Proof. As the preimage of every $(\psi_* \mu)$-null set under ϕ is μ-null, it follows that the preimage of every $(\psi_* \mu)$-measurable set under ϕ is μ-measurable, thus $\phi \circ \psi$ is μ-measurable.

By Proposition 2.1, there are $(\psi_* \mu)$-measurable sets $B_n \subseteq X$ and real numbers $r_n > 0$ with the property that $\phi = \sum_{n \in \mathbb{N}} r_n \chi_{B_n}$. Then $\int \phi \circ \psi \, d\mu = \int \sum_{n \in \mathbb{N}} r_n \chi_{\psi^{-1}(B_n)} \, d\mu = \int \phi \, d(\psi_* \mu)$ by Proposition 2.5. ⊠

3. Product measures

The *product* of Borel spaces X and Y is the Borel space whose underlying set is $X \times Y$ and whose distinguished σ-algebra is that generated by Borel rectangles.

Proposition 3.1. *Suppose that X and Y are Borel spaces and μ and ν are σ-finite Borel measures on X and Y. Then there is a unique Borel measure $\mu \times \nu$ on $X \times Y$ such that $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$ for all Borel sets $A \subseteq X$ and $B \subseteq Y$.*

Proof. Let λ denote the function on the family of Borel rectangles given by $\lambda(A \times B) = \mu(A)\nu(B)$. If $A \subseteq X$ and $B \subseteq Y$ are Borel and

$(A_n \times B_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint Borel rectangles whose union is $A \times B$, then Proposition 2.5 ensures that

$$\begin{aligned}\mu(A)\nu(B) &= \int\int \chi_A(x)\chi_B(y)\ d\mu(x)\ d\nu(y)\\ &= \int\int \textstyle\sum_{n\in\mathbb{N}} \chi_{A_n}(x)\chi_{B_n}(y)\ d\mu(x)\ d\nu(y)\\ &= \textstyle\sum_{n\in\mathbb{N}} \int\int \chi_{A_n}(x)\chi_{B_n}(y)\ d\mu(x)\ d\nu(y)\\ &= \textstyle\sum_{n\in\mathbb{N}} \mu(A_n)\nu(B_n),\end{aligned}$$

so $\lambda(A \times B) = \sum_{n\in\mathbb{N}} \lambda(A_n \times B_n)$, thus λ is a measure.

Suppose now that $A, A' \subseteq X$ and $B, B' \subseteq Y$. As $(A\times B)\cap(A'\times B')$ is $(A \cap A') \times (B \cap B')$, it follows that the family of Borel rectangles is closed under finite intersections. As $(A \times B) \setminus (A' \times B')$ is the disjoint union of $(A\setminus A')\times(B\setminus B')$, $(A\cap A')\times(B\setminus B')$, and $(A\setminus A')\times(B\cap B')$, it follows that differences of Borel rectangles are finite disjoint unions of Borel rectangles. An appeal to Remark 1.8 therefore yields the existence of a unique extension of λ to a Borel measure on $X \times Y$. ⊠

Proposition 3.2. *Suppose that $\mathcal{U} \subseteq \mathcal{P}(X)$ is closed under finite intersections. Then the closure of $\mathcal{U}$ under complements and countable disjoint unions is a σ-algebra.*

Proof. It is sufficient to show that the closure $\mathcal{B}$ of $\mathcal{U}$ under complements and countable disjoint unions is itself closed under finite intersections (and therefore countable unions).

Lemma 3.3. *Suppose that $B \in \mathcal{B}$ and $U \in \mathcal{U}$. Then $B \cap U \in \mathcal{B}$.*

Proof. The closure of $\mathcal{U}$ under finite intersections yields the special case where $B \in \mathcal{U}$. If $B\cap U \in \mathcal{B}$, then $({\sim}B)\cap U = {\sim}((B\cap U)\cup{\sim}U)$, so $({\sim}B)\cap U \in \mathcal{B}$. And if $(B_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint subsets of X with the property that $B_n \cap U \in \mathcal{B}$ for all $n \in \mathbb{N}$, then $(\bigcup_{n\in\mathbb{N}} B_n) \cap U = \bigcup_{n\in\mathbb{N}}(B_n \cap U)$, so $(\bigcup_{n\in\mathbb{N}} B_n) \cap U \in \mathcal{B}$. ⊠

If $B \in \mathcal{B}$ has the property that $B \cap C \in \mathcal{B}$ for all $C \in \mathcal{B}$, then $({\sim}B) \cap C = {\sim}((B \cap C) \cup {\sim}C)$ for all $C \in \mathcal{B}$, so $({\sim}B) \cap C \in \mathcal{B}$ for all $C \in \mathcal{B}$. And if $(B_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint subsets of X with the property that $B_n \cap C \in \mathcal{B}$ for all $C \in \mathcal{B}$ and $n \in \mathbb{N}$, then $(\bigcup_{n\in\mathbb{N}} B_n) \cap C = \bigcup_{n\in\mathbb{N}}(B_n \cap C)$ for all $C \in \mathcal{B}$, so $(\bigcup_{n\in\mathbb{N}} B_n) \cap C \in \mathcal{B}$ for all $C \in \mathcal{B}$. ⊠

Theorem 3.4 (Fubini). *Suppose that X and Y are Borel spaces, μ and ν are σ-finite Borel measures on X and Y, and $R \subseteq X \times Y$ is Borel. Then the function $\phi_R\colon X \to [0, \infty]$ given by $\phi_R(x) = \nu(R_x)$ is Borel and $(\mu \times \nu)(R) = \int \phi_R\ d\mu$.*

PROOF. It is clearly sufficient to take care of the special case that μ and ν are both finite.

If there are Borel sets $A \subseteq X$ and $B \subseteq Y$ with $R = A \times B$, then $\phi_R = \nu(B)\chi_A$, so ϕ_R is Borel and $(\mu \times \nu)(R) = \mu(A)\nu(B) = \int \phi_R \, d\mu$.

If $\phi_{\sim R}$ is Borel and $(\mu \times \nu)(\sim R) = \int \phi_{\sim R} \, d\mu$, then the fact that $\phi_R + \phi_{\sim R} = \nu(Y)$ ensures that ϕ_R is Borel, and Proposition 2.5 implies that $\int \phi_R \, d\mu + \int \phi_{\sim R} \, d\mu = \mu(X)\nu(Y)$, thus $(\mu \times \nu)(R) = \int \phi_R \, d\mu$.

If $(R_n)_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint Borel subsets of $X \times Y$, whose union is R, such that ϕ_{R_n} is Borel and $(\mu \times \nu)(R_n) = \int \phi_{R_n} \, d\mu$ for all $n \in \mathbb{N}$, then $\phi_R = \sum_{n \in \mathbb{N}} \phi_{R_n}$, so ϕ_R is Borel. Proposition 2.5 yields that $(\mu \times \nu)(R) = \sum_{n \in \mathbb{N}} (\mu \times \nu)(R_n) = \int \sum_{n \in \mathbb{N}} \phi_{R_n} \, d\mu = \int \phi_R \, d\mu$. ⊠

4. Absolute continuity

A Borel measure μ is *ccc* if there is no uncountable sequence of pairwise disjoint μ-positive sets.

PROPOSITION 4.1. *Suppose that X is a Borel space, μ is a ccc Borel measure on X, and $\mathcal{B} \subseteq \mathcal{P}(X)$ is closed under countable disjoint unions. Then there is a set in $\mathcal{B}$ whose complement does not contain a μ-positive set in $\mathcal{B}$.*

PROOF. Fix a maximal family $\mathcal{A} \subseteq \mathcal{B}$ of pairwise disjoint μ-positive sets in $\mathcal{B}$. As the assumption that μ is ccc ensures that $\mathcal{A}$ is countable, it follows that the set $B = \bigcup \mathcal{A}$ is as desired. ⊠

PROPOSITION 4.2. *Suppose that X is a Borel space, μ is a Borel measure on X, ν is a ccc Borel measure on X, $r > 0$, and $B \subseteq X$ is a Borel set with the property that $\mu(A) \leq r\nu(A)$ for all ν-positive Borel sets $A \subseteq B$, but no ν-positive Borel subset of $\sim B$ has this property. Then $\mu(C) > r\nu(C)$ for all ν-positive Borel sets $C \subseteq \sim B$.*

PROOF. If $C \subseteq \sim B$ is a ν-positive Borel set, then Proposition 4.1 yields a Borel set $D \subseteq C$ such that $\nu(D) > 0 \Longrightarrow \mu(D) > r\nu(D)$ but no ν-positive Borel subset of $C \setminus D$ has this property. As our assumption on B ensures that $C \setminus D$ is ν-null, it follows that D is ν-positive, thus $\mu(C) \geq \mu(D) > r\nu(D) = r\nu(C)$. ⊠

PROPOSITION 4.3. *Suppose that X is a Borel space, μ is a σ-finite Borel measure on X, and ν is a ccc Borel measure on X. Then there is a sequence $(B_n)_{n \in \mathbb{N}}$ of Borel subsets of X, whose union is ν-conull, such that $\mu(B) \leq n\nu(B)$ for all $n \in \mathbb{N}$ and ν-positive Borel sets $B \subseteq B_n$.*

PROOF. It is sufficient to take care of the special case that μ is finite. By Proposition 4.1, there are Borel sets $B_n \subseteq X$ with the property that $\mu(B) \leq n\nu(B)$ for all ν-positive Borel sets $B \subseteq B_n$, but no ν-positive Borel subset of $\sim B_n$ has this property. But Proposition 4.2 ensures that if $\nu(\sim\bigcup_{n\in\mathbb{N}} B_n) > 0$, then $\mu(\sim\bigcup_{n\in\mathbb{N}} B_n) > m\nu(\sim\bigcup_{n\in\mathbb{N}} B_n)$ for all $m \in \mathbb{N}$, contradicting the finiteness of μ. ⊠

A Borel measure μ on X is *absolutely continuous* with respect to a Borel measure ν on X, or $\mu \ll \nu$, if $\mu(B) > 0 \Longrightarrow \nu(B) > 0$ for all Borel sets $B \subseteq X$.

Proposition 4.4. *Suppose that X is a Borel space, μ is a finite Borel measure on X, ν is a ccc Borel measure on X, and $\mu \ll \nu$. Then $\forall\delta > 0 \exists\epsilon > 0 \forall B \subseteq X$ Borel $(\mu(B) \geq \delta \Longrightarrow \nu(B) \geq \epsilon)$.*

Proof. The *restriction* of μ below a Borel set $B \subseteq X$ is the corresponding Borel measure $\mu \restriction B$ on the Borel subspace $B \subseteq X$. By Proposition 4.3, there are Borel sets $B_n \subseteq X$, whose union is ν-conull, such that $\mu \restriction B_n \leq n\nu \restriction B_n$ for all $n \in \mathbb{N}$. Given $\delta > 0$, fix any $\delta' < \delta$, as well as $N \in \mathbb{N}$ with $\mu(\bigcup_{n\leq N} B_n) \geq \mu(X) - \delta'$. Set $\epsilon = (\delta - \delta')/N$, and note that if $B \subseteq X$ is Borel and $\mu(B) \geq \delta$, then $\nu(B) \geq \nu(B \cap \bigcup_{n\leq N} B_n) \geq \mu(B \cap \bigcup_{n\leq N} B_n)/N \geq (\delta - \delta')/N = \epsilon$. ⊠

A *Radon-Nikodým derivative* of a Borel measure μ with respect to a Borel measure ν is a Borel function $\phi\colon X \to [0, \infty)$ satisfying the conclusion of the following theorem.

Theorem 4.5 (Radon-Nikodým). *Suppose that X is a Borel space and $\mu \ll \nu$ are σ-finite Borel measures on X. Then there is a Borel function $\phi\colon X \to [0, \infty)$ with $\mu(B) = \int_B \phi \, d\nu$ for all Borel sets $B \subseteq X$.*

Proof. It is clearly sufficient to take care of the special case that ν is finite. By Proposition 4.3, we can assume that $\mu \leq \nu$.

Define $r\colon 2^{<\mathbb{N}} \to [0, 1)$ by $r(s) = \sum_{n\in\mathrm{supp}(s)} 1/2^{n+1}$. A straightforward recursive construction utilizing Propositions 4.1 and 4.2 yields a sequence $(B_s)_{s\in 2^{<\mathbb{N}}}$ of Borel sets with the property that $B_\emptyset = X$, B_s is the disjoint union of $B_{s\frown(0)}$ and $B_{s\frown(1)}$ for all $s \in 2^{<\mathbb{N}}$, and $r(s)\nu \restriction B_s \leq \mu \restriction B_s \leq (r(s) + 1/2^{|s|})\nu \restriction B_s$ for all $s \in 2^{<\mathbb{N}}$. For each $n \in \mathbb{N}$, let $s_n(x)$ be the unique $s \in 2^n$ with $x \in B_s$, and define $\phi_n\colon X \to \{m/2^n \mid m < 2^n\}$ by $\phi_n(x) = r(s_n(x))$. Now define $\phi\colon X \to [0, 1]$ by $\phi(x) = \lim_{n\to\infty} \phi_n(x)$.

To see that ϕ is as desired, observe that if $B \subseteq X$ is Borel and $n \in \mathbb{N}$, then $\int_B \phi_n \, d\nu = \sum_{s\in 2^n} r(s)\nu(B \cap B_s)$, from which it follows that $\int_B \phi_n \, d\nu \leq \mu(B) \leq \int_B \phi_n + 1/2^n \, d\nu$, and therefore the fact that $\phi_n \leq \phi \leq \phi_n + 1/2^n$ ensures that $|\mu(B) - \int_B \phi \, d\mu| \leq (1/2^n)\nu(B)$, thus $\mu(B) = \int_B \phi \, d\mu$. ⊠

Part II

Measures on Polish spaces

5. Lebesgue measure

The *Lebesgue measure* is the Borel measure given by the following.

PROPOSITION 5.1. *There is a unique Borel measure m on $\mathbb{R}$ with the property that $m([r,s)) = s - r$ for all real numbers $r \leq s$.*

PROOF. Let $\mathcal{U}$ denote the family of all sets of the form $[r,s)$, where $r \leq s$ are real numbers. Clearly $\mathcal{U}$ is closed under intersections, and differences of sets in $\mathcal{U}$ are in the closure of $\mathcal{U}$ under finite disjoint unions. Define $\mu \colon \mathcal{U} \to [0,\infty]$ given by $\mu([r,s)) = s - r$.

LEMMA 5.2. *The function μ is a measure on $\mathcal{U}$.*

PROOF. Suppose that $\mathcal{V} \subseteq \mathcal{U}$ is a countable family of non-empty pairwise disjoint sets whose union is also in $\mathcal{U}$. Then the restriction of $\leq$ to $\{\min(V) \mid V \in \mathcal{V}\}$ is well-founded, for if $(V_n)_{n\in\mathbb{N}}$ is a sequence of sets in $\mathcal{V}$ whose left endpoints r_n are strictly decreasing, then the unique set $V \in \mathcal{V}$ containing the point $r = \lim_{n\to\infty} r_n$ intersects V_n for all but finitely many $n \in \mathbb{N}$, contradicting the fact that the sets in $\mathcal{V}$ are pairwise disjoint. Fix an ordinal $\gamma < \omega_1$ and an injective enumeration $(V_\alpha)_{\alpha<\gamma}$ of $\mathcal{V}$ for which the corresponding left endpoints r_α are strictly increasing. Clearly $V_\alpha = [r_\alpha, r_{\alpha+1})$ whenever $\alpha + 1 < \gamma$, and $r_\lambda = \lim_{\alpha\to\lambda} r_\alpha$ for all limit ordinals $\lambda < \alpha$. As a straightforward induction shows that $r_\beta - r_0 = \sum_{\alpha<\beta} r_{\alpha+1} - r_\alpha$ for all $\beta < \gamma$, it follows that $\mu(\bigcup_{\alpha<\gamma} V_\alpha) = \sum_{\alpha<\gamma} \mu(V_\alpha)$, thus μ is a measure on $\mathcal{U}$. ⊠

Remark 1.8 therefore ensures that μ has a unique extension to a Borel measure on $\mathbb{R}$. ⊠

A *Borel probability measure* μ on a Borel space X is a Borel measure μ on X with the property that $\mu(X) = 1$. When X is a Borel space with respect to which every singleton is Borel, we say that a Borel measure μ on X is *continuous* if every singleton is μ-null.

THEOREM 5.3. *Suppose that X is a standard Borel space and μ is a continuous Borel probability measure on X. Then there is a Borel isomorphism $\pi \colon X \to [0,1)$ with the property that $\pi_*\mu$ is the Lebesgue measure on $[0,1)$.*

PROOF. By the isomorphism theorem for standard Borel spaces, we can assume that $X = [0,1)$. Define $\phi \colon X \to [0,1)$ by $\phi(x) = \mu([0,x))$.

LEMMA 5.4. *The function ϕ is continuous.*

PROOF. To see that ϕ is continuous at a point $x \in X$, note that if $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence of real numbers converging to x, then $[0,x) = \bigcup_{n\in\mathbb{N}}[0,x_n)$, so $\mu([0,x_n)) \to \mu([0,x))$, thus $\phi(x_n) \to \phi(x)$. Similarly, if $(x_n)_{n\in\mathbb{N}}$ is a decreasing sequence of real numbers converging to x, then $[0,x] = \bigcap_{n\in\mathbb{N}}[0,x_n)$, so $\mu([0,x_n)) \to \mu([0,x])$, and since the continuity of μ ensures that $\mu([0,x)) = \mu([0,x])$, it follows that $\phi(x_n) \to \phi(x)$. In particular, these observations ensure that every sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers converging to x has a subsequence $(y_n)_{n\in\mathbb{N}}$ for which $\phi(y_n) \to \phi(x)$, thus $\phi(x_n) \to \phi(x)$. ⊠

In conjunction with the intermediate value theorem and the facts that $\phi(0) = 0$ and $\phi(x) \to 1$ as $x \to 1$, Lemma 5.4 ensures that ϕ is surjective, and therefore that $\phi^{-1}(\{r\})$ is a non-empty closed interval for all $r \in [0,1)$.

One consequence of this observation is that if $r \in [0,1)$, then $(\phi_*\mu)([0,r)) = \mu(\phi^{-1}([0,r))) = \mu([0,\min\phi^{-1}(\{r\}))) = r$, so if $r \leq s$ are in $[0,1)$, then $(\phi_*\mu)([r,s)) = (\phi_*\mu)([0,s)) - (\phi_*\mu)([0,r)) = s - r$, thus $\phi_*\mu$ is the Lebesgue measure on $[0,1)$.

Another consequence of the previous observation is that the set $C = \{r \in [0,1) \mid |\phi^{-1}(\{r\})| > 1\}$ is countable, since $\leq$ is ccc. As the perfect set theorem ensures the existence of a continuous injection of $2^{\mathbb{N}}$ into $[0,1) \setminus C$, and therefore the existence of a continuous injection of $2^{\mathbb{N}}$ into a Lebesgue-null subset of $[0,1) \setminus C$, it follows that there is an uncountable Lebesgue-null Borel superset $N \subseteq [0,1)$ of C. One more application of the isomorphism theorem for standard Borel spaces yields a Borel isomorphism $\psi\colon \phi^{-1}(N) \to N$, in which case the function $\pi = (\phi \restriction {\sim}\phi^{-1}(N)) \cup \psi$ is a Borel automorphism of $[0,1)$ with the property that $\pi_*\mu$ is the Lebesgue measure on $[0,1)$, since the fact that $\pi = \phi$ off of the μ-null set $\phi^{-1}(N)$ ensures that $\pi_*\mu = \phi_*\mu$. ⊠

6. Regularity

We say that a Borel measure μ on a topological space X is *strongly regular* if every μ-measurable set $B \subseteq X$ is contained in a G_δ set $G \subseteq X$ for which $\mu(G \setminus B) = 0$, or equivalently, if every μ-measurable set $B \subseteq X$ contains an F_σ set $F \subseteq X$ for which $\mu(B \setminus F) = 0$.

Proposition 6.1. *Suppose that X is a metric space and μ is a sum of countably-many finite Borel measures on X. Then μ is strongly regular.*

Proof. We will show that every μ-measurable set $B \subseteq X$ contains an F_σ set $F \subseteq X$ for which $\mu(B \setminus F) = 0$. It is sufficient to handle the special case that B is Borel and μ is finite.

Let $\mathcal{B}$ denote the family of all sets $B \subseteq X$ with the property that for all $\epsilon > 0$, there is a closed set $C \subseteq X$ contained in B for which

$\mu(B \setminus C) \leq \epsilon$. As $\mathcal{B}$ trivially contains the closed subsets of X and every open subset of X is F_σ, we need only show that $\mathcal{B}$ is closed under countable intersections and countable unions. Towards this end, suppose that $(B_n)_{n\in\mathbb{N}}$ is a sequence of sets in $\mathcal{B}$.

To see that $\bigcap_{n\in\mathbb{N}} B_n \in \mathcal{B}$, suppose that $\epsilon > 0$, fix a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive real numbers whose sum is at most ϵ, and fix closed sets $C_n \subseteq X$ contained in B_n such that $\mu(B_n \setminus C_n) \leq \epsilon_n$ for all $n \in \mathbb{N}$. Then the set $C = \bigcap_{n\in\mathbb{N}} C_n$ is closed and contained in the set $B = \bigcap_{n\in\mathbb{N}} B_n$. As $B \setminus C \subseteq \bigcup_{n\in\mathbb{N}} B_n \setminus C_n$, it follows that $\mu(B \setminus C) \leq \epsilon$.

To see that $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$, suppose that $\epsilon > 0$, fix any positive real number $\delta < \epsilon$ and $N \in \mathbb{N}$ with $\mu(\bigcup_{n\in\mathbb{N}} B_n \setminus \bigcup_{n\leq N} B_n) \leq \epsilon - \delta$, fix a sequence $(\delta_n)_{n\leq N}$ of positive real numbers whose sum is at most δ, and fix closed sets $C_n \subseteq X$ contained in B_n with $\mu(B_n \setminus C_n) \leq \delta_n$ for all $n \leq N$. Then the set $C = \bigcup_{n\leq N} C_n$ is closed and contained in the set $B = \bigcup_{n\in\mathbb{N}} B_n$. As $B \setminus C \subseteq (\bigcup_{n\leq N} B_n \setminus C_n) \cup (\bigcup_{n\in\mathbb{N}} B_n \setminus \bigcup_{n\leq N} B_n)$, it follows that $\mu(B \setminus C) \leq \epsilon$. ⊠

A set $B \subseteq X$ is a μ-*envelope* for a set $Y \subseteq X$ if $Y \subseteq B$ and every μ-measurable subset of $B \setminus Y$ is μ-null.

Proposition 6.2. *Suppose that X is a metric space and μ is a sum of countably-many finite Borel measures on X. Then every set $Y \subseteq X$ has a G_δ μ-envelope.*

Proof. It is sufficient to show that every set $Y \subseteq X$ is the μ-envelope of an F_σ set. Towards this end, appeal to Proposition 4.1 to obtain an F_σ set $F \subseteq X$ contained in Y with the property that no F_σ subset of X contained in $Y \setminus F$ is μ-positive, and note that if $Y \setminus F$ contains a μ-positive set, then Proposition 6.1 ensures that it contains a μ-positive F_σ subset of X, a contradiction. ⊠

We say that a Borel measure μ on a topological space is *strongly tight* if every μ-measurable set $B \subseteq X$ contains a K_σ set $K \subseteq X$ for which $\mu(B \setminus K) = 0$.

Proposition 6.3. *Suppose that X is a Polish metric space and μ is a sum of countably-many finite Borel measures on X. Then μ is strongly tight.*

Proof. It is sufficient to handle the special case that μ is finite. By Proposition 6.1, we need only show that if $C \subseteq X$ is closed and $\epsilon > 0$, then there is a compact set $K \subseteq C$ for which $\mu(C \setminus K) \leq \epsilon$. Towards this end, fix sequences $(\delta_n)_{n\in\mathbb{N}}$ and $(\epsilon_n)_{n\in\mathbb{N}}$ of positive real numbers for which $\delta_n \to 0$ and $\sum_{n\in\mathbb{N}} \epsilon_n \leq \epsilon$. For each $n \in \mathbb{N}$, fix a cover $(C_{m,n})_{m\in\mathbb{N}}$ of C by closed subsets of diameter at most δ_n, and fix $M_n \in \mathbb{N}$ with

$\mu(C \setminus \bigcup_{m<M_n} C_{m,n}) \leq \epsilon_n$. Then the set $K = \bigcap_{n \in \mathbb{N}} \bigcup_{m<M_n} C_{m,n}$ is compact and $C \setminus K \subseteq \bigcup_{n \in \mathbb{N}} C \setminus \bigcup_{m<M_n} C_{m,n}$, thus $\mu(C \setminus K) \leq \epsilon$. ⊠

Clinton Conley has pointed out that one can also establish Proposition 6.3 by simply appealing to the fact that every Polish space is a G_δ subspace of a compact Polish space, since strong regularity and strong tightness are equivalent for Borel subsets of K_σ spaces.

Proposition 6.4 (Lusin). *Suppose that X is a Polish space and μ is a finite Borel measure on X. Then for every $\epsilon > 0$ and μ-measurable function $\phi \colon X \to Y$ to a second countable topological space, there is a compact set $K \subseteq X$ such that $\mu({\sim}K) \leq \epsilon$ and $\phi \upharpoonright K$ is continuous.*

Proof. Fix a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ of positive real numbers for which $\sum_{n \in \mathbb{N}} \epsilon_n \leq \epsilon$, as well as an enumeration $(V_n)_{n \in \mathbb{N}}$ of a basis for Y. For each $n \in \mathbb{N}$, Proposition 6.3 yields compact sets $K_n \subseteq \phi^{-1}(V_n)$ and $K_n' \subseteq {\sim}\phi^{-1}(V_n)$ for which $\mu({\sim}(K_n \cup K_n')) \leq \epsilon_n$. Then the set $K = \bigcap_{n \in \mathbb{N}} K_n \cup K_n'$ is as desired. ⊠

A measure μ is *semifinite* if every μ-positive set contains a μ-finite μ-positive set. The following observation ensures that every semifinite Borel measure μ on a Polish space, satisfying the weakening of the conclusion of Proposition 6.4 where K is σ-compact, is necessarily σ-finite, and while σ-finiteness is insufficient to obtain this weakening, it is equivalent to the existence of a finer Polish topology, compatible with the underlying Borel structure, for which the weakening holds.

Proposition 6.5. *Suppose that X is a Polish space and μ is a semifinite Borel measure on X. Then the following are equivalent:*

1. *The union of all μ-finite open sets is μ-conull.*
2. *For every $\epsilon > 0$ and μ-measurable function $\phi \colon X \to Y$ to a second countable topological space, there is a σ-compact set $K \subseteq X$ such that $\mu({\sim}K) \leq \epsilon$ and $\phi \upharpoonright K$ is continuous.*
3. *For every $\epsilon > 0$ and compact set $K \subseteq X$, there is a μ-measurable set $B \subseteq X$ such that $\mu({\sim}B) \leq \epsilon$ and $\chi_K \upharpoonright B$ is continuous.*

Proof. To see (1) $\Longrightarrow$ (2), note that condition (1) yields a sequence $(U_n)_{n \in \mathbb{N}}$ of μ-finite open subsets of X whose union is μ-conull, and suppose that $\epsilon > 0$ and $\phi \colon X \to Y$ is a μ-measurable function to a second countable topological space. We will recursively construct pairwise disjoint open sets $V_N \subseteq U_N$ and compact sets $K_N \subseteq V_N$ such that $\mu(\bigcup_{M \leq N} U_M \setminus \bigcup_{m \leq M} K_m) < \epsilon$ and $\phi \upharpoonright K_N$ is continuous for all $N \in \mathbb{N}$. In order to facilitate the construction, we will ensure that $\overline{V_n} \subseteq U_n$ for all $n \in \mathbb{N}$ as well. Suppose that $N \in \mathbb{N}$ and we have already found

$(K_n)_{n<N}$ and $(V_n)_{n<N}$. As $U_N \setminus \bigcup_{n<N} \overline{V_n}$ is open and therefore F_σ, there is a closed set $C_N \subseteq U_N \setminus \bigcup_{n<N} \overline{V_n}$ for which $\mu((U_N \setminus \bigcup_{n<N} \overline{V_n}) \setminus C_N)$ is strictly less than $\epsilon - \mu(\bigcup_{M<N} U_M \setminus \bigcup_{m \leq M} K_m)$. Proposition 6.4 then yields a compact set $K_N \subseteq C_N$ for which $\mu((U_N \setminus \bigcup_{n<N} \overline{V_n}) \setminus K_N)$ is strictly less than $\epsilon - \mu(\bigcup_{M<N} U_M \setminus \bigcup_{m \leq M} K_m)$ and $\phi \restriction K_N$ is continuous. As $\bigcup_{M \leq N} U_M \setminus \bigcup_{m \leq M} K_m$ is the union of $(U_N \setminus \bigcup_{n<N} \overline{V_n}) \setminus K_N$ and $\bigcup_{M<N} U_M \setminus \bigcup_{m \leq M} K_m$, it follows that $\mu(\bigcup_{M \leq N} U_M \setminus \bigcup_{m \leq M} K_m) < \epsilon$. The compactness of K_N then yields an open set $V_N \supseteq K_N$ for which $\overline{V_N} \subseteq U_N \setminus \bigcup_{n<N} \overline{V_n}$, which completes the recursive construction. It follows that the σ-compact set $K = \bigcup_{n \in \mathbb{N}} K_n$ has the property that $\mu(\sim K) = \mu(\bigcup_{n \in \mathbb{N}} U_n \setminus \bigcup_{n \in \mathbb{N}} K_n) \leq \mu(\bigcup_{N \in \mathbb{N}} U_N \setminus \bigcup_{n \leq N} K_n) \leq \epsilon$ and $\phi \restriction K$ is continuous, since if $W \subseteq Y$ is open, then there are open sets $V_n' \subseteq V_n$ with $(\phi \restriction K_n)^{-1}(W) = K_n \cap V_n'$ for all $n \in \mathbb{N}$, in which case $(\phi \restriction K)^{-1}(W) = \bigcup_{n \in \mathbb{N}} K_n \cap V_n' = K \cap \bigcup_{n \in \mathbb{N}} V_n'$, thus $(\phi \restriction K)^{-1}(W)$ is a relatively open subset of K.

As (2) $\Longrightarrow$ (3) is trivial, it only remains to show $\neg$(1) $\Longrightarrow$ $\neg$(3). Towards this end, appeal to the semifiniteness of μ to obtain a μ-finite μ-positive set $A \subseteq X$ consisting solely of points without μ-finite open neighborhoods, and appeal to Proposition 6.3 to obtain a compact μ-positive set $K \subseteq A$. Suppose now that $\epsilon < \mu(K)$ and $B \subseteq X$ is a μ-measurable set for which $\mu(\sim B) \leq \epsilon$. Then there exists $x \in B \cap K$, and since every open neighborhood of x contains points of $B \setminus K$, it follows that $\chi_K \restriction B$ is discontinuous at x. ⊠

The following observation explains the necessity of ϵ in the statement of Proposition 6.4.

Proposition 6.6. *Suppose that X is a Polish space and μ is a semifinite Borel measure on X. Then the following are equivalent:*

1. *There is a discrete μ-conull set.*
2. *For every function $\phi \colon X \to Y$ to a topological space, there is a discrete μ-conull set $D \subseteq X$ such that $\phi \restriction D$ is continuous.*
3. *For every compact set $K \subseteq X$, there is a μ-conull set $B \subseteq X$ such that $\chi_K \restriction B$ is continuous.*

Proof. As (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) is trivial, we need only show $\neg$(1) $\Longrightarrow$ $\neg$(3). Towards this end, suppose first that there exist $x \in X$ and a sequence $(x_n)_{n \in \mathbb{N}}$ of points of X with $\mu(\{x\}) > 0$, $\mu(\{x_n\}) > 0$ for all $n \in \mathbb{N}$, and $x_n \to x$, and observe that if $B \subseteq X$ is μ-conull, then all of these points are in B, thus $\chi_{\{x\}} \restriction B$ is discontinuous at x. If such points do not exist, then $\{x \in X \mid \mu(\{x\}) > 0\}$ is discrete, so $\{x \in X \mid \mu(\{x\}) = 0\}$ is μ-positive, thus the semifiniteness of μ yields a μ-finite μ-positive set $A \subseteq X$ on which μ is continuous, and

Proposition 6.3 yields a compact μ-positive set $K \subseteq A$. By deleting the union of the μ-null relatively open subsets of K, we can assume that every relatively open subset of K is μ-positive.

LEMMA 6.7. *There is a relatively dense open set $U \subseteq K$ with the property that $\mu(U) < \mu(K)$.*

PROOF. Fix a sequence $(k_n)_{n\in\mathbb{N}}$ of positive natural numbers for which $\sum_{n\in\mathbb{N}} 1/k_n < 1$, as well as an enumeration $(U_n)_{n\in\mathbb{N}}$ of non-empty relatively open subsets of K forming a basis. Observe that for each $n \in \mathbb{N}$, the set U_n necessarily contains k_n distinct points, so there is a sequence of k_n non-empty pairwise disjoint relatively open subsets of U_n, and therefore a relatively open set $U'_n \subseteq U_n$ with the property that $\mu(U'_n) \leq (1/k_n)\mu(U_n) \leq (1/k_n)\mu(K)$. But then the set $U = \bigcup_{n\in\mathbb{N}} U'_n$ is as desired. ⊠

It only remains to observe that if $B \subseteq X$ is μ-conull, then there exists $x \in B \cap (K \setminus U)$, and since every open neighborhood of x intersects $B \cap U$, it follows that $\chi_{K\setminus U} \restriction B$ is discontinuous at x. ⊠

7. Density

We say that a point x of a metric space X is a *μ-density point* of a μ-measurable set $B \subseteq X$ if $\mu(B \cap \mathcal{B}(x,\epsilon))/\mu(\mathcal{B}(x,\epsilon)) \to 1$ as $\epsilon \to 0$.

THEOREM 7.1 (Lebesgue). *Suppose that X is a Polish ultrametric space, μ is a finite Borel measure on X, and $B \subseteq X$ is μ-measurable. Then μ-almost every point of B is a μ-density point of B.*

PROOF. By throwing out the maximal μ-null open set, we can assume that every non-empty open set is μ-positive. For each $\epsilon > 0$ and $r < 1$, the fact that X is an ultrametric space ensures that the set

$$U_{\epsilon,r} = \bigcup_{\delta\leq\epsilon} \{x \in X \mid \mu(B \cap \mathcal{B}(x,\delta))/\mu(\mathcal{B}(x,\delta)) \leq r\}$$

is open, so the set $G_r = \bigcap_{\epsilon>0} U_{\epsilon,r}$ is G_δ. Suppose, towards a contradiction, that there exists $r < 1$ for which $\mu(B \cap G_r) > 0$. Then Proposition 6.3 yields a μ-positive compact set $K \subseteq B \cap G_r$, and Proposition 6.1 yields an open set $U \supseteq K$ with $\mu(K) > r\mu(U)$. For each $x \in K$, fix $\epsilon_x > 0$ such that $\mathcal{B}(x,\epsilon_x) \subseteq U$ and $\mu(B \cap \mathcal{B}(x,\epsilon_x))/\mu(\mathcal{B}(x,\epsilon_x)) \leq r$. Let N be the least natural number for which there is a sequence $(x_n)_{n\leq N}$ of points of K with the property that the set $V = \bigcup_{n\leq N} \mathcal{B}(x_n,\epsilon_{x_n})$ contains K. As the minimality of N ensures that the sets $\mathcal{B}(x_n,\epsilon_{x_n})$ are pairwise disjoint, it follows that $\mu(K)/\mu(U) \leq \mu(B \cap V)/\mu(V) \leq r$, the desired contradiction. ⊠

A function $\phi\colon X \to [0,\infty]$ is μ-*integrable* if it is μ-measurable and $\int \phi \; d\mu < \infty$. Let $\overline{\phi}^{\mu}(B)$ denote $\int_B \phi \; d\mu/\mu(B)$.

Proposition 7.2 (Lebesgue). *Suppose that X is a Polish ultrametric space, μ is a finite Borel measure on X, and $\phi\colon X \to [0,\infty]$ is μ-integrable. Then $\phi(x) = \lim_{\epsilon\to 0} \overline{\phi}^{\mu}(\mathcal{B}(x,\epsilon))$ for μ-almost every $x \in X$.*

Proof. By throwing out the maximal μ-null open subset of X, we can assume that every non-empty open subset of X is μ-positive. By Proposition 2.1, there are μ-measurable sets $B_n \subseteq X$ and real numbers $r_n > 0$ for which $\phi = \sum_{n\in\mathbb{N}} r_n \chi_{B_n}$. For each $N \in \mathbb{N}$, define $\phi_N = \sum_{n\leq N} r_n \chi_{B_n}$, and observe that Theorem 7.1 yields that $\phi_N(x) = \sum_{n\leq N} r_n \chi_{B_n}(x) = \sum_{n\leq N} r_n \lim_{\epsilon\to 0} \overline{\chi_{B_n}}^{\mu}(\mathcal{B}(x,\epsilon)) = \lim_{\epsilon\to 0} \overline{\phi_N}^{\mu}(\mathcal{B}(x,\epsilon))$ for μ-almost every $x \in X$, thus $\lim_{\epsilon\to 0} \overline{\phi}^{\mu}(\mathcal{B}(x,\epsilon)) \geq \phi(x)$ for μ-almost every $x \in X$. To show that $\lim_{\epsilon\to 0} \overline{\phi}^{\mu}(\mathcal{B}(x,\epsilon)) \leq \phi(x)$ for μ-almost every $x \in X$, it is sufficient to show that if $0 < \delta < \mu(X)$, then the set of $x \in X$ with $\lim_{\epsilon\to 0} \overline{\phi}^{\mu}(\mathcal{B}(x,\epsilon)) < \delta + \phi(x)$ has μ-measure at least $\mu(X) - \delta$. Towards this end, note that Proposition 2.3, in conjunction with the μ-integrability of ϕ, yields $N \in \mathbb{N}$ sufficiently large that $\int \phi - \phi_N \; d\mu \leq \delta^2$. Define $U = \bigcup_{\epsilon>0} \{x \in X \mid \overline{\phi - \phi_N}^{\mu}(\mathcal{B}(x,\epsilon)) \geq \delta\}$.

Lemma 7.3. *The set U is open and $\mu(U) \leq \delta$.*

Proof. For all $x \in U$, let ϵ_x denote the maximal real number for which $\overline{\phi - \phi_N}^{\mu}(\mathcal{B}(x,\epsilon)) \geq \delta$. Then the sets $\mathcal{B}(x,\epsilon_x)$ yield a partition of U, so U is open and $\delta^2 \geq \int_U \phi - \phi_N \; d\mu \geq \delta\mu(U)$, thus $\mu(U) \leq \delta$. ⊠

But $\lim_{\epsilon\to 0} \overline{\phi}^{\mu}(\mathcal{B}(x,\epsilon)) = \lim_{\epsilon\to 0} \overline{\phi - \phi_N}^{\mu}(\mathcal{B}(x,\epsilon)) + \overline{\phi_N}^{\mu}(\mathcal{B}(x,\epsilon)) < \delta + \phi_N(x)$ for μ-almost every $x \in {\sim}U$, by Lemma 7.3. ⊠

This, in turn, allows us to compute Radon-Nikodým derivatives.

Proposition 7.4. *Suppose that X is a Polish ultrametric space, μ is a finite Borel measure on X, ν is a Borel measure on X, and $\phi\colon X \to [0,\infty]$ is a Radon-Nikodým derivative of μ with respect to ν. Then $\phi(x) = \lim_{\epsilon\to 0} \mu(\mathcal{B}(x,\epsilon))/\nu(\mathcal{B}(x,\epsilon))$ for ν-almost all $x \in X$.*

Proof. As $\phi(x) = \lim_{\epsilon\to 0} \int_{\mathcal{B}(x,\epsilon)} \phi(y) \; d\nu(y)/\nu(\mathcal{B}(x,\epsilon))$ for ν-almost all $x \in X$ by Proposition 7.2, the desired result follows from the fact that $\mu(\mathcal{B}(x,\epsilon)) = \int_{\mathcal{B}(x,\epsilon)} \phi(y) \; d\nu(y)$ for all $\epsilon > 0$ and $x \in X$. ⊠

It is not difficult to verify that the results of this section hold for a given semifinite Borel measure μ on a Polish ultrametric space if and only if the union of all μ-finite open sets is μ-conull.

8. Extensions

When X is a metric space and $\mathcal{U} \subseteq \mathcal{P}(X)$, we say that $U \in \mathcal{U}$ is *approximately bounded* with respect to a finitely-additive measure μ on $\mathcal{U}$ if $\mu(U) = \sup\{\mu(V) \mid V \in \mathcal{U}$ is δ-bounded and $\overline{V} \subseteq U\}$ for all $\delta > 0$.

PROPOSITION 8.1. *Suppose that X is a complete metric space, $\mathcal{U}$ is an algebra of clopen subsets of X, and μ is a finitely-additive measure on $\mathcal{U}$ with respect to which every set in $\mathcal{U}$ is approximately bounded. Then μ is a measure.*

PROOF. By Proposition 1.1, we need only show that μ is σ-subadditive. Suppose, towards a contradiction, that there is a sequence $(U_n)_{n \in \mathbb{N}}$ of sets in $\mathcal{U}$ with $\bigcup_{n \in \mathbb{N}} U_n \in \mathcal{U}$ and $\mu(\bigcup_{n \in \mathbb{N}} U_n) > \sum_{n \in \mathbb{N}} \mu(U_n)$. Fix a sequence $(\delta_m)_{m \in \mathbb{N}}$ of positive real numbers converging to zero, as well as δ_m-bounded sets $V_m \in \mathcal{U}$ such that $V_0 \subseteq \bigcup_{n \in \mathbb{N}} U_n$, $V_{m+1} \subseteq V_m$, and $\mu(V_m) > \sum_{n \in \mathbb{N}} \mu(U_n)$ for all $m \in \mathbb{N}$. As $(U_n)_{n \in \mathbb{N}}$ covers the compact set $K = \bigcap_{m \in \mathbb{N}} \overline{V_m}$, so too does $(U_n)_{n \leq N}$ for some $N \in \mathbb{N}$.

LEMMA 8.2. *There exists $m \in \mathbb{N}$ for which $V_m \subseteq \bigcup_{n \leq N} U_n$.*

PROOF. For each $m \in \mathbb{N}$, fix $I_m \in \mathbb{N}$ and a sequence $(V_{i,m})_{i < I_m}$ of open sets of diameter at most $2\delta_m$ whose union is V_m. Let T be the tree of all $t \in \bigcup_{M \in \mathbb{N}} \prod_{m < M} I_m$ for which $\bigcap_{m < |t|} V_{t(m),m} \not\subseteq \bigcup_{n \leq N} U_n$, and note that T is necessarily well-founded, since any branch b through T would give rise to a singleton $\bigcap_{m \in \mathbb{N}} V_{b(m),m}$ contained in $K \setminus \bigcup_{n \leq N} U_n$. König's Lemma therefore yields $M \in \mathbb{N}$ for which $T \subseteq \bigcup_{L \leq M} \prod_{\ell < L} I_\ell$, in which case $\overline{V_M} \subseteq \bigcup_{n \leq N} U_n$. ⊠

As $\mu(V_m) > \sum_{n \leq N} \mu(U_n) \geq \mu(\bigcup_{n \leq N} U_n)$ for all $m \in \mathbb{N}$, Lemma 8.2 contradicts the monotonicity of μ. ⊠

Proposition 8.1 ensures that if $\mathcal{U}$ is a basis for X, then every finitely-additive σ-finite measure μ on $\mathcal{U}$ has a unique extension to a Borel measure on X. As every zero-dimensional Polish space is homeomorphic to a closed subset of $\mathbb{N}^{\mathbb{N}}$, the following observation shows that, by choosing $\mathcal{U}$ with more care, one can obtain an even more concrete representation of Borel measures on such spaces.

PROPOSITION 8.3. *Suppose that $\mathcal{U} = \{\mathcal{N}_s \mid s \in \mathbb{N}^{<\mathbb{N}}\}$ is the family of basic clopen neighborhoods of $\mathbb{N}^{\mathbb{N}}$ and $\mu \colon \mathcal{U} \to [0, \infty]$ has the property that $\forall s \in \mathbb{N}^{<\mathbb{N}}\ \mu(\mathcal{N}_s) = \sum_{n \in \mathbb{N}} \mu(\mathcal{N}_{s \frown (n)})$. Then there is a unique extension of μ to a measure on the algebra generated by $\mathcal{U}$.*

PROOF. The *external boundary* of a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is the set $\partial_{\text{ext}}(T)$ of all $\sqsubseteq$-minimal elements of $\mathbb{N}^{<\mathbb{N}} \setminus T$.

Lemma 8.4. *Suppose that $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a well-founded tree. Then $\mu(\mathcal{N}_s) = \sum_{t \in \partial_{ext}(T)} \mu(\mathcal{N}_{s \frown t})$ for all $s \in \mathbb{N}^{<\mathbb{N}}$.*

Proof. By induction on the pruning rank of T. Suppose that $0 < \alpha < \omega_1$, the lemma holds for well-founded trees with pruning rank strictly less than α, and the pruning rank of T is α. For all $n \in \mathbb{N}$, set $T_n = \{t \in \mathbb{N}^{<\mathbb{N}} \mid (n) \frown t \in T\}$, and note that if $s \in \mathbb{N}^{<\mathbb{N}}$, then

$$\begin{aligned}\mu(\mathcal{N}_s) &= \textstyle\sum_{n \in \mathbb{N}} \mu(\mathcal{N}_{s \frown (n)}) \\ &= \textstyle\sum_{n \in \mathbb{N}} \sum_{t \in \partial_{\mathrm{ext}}(T_n)} \mu(\mathcal{N}_{s \frown (n) \frown t}) \\ &= \textstyle\sum_{t \in \partial_{\mathrm{ext}}(T)} \mu(\mathcal{N}_{s \frown t}),\end{aligned}$$

since $\partial_{\mathrm{ext}}(T) = \{(n) \frown t \mid n \in \mathbb{N} \text{ and } t \in \partial_{\mathrm{ext}}(T_n)\}$. ⊠

It follows that μ is a measure.

Lemma 8.5. *Suppose that X is an ultrametric space. Then the algebra generated by the open balls is contained in the closure of the open balls under disjoint unions.*

Proof. Note that if A is in the algebra generated by the open balls, then the fact that the intersection of any two open balls is again an open ball ensures that A is of the form $\bigcup_{m<M} A_m \setminus \bigcup_{n<N_m} B_{m,n}$, where each M and N_m is a natural number, each A_m is an open ball or X, and each $B_{m,n}$ is an open ball strictly contained in A_m. Set $\delta = \min_{m,n \in \mathbb{N}} \operatorname{diam}(B_{m,n})$, and observe that A is the union of the open balls of diameter δ that intersect A. ⊠

It follows that the algebra generated by $\mathcal{U}$ is contained in the closure of $\mathcal{U}$ under countable disjoint unions. As $\mathcal{U}$ is closed under finite intersections, Proposition 1.7 therefore ensures the existence of a unique extension of μ to a measure on the algebra generated by $\mathcal{U}$. ⊠

9. The space of probability measures

We begin with a simple observation that allows one to view the result thereafter as a generalization of a part of Fubini's theorem.

Proposition 9.1. *Suppose that X is a set, Y is a Borel space, and $R \subseteq X \times Y$ is in the σ-algebra generated by the sets of the form $A \times B$, where $A \subseteq X$ and $B \subseteq Y$ is Borel. Then R_x is Borel for all $x \in X$.*

Proof. If $R = A \times B$, where $A \subseteq X$ and $B \subseteq Y$ is Borel, then the fact that $R_x \in \{\emptyset, B\}$ for all $x \in X$ ensures that R_x is Borel for all $x \in X$. If $R \subseteq X \times Y$ has the property that R_x is Borel for all $x \in X$, then the fact that $(\sim R)_x = \sim(R_x)$ for all $x \in X$ ensures that $(\sim R)_x$ is Borel for all $x \in X$. And if $(R_n)_{n \in \mathbb{N}}$ is a sequence of subsets of

$X \times Y$ with the property that $(R_n)_x$ is Borel for all $n \in \mathbb{N}$ and $x \in X$, then the fact that $(\bigcup_{n\in\mathbb{N}} R_n)_x = \bigcup_{n\in\mathbb{N}}(R_n)_x$ for all $x \in X$ ensures that $(\bigcup_{n\in\mathbb{N}} R_n)_x$ is Borel for all $x \in X$. ⊠

We endow the set $P(X)$ of all Borel probability measures on a Borel space X with the smallest σ-algebra rendering the functions $\mu \mapsto \mu(B)$ Borel, where B varies over all Borel subsets of X.

Proposition 9.2. *Suppose that X and Y are Borel spaces and $R \subseteq X \times Y$ is Borel. Then the function $\phi_R \colon P(Y) \times X \to [0, 1]$ given by $\phi_R(\mu, x) = \mu(R_x)$ is Borel.*

Proof. If $R = A \times B$, where $A \subseteq X$ and $B \subseteq Y$ are Borel, then $\phi_R(\mu, x) = \mu(B)\chi_A(x)$, thus ϕ_R is Borel. If $R \subseteq X \times Y$ has the property that ϕ_R is Borel, then the fact that $\phi_{\sim R} = 1 - \phi_R$ ensures that $\phi_{\sim R}$ is Borel. And if $(R_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint subsets of $X \times Y$ with the property that ϕ_{R_n} is Borel for all $n \in \mathbb{N}$, then the fact that $\phi_{\bigcup_{n\in\mathbb{N}} R_n} = \sum_{n\in\mathbb{N}} \phi_{R_n}$ ensures that $\phi_{\bigcup_{n\in\mathbb{N}} R_n}$ is Borel. ⊠

When X is a zero-dimensional Polish space, we also endow $P(X)$ with the smallest topology rendering the functions $\mu \mapsto \mu(U)$ continuous, where U ranges over all clopen subsets of X.

Proposition 9.3. *Suppose that X is a zero-dimensional Polish space, τ is the topology on $P(X)$, and $B \subseteq X$ is Borel. Then the function $\mu \mapsto \mu(B)$ is τ-Borel.*

Proof. If $B \subseteq X$ has the property that the function $\mu \mapsto \mu(B)$ is τ-Borel, then the fact that $\mu({\sim}B) = 1 - \mu(B)$ ensures that the function $\mu \mapsto \mu({\sim}B)$ is τ-Borel. And if $(B_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint subsets of X with the property that the function $\mu \mapsto \mu(B_n)$ is τ-Borel for all $n \in \mathbb{N}$, then the fact that $\mu(\bigcup_{n\in\mathbb{N}} B_n) = \sum_{n\in\mathbb{N}} \mu(B_n)$ ensures that the function $\mu \mapsto \mu(\bigcup_{n\in\mathbb{N}} B_n)$ is τ-Borel. ⊠

It follows that a subset of $P(X)$ is Borel if and only if it is τ-Borel.

Proposition 9.4. *Suppose that X is a zero-dimensional Polish space. Then $P(X)$ is a Polish space.*

Proof. Fix a countable algebra $\mathcal{U} \subseteq \mathcal{P}(X)$ of sets forming a basis for X. A *finitely-additive probability measure* on $\mathcal{U}$ is a finitely-additive measure μ on $\mathcal{U}$ for which $\mu(X) = 1$. As the set $C \subseteq [0, 1]^{\mathcal{U}}$ of finitely-additive probability measures on $\mathcal{U}$ is closed, it follows that the set $G \subseteq C$ of finitely-additive probability measures on $\mathcal{U}$ with respect to which every set in $\mathcal{U}$ is approximately bounded is G_δ, thus Polish.

As Proposition 8.1 ensures that each $\mu \in G$ is a measure, Theorem 1.5 implies that each $\mu \in G$ has a unique extension to a Borel probability measure on X. We therefore obtain a bijection $\pi\colon G \to P(X)$ by letting $\pi(\mu)$ be this unique extension.

To see that π is open, note that if $U \in \mathcal{U}$ and $V \subseteq [0,1]$ is open, then $\pi(\{\mu \in G \mid \mu(U) \in V\}) = \{\mu \in P(X) \mid \mu(U) \in V\}$. To see that π is continuous, note first that if $\mu \in G$, $V \subseteq X$ is clopen, $0 \leq r \leq 1$, and $\pi(\mu)(V) > r$, then there exists $U \subseteq V$ in $\mathcal{U}$ with the property that $\mu(U) > r$, so the π-image of the open neighborhood $\{\nu \in G \mid \nu(U) > r\}$ of μ is contained in $\{\nu \in P(X) \mid \nu(V) > r\}$, thus $\pi^{-1}(\{\nu \in P(X) \mid \nu(V) > r\})$ is open. But then the sets of the form $\pi^{-1}(\{\nu \in P(X) \mid \nu(V) < r\})$ are also open, since $\pi(\mu)(V) < r$ if and only if $\pi(\mu)(\sim V) > 1 - r$. It follows that the preimage of every open subset of $P(X)$ under π is open, thus π is continuous. ⊠

Remark 9.5. In the special case that X is compact, the sets C and G coincide, thus $P(X)$ is compact.

Remark 9.6. Proposition 9.4 and Remark 9.5 can be similarly established using Proposition 8.3 in place of Proposition 8.1.

10. Analytic sets

Recall that a non-empty topological space is *analytic* if it is a continuous image of $\mathbb{N}^\mathbb{N}$.

Proposition 10.1 (Lusin). *Suppose that X is a metric space and μ is a sum of countably-many finite Borel measures on X. Then every analytic set $A \subseteq X$ is μ-measurable.*

Proof. Suppose that $\phi\colon \mathbb{N}^\mathbb{N} \to A$ is a continuous surjection. For each sequence $t \in \mathbb{N}^{<\mathbb{N}}$, define $A_t = \phi(\mathcal{N}_t)$ and appeal to Proposition 6.2 to obtain a Borel μ-envelope B_t for A_t contained in $\overline{A_t}$. The fact that $A = \bigcup_{b \in \mathbb{N}^\mathbb{N}} \bigcap_{n \in \mathbb{N}} A_{b \restriction n} = \bigcup_{b \in \mathbb{N}^\mathbb{N}} \bigcap_{n \in \mathbb{N}} B_{b \restriction n} \subseteq \bigcap_{n \in \mathbb{N}} \bigcup_{t \in \mathbb{N}^n} B_t$ ensures that to establish the μ-measurability of A, it is sufficient to show that $\bigcap_{n \in \mathbb{N}} \bigcup_{t \in \mathbb{N}^n} B_t \setminus \bigcup_{b \in \mathbb{N}^\mathbb{N}} \bigcap_{n \in \mathbb{N}} B_{b \restriction n}$ is μ-null. And for this, it is enough to show that $B_t \setminus \bigcup_{n \in \mathbb{N}} B_{t \frown (n)}$ is μ-null for all $t \in \mathbb{N}^{<\mathbb{N}}$. But $B_t \setminus \bigcup_{n \in \mathbb{N}} B_{t \frown (n)} \subseteq B_t \setminus \bigcup_{n \in \mathbb{N}} A_{t \frown (n)} = B_t \setminus A_t$, and is therefore μ-null by the definition of μ-envelope. ⊠

Define $\preceq_n$ on $\mathbb{N}^n$ by $s \preceq_n t \iff \forall m < n \; s(m) \leq t(m)$, and define $\preceq$ on $\mathbb{N}^\mathbb{N}$ by $a \preceq b \iff \forall n \in \mathbb{N} \; a(n) \leq b(n)$.

Proposition 10.2. *Suppose that $b \in \mathbb{N}^\mathbb{N}$, X is a topological space, and $\phi\colon \mathbb{N}^\mathbb{N} \to X$ is closed-to-one. Then $\phi(\preceq^b) = \bigcap_{n \in \mathbb{N}} \bigcup_{s \preceq_n b \restriction n} \phi(\mathcal{N}_s)$.*

Proof. Suppose that $x \in \bigcap_{n \in \mathbb{N}} \bigcup_{s \preceq_n b \restriction n} \phi(\mathcal{N}_s)$, and fix sequences $a_n \in \mathbb{N}^{\mathbb{N}}$ such that $a_n \restriction n \preceq_n b \restriction n$ and $\phi(a_n) = x$ for all $n \in \mathbb{N}$. The former condition ensures the existence of a limit point a of $\{a_n \mid n \in \mathbb{N}\}$, in which case $a \preceq b$ and the latter condition implies that $\phi(a) = x$. ⊠

Proposition 10.3. *Suppose that X is a metric space, μ is a finite Borel measure on X, and $\phi \colon \mathbb{N}^{\mathbb{N}} \to X$ is a continuous function. Then $\mu(\phi(\mathbb{N}^{\mathbb{N}})) = \sup_{b \in \mathbb{N}^{\mathbb{N}}} \mu(\phi(\preceq^b))$.*

Proof. Given $\epsilon > 0$, recursively construct $b \in \mathbb{N}^{\mathbb{N}}$ such that $\mu(\bigcup_{s \preceq_n b \restriction n} \phi(\mathcal{N}_s)) > \mu(\phi(\mathbb{N}^{\mathbb{N}})) - \epsilon$ for all $n \in \mathbb{N}$. It then follows that $\mu(\bigcap_{n \in \mathbb{N}} \bigcup_{s \preceq_n b \restriction n} \phi(\mathcal{N}_s)) \geq \mu(\phi(\mathbb{N}^{\mathbb{N}})) - \epsilon$, in which case Proposition 10.2 ensures that $\mu(\phi(\preceq^b)) \geq \mu(\phi(\mathbb{N}^{\mathbb{N}})) - \epsilon$. ⊠

Remark 10.4. More generally, if $A \subseteq \phi(\mathbb{N}^{\mathbb{N}})$ is μ-measurable, then for all $\epsilon > 0$, Proposition 6.1 yields a closed set $C \subseteq X$ contained in A for which $\mu(A \setminus C) < \epsilon$. As $\phi^{-1}(C)$ is also closed, and is therefore the set of branches through a tree T_C on $\mathbb{N}$, the above argument can be applied to obtain a locally finite subtree T of T_C for which $\mu(A \setminus \phi([T])) < \epsilon$.

We next note that analytic sets are vertical projections of closed sets with compact horizontal sections.

Proposition 10.5. *Suppose that X is a metric space and $\phi \colon \mathbb{N}^{\mathbb{N}} \to X$ is continuous. Then $(\phi \times \mathrm{id})(\preceq)$ is a closed subset of $X \times \mathbb{N}^{\mathbb{N}}$.*

Proof. Suppose that $(x, b) \in \overline{(\phi \times \mathrm{id})(\preceq)}$, and fix $a_n \preceq b_n$ for all $n \in \mathbb{N}$ such that $(\phi(a_n), b_n) \to (x, b)$. Note that if $k \in \mathbb{N}$, then $a_n \restriction k \preceq_k b \restriction k$ for all but finitely many $n \in \mathbb{N}$, so there is a limit point a of $\{a_n \mid n \in \mathbb{N}\}$. Then $a \preceq b$ and $(\phi \times \mathrm{id})(a, b) = (x, b)$. ⊠

The following observation generalizes Proposition 9.2.

Proposition 10.6 (Kondô-Tugué). *Suppose that X is a metric space, Y is a zero-dimensional Polish space, and $R \subseteq X \times Y$ is analytic. Then so too is the set $S = \{(\mu, x, r) \in P(Y) \times X \times [0, 1] \mid \mu(R_x) > r\}$.*

Proof. Suppose that $\phi \colon \mathbb{N}^{\mathbb{N}} \to R$ is a continuous surjection. For $\mu \in P(Y)$ and $x \in X$, let μ_x be the Borel probability measure on $X \times Y$ given by $\mu_x(S) = \mu(S_x)$. Proposition 10.3 then ensures that $\mu(\phi(\mathbb{N}^{\mathbb{N}})_x) = \mu_x(\phi(\mathbb{N}^{\mathbb{N}})) = \sup_{b \in \mathbb{N}^{\mathbb{N}}} \mu_x(\phi(\preceq^b)) = \sup_{b \in \mathbb{N}^{\mathbb{N}}} \mu(\phi(\preceq^b)_x)$, and the set $C = \{((b, x), y) \in (\mathbb{N}^{\mathbb{N}} \times X) \times Y \mid ((x, y), b) \in (\phi \times \mathrm{id})(\preceq)\}$ is closed by Proposition 10.5. As $C_{(b,x)} = \phi(\preceq^b)_x$ for all $b \in \mathbb{N}^{\mathbb{N}}$, it follows that $\mu(R_x) > r \iff \exists b \in \mathbb{N}^{\mathbb{N}}\ \mu(C_{(b,x)}) > r$ for all $r \in [0, 1]$, so Proposition 9.2 yields the desired result. ⊠

11. Ergodic decomposition

The following fact is the main observation underlying the results of this section.

THEOREM 11.1. *Suppose that X is a standard Borel space, E is an equivalence relation on X, and μ is a Borel probability measure on X. Then there is an E-invariant Borel function $\phi\colon X \to P(X)$ such that $\mu(A \cap B) = \int_A \phi(x)(B)\ d\mu(x)$ for all E-invariant Borel sets $A \subseteq X$ and all Borel sets $B \subseteq X$.*

PROOF. By the isomorphism theorem for standard Borel spaces, we can assume that X is a compact zero-dimensional metric space. Fix a countable algebra $\mathcal{U}$ of clopen subsets of X forming a basis.

For each $U \in \mathcal{U}$, let μ_U denote the finite Borel measure on X given by $\mu_U(B) = \mu(B \cap U)$ for all Borel sets $B \subseteq X$.

LEMMA 11.2. *For each $U \in \mathcal{U}$, there is an E-invariant Borel function $\psi_U\colon X \to [0,1]$ with the property that $\mu_U(A) = \int_A \psi_U\ d\mu$ for all E-invariant Borel sets $A \subseteq X$.*

PROOF. Theorem 4.5 yields a Borel function $\psi'_U\colon X/E \to [0,1]$ such that $(\mu_U/E)(A/E) = \int_{A/E} \psi'_U\ d(\mu/E)$ for all E-invariant Borel sets $A \subseteq X$. Note that the E-invariant function $\psi_U\colon X \to [0,1]$ given by $\psi_U(x) = \psi'_U([x]_E)$ is Borel. By Proposition 2.1, there are E-invariant Borel sets $A_n \subseteq X$ and real numbers $r_n > 0$ such that $\psi'_U = \sum_{n\in\mathbb{N}} r_n \chi_{A_n/E}$, and therefore $\psi_U = \sum_{n\in\mathbb{N}} r_n \chi_{A_n}$. If $A \subseteq X$ is an E-invariant Borel set, then Proposition 2.3 ensures that

$$\begin{aligned}\int_{A/E} \psi'_U\ d(\mu/E) &= \sum_{n\in\mathbb{N}} r_n (\mu/E)((A \cap A_n)/E) \\ &= \sum_{n\in\mathbb{N}} r_n \mu(A \cap A_n) \\ &= \int_A \psi_U\ d\mu,\end{aligned}$$

so $\mu_U(A) = (\mu_U/E)(A/E) = \int_{A/E} \psi'_U\ d(\mu/E) = \int_A \psi_U\ d\mu$. ⊠

Define $\psi\colon X \to [0,1]^{\mathcal{U}}$ by $\psi(x)(U) = \psi_U(x)$.

LEMMA 11.3. *For μ-almost all $x \in X$, the function $\psi(x)$ is a finitely-additive probability measure on $\mathcal{U}$.*

PROOF. As $\int \psi_X\ d\mu = \mu_X(X) = 1$, it follows that $\psi_X(x) = 1$ for μ-almost all $x \in X$. As $\int_A \psi_U\ d\mu = \mu(A \cap U)$ for all E-invariant Borel sets $A \subseteq X$ and $U \in \mathcal{U}$, it follows that if $U, V \in \mathcal{U}$ are disjoint sets whose union is also in $\mathcal{U}$, then $\int_A \psi_{U\cup V}\ d\mu = \int_A \psi_U + \psi_V\ d\mu$ for all E-invariant Borel sets $A \subseteq X$, thus $\psi_{U\cup V}(x) = \psi_U(x) + \psi_V(x)$ for μ-almost all $x \in X$. ⊠

As every finitely-additive measure on $\mathcal{U}$ is a measure on $\mathcal{U}$, Theorem 1.5 yields an E-invariant Borel function $\phi\colon X \to P(X)$ with the property that for all $x \in X$, if $\psi(x)$ is a finitely-additive measure on $\mathcal{U}$, then $\phi(x)$ is the unique extension of $\psi(x)$ to a Borel probability measure on X.

It only remains to note that if $A \subseteq X$ is an E-invariant Borel set, then the functions $B \mapsto \mu(A \cap B)$ and $B \mapsto \int_A \phi(x)(B)\, d\mu(x)$ are finite Borel measures on X agreeing on each set in $\mathcal{U}$, so on all open subsets of X, and therefore on all Borel subsets of X, by Proposition 6.1. ⊠

Remark 11.4. If $A \subseteq X$ is an E-invariant Borel set, then the fact that $\int_A \phi(x)(\sim A)\, d\mu(x) = \mu(A \cap (\sim A)) = 0$ ensures that $\phi(x)(\sim A) = 0$ for μ-almost all $x \in A$, thus $\phi(x)(A) = 1$ for μ-almost all $x \in A$.

Remark 11.5. Suppose that F is a Borel superequivalence relation of E that is *smooth*, in the sense that there are Borel sets $A_n \subseteq X$ such that $x \mathrel{F} y \iff \forall n \in \mathbb{N}\ \chi_{A_n}(x) = \chi_{A_n}(y)$ for all $x, y \in X$. By Remark 11.4, the sets $C_n = \{x \in X \mid \phi(x)(A_n) = \chi_{A_n}(x)\}$ are μ-conull, thus so too is the set $C = \bigcap_{n \in \mathbb{N}} C_n$. As $[x]_F$ is $\phi(x)$-conull for all $x \in C$, it follows that $[x]_F$ is $\phi(x)$-conull for μ-almost all $x \in X$.

Remark 11.6. The special case of Remark 11.5 for the equivalence relation on X given by $x \mathrel{F} y \iff \phi(x) = \phi(y)$ ensures that if ϕ satisfies the conclusion of Theorem 11.1, then $\phi^{-1}(\phi(x))$ is $\phi(x)$-conull for μ-almost all $x \in X$. By altering ϕ off of this μ-conull Borel set, we can therefore ensure that $\phi^{-1}(\phi(x))$ is $\phi(x)$-conull for all $x \in X$.

A *Borel disintegration* of μ through a Borel function $\phi\colon X \to Y$ is a Borel function $\psi\colon Y \to P(X)$ such that $\mu(B) = \int \psi(y)(B)\, d(\phi_*\mu)(y)$ for all Borel sets $B \subseteq X$, and $\phi^{-1}(y)$ is $\psi(y)$-conull for $(\phi_*\mu)$-almost all $y \in Y$.

Theorem 11.7. *Suppose that X and Y are standard Borel spaces, μ is a Borel probability measure on X, and $\phi\colon X \to Y$ is Borel. Then there is a Borel disintegration of μ through ϕ.*

Proof. Let E be the smooth Borel equivalence relation on X given by $w \mathrel{E} x \iff \phi(w) = \phi(x)$. By Theorem 11.1, there is an E-invariant Borel function $\phi'\colon X \to P(X)$ such that $\mu(A \cap B) = \int_A \phi'(x)(B)\, d\mu(x)$ for all E-invariant Borel sets $A \subseteq X$ and all Borel sets $B \subseteq X$. As Remark 11.5 ensures that the set $C = \{x \in X \mid \phi'(x)([x]_E) = 1\}$ is μ-conull, there is a $(\phi_*\mu)$-conull Borel set $D \subseteq \phi(C)$. Fix a Borel function $\psi\colon Y \to P(X)$ with $\psi(y) = \nu \iff \exists x \in X\ (\phi(x) = y \text{ and } \phi'(x) = \nu)$ for all $\nu \in P(X)$ and $y \in D$.

To see that ϕ is as desired, note that if $B \subseteq X$ is Borel, then

$$\begin{aligned}\mu(B) &= \int \phi'(x)(B)\ d\mu(x) \\ &= \int (\psi \circ \phi)(x)(B)\ d\mu(x) \\ &= \int \psi(y)(B)\ d(\phi_*\mu)(y),\end{aligned}$$

and if $y \in D$, then there exists $x \in C$ such that $\phi(x) = y$, so $\psi(y)(\phi^{-1}(y)) = \phi'(x)([x]_E) = 1$, thus $\phi^{-1}(y)$ is $\psi(y)$-conull for $(\phi_*\mu)$-almost all $y \in Y$. ⊠

A Borel measure μ on X is *ergodic* with respect to an equivalence relation E on X if every E-invariant μ-measurable set is μ-conull or μ-null. A *Borel decomposition* of μ is a Borel function $\phi\colon X \to P(X)$ such that $\mu(B) = \int \phi(x)(B)\ d\mu(x)$ for all Borel sets $B \subseteq X$, and $\phi^{-1}(\phi(x))$ is $\phi(x)$-conull for all $x \in X$.

Theorem 11.8 (Kechris, Louveau-Mokobodzki). *Suppose that X is a Polish space, E is a K_σ equivalence relation on X, and μ is a Borel probability measure on X. Then there is an E-invariant Borel decomposition $\phi\colon X \to P(X)$ of μ into E-ergodic measures.*

Proof. By Theorem 11.1, there is an E-invariant Borel function $\phi\colon X \to P(X)$ such that $\mu(A \cap B) = \int_A \phi(x)(B)\ d\mu(x)$ for all E-invariant Borel sets $A \subseteq X$ and all Borel sets $B \subseteq X$. By Remark 11.6, we can ensure that $\phi^{-1}(\phi(x))$ is $\phi(x)$-conull for all $x \in X$, so it only remains to show that $\phi(x)$ is E-ergodic for μ-almost all $x \in X$.

Fix a K_σ set $K \subseteq 2^{\mathbb{N}} \times X$ whose vertical sections are exactly the K_σ subsets of X. Then the set $K_E = \{(c, y) \in 2^{\mathbb{N}} \times X \mid \exists x \in X\ c\ K\ x\ E\ y\}$ is also K_σ, and its vertical sections are exactly the E-invariant K_σ subsets of X. As the set $R = \{(\nu, c) \in P(X) \times 2^{\mathbb{N}} \mid 0 < \nu((K_E)_c) < 1\}$ is Borel by Proposition 9.2, the Jankov-von Neumann uniformization theorem yields a $\sigma(\mathbf{\Sigma}^1_1)$-measurable function $\psi\colon \mathrm{proj}_{P(X)}(R) \to 2^{\mathbb{N}}$ whose graph is contained in R. Proposition 10.1 ensures that every such function is μ-measurable.

Suppose, towards a contradiction, that $(\phi_*\mu)(\mathrm{proj}_{P(X)}(R)) > 0$. By Proposition 6.4, there is a $(\phi_*\mu)$-positive Borel set $B \subseteq \mathrm{proj}_{P(X)}(R)$ for which $\psi \restriction B$ is Borel. Then the set $A = \phi^{-1}(B) \cap ((\psi \circ \phi) \times \mathrm{id})^{-1}(K_E)$ is Borel and E-invariant, and $0 < \phi(x)(A) < 1$ for all $x \in \phi^{-1}(B)$, so $\mu(A) = \int \phi(x)(A)\ d\mu(x) > 0$, contradicting Remark 11.4. ⊠

Remark 11.9. Kechris has established the generalization of Theorem 11.8 to analytic equivalence relations, assuming that continuous images of co-analytic subsets of Polish spaces are μ-measurable. A modification of the above argument can be used to establish this:

Fix a closed set $C \subseteq \mathbb{N}^\mathbb{N} \times X$ whose vertical sections are exactly the closed subsets of X, and note that while the corresponding set C_E is merely analytic, Proposition 9.2 nevertheless ensures that the set R of $(\mu, b) \in P(X) \times (\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N})$ for which $\mu(C_{b(0)}), \mu(C_{b(1)}) > 0$ and $(C_E)_{b(0)} \cap (C_E)_{b(1)} = \emptyset$ is co-analytic. The Kondô-Novikov uniformization theorem therefore yields a function $\psi \colon \mathrm{proj}_{P(X)}(R) \to \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ whose graph is co-analytic and contained in R, and the measurability assumption ensures that ψ is μ-measurable, in which case we can define B exactly as before. But this time, fix an E-invariant Borel set $A \subseteq X$ separating $\phi^{-1}(B) \cap ((\mathrm{proj}_0 \circ \psi \circ \phi) \times \mathrm{id})^{-1}(K_E)$ from $\phi^{-1}(B) \cap ((\mathrm{proj}_1 \circ \psi \circ \phi) \times \mathrm{id})^{-1}(K_E)$, and proceed as before.

Remark 11.10. Louveau-Mokobodzki established the generalization of Theorem 11.8 to analytic equivalence relations in ZFC by showing that if X is a Polish space, E is an analytic equivalence relation on X, and μ is a Borel probability measure on X, then there is a K_σ subequivalence relation F of E with the property that for every F-invariant Borel set $A \subseteq X$, there is an E-invariant Borel set $A' \subseteq X$ for which $\mu(A \triangle A') = 0$. To obtain the former result from the latter, simply use F instead of E in the original proof, fix an E-invariant Borel set $A' \subseteq X$ for which $\mu(A \triangle A') = 0$, and note that $\mu(A \triangle A') = \int \phi(x)(A \triangle A') \, d\mu(x)$, so $0 < \phi(x)(A') < 1$ for μ-almost all $x \in \phi^{-1}(B)$, yielding the same contradiction as before.

Part III

Countable Borel equivalence relations

12. Smoothness

An equivalence relation is *finite* if its classes are all finite. A *reduction* of an equivalence relation E on X to an equivalence relation F on Y is function $\pi \colon X \to Y$ with the property that two points are E-related if and only if their images are F-related. Note that a Borel equivalence relation on a standard Borel space is smooth if and only if it is Borel reducible to equality on a standard Borel space.

PROPOSITION 12.1. *Suppose that X is a standard Borel space and E is a finite Borel equivalence relation on X. Then E is smooth.*

PROOF. Fix a Borel linear ordering $\leq$ of X. Then the Lusin-Novikov uniformization theorem ensures that the function $\phi \colon X \to X$, sending each point of X to the $\leq$-minimal element of its E-class, is a Borel reduction of E to equality. ⊠

An equivalence relation is *countable* if its classes are all countable. A set $B \subseteq X$ is *E-complete* if it intersects every E-class. A *partial transversal* of E is a set $B \subseteq X$ intersecting every E-class in at most one point, and such a set is a *transversal* of E if it is also E-complete. A *selector* for E is a reduction $\phi \colon X \to X$ of E to equality for which $\text{graph}(\phi) \subseteq E$.

The proof of Proposition 12.1 yields the stronger fact that every finite Borel equivalence relation on a standard Borel space admits a Borel selector. But this is a special case of a more general fact.

PROPOSITION 12.2. *Suppose that X is a standard Borel space and E is a countable Borel equivalence relation on X. Then the following are equivalent:*

1. *The relation E is smooth.*
2. *There is a Borel selector for E.*
3. *There is a Borel transversal of E.*
4. *There is a sequence $(B_n)_{n \in \mathbb{N}}$ of Borel partial transversals of E such that $X = \bigcup_{n \in \mathbb{N}} B_n$.*
5. *There is a sequence $(B_n)_{n \in \mathbb{N}}$ of Borel transversals of E such that $X = \bigcup_{n \in \mathbb{N}} B_n$.*
6. *There is a sequence $(\phi_n)_{n \in \mathbb{N}}$ of Borel selectors for E such that $E = \bigcup_{n \in \mathbb{N}} \text{graph}(\phi_n)$.*

PROOF. To see (2) $\implies$ (1), note that every selector for E is a reduction of E to equality.

To see (3) $\implies$ (2), note that if $B \subseteq X$ is a Borel transversal of E, and $\phi \colon X \to B$ is the unique function with $\text{graph}(\phi) \subseteq E$, then $\text{graph}(\phi)$ is Borel, thus so too is ϕ, hence ϕ is a Borel selector for E.

To see (4) $\Longrightarrow$ (3), note that if $(B_n)_{n \in \mathbb{N}}$ is a sequence of Borel partial transversals of E for which $X = \bigcup_{n \in \mathbb{N}} B_n$, then the Lusin-Novikov uniformization theorem ensures that the sets $B'_n = B_n \setminus \bigcup_{m<n} [B_m]_E$ are Borel for all $n \in \mathbb{N}$, thus $\bigcup_{n \in \mathbb{N}} B'_n$ is a Borel transversal of E.

To see (5) $\Longrightarrow$ (4), note that every transversal of E is also a partial transversal of E.

To see (6) $\Longrightarrow$ (5), note that if $(\phi_n)_{n \in \mathbb{N}}$ is a sequence of Borel selectors for E such that $E = \bigcup_{n \in \mathbb{N}} \text{graph}(\phi_n)$, then the Lusin-Novikov uniformization theorem ensures that the sets $B_n = \phi_n(X)$ are Borel. But each B_n is a transversal of E and $X = \bigcup_{n \in \mathbb{N}} B_n$.

To see (1) $\Longrightarrow$ (6), note that if $\pi \colon X \to Y$ is a Borel reduction of E to equality on a standard Borel space, then the Lusin-Novikov uniformization theorem ensures that $\pi(X)$ is Borel. As graphs of Borel functions are themselves Borel, it also yields Borel functions $\pi_n \colon \pi(X) \to X$ with $\text{graph}(\pi^{-1}) = \bigcup_{n \in \mathbb{N}} \text{graph}(\pi_n)$. Then the functions $\phi_n = \pi_n \circ \pi$ are Borel selectors for E and $E = \bigcup_{n \in \mathbb{N}} \text{graph}(\phi_n)$. ⊠

An equivalence relation is *aperiodic* if its classes are all infinite.

Proposition 12.3. *Suppose that X is a standard Borel space and E is an aperiodic countable smooth Borel equivalence relation on X. Then there is a sequence $(B_n)_{n \in \mathbb{N}}$ of pairwise disjoint Borel transversals of E such that $X = \bigcup_{n \in \mathbb{N}} B_n$.*

Proof. By Proposition 12.2, there is a sequence $(A_n)_{n \in \mathbb{N}}$ of Borel partial transversals of E with the property that $X = \bigcup_{n \in \mathbb{N}} A_n$. Define $k_n \colon X \to \mathbb{N}$ by $k_n(x) = \min\{k \in \mathbb{N} \mid A_k \cap [x]_E \not\subseteq \bigcup_{m<n} A_{k_m(x)}\}$ for all $n \in \mathbb{N}$. As the Lusin-Novikov uniformization theorem ensures that each k_n is Borel, so too are the sets $B_n = \bigcup_{k \in \mathbb{N}} \{x \in A_k \mid k = k_n(x)\}$. But these sets are pairwise disjoint transversals of E and $X = \bigcup_{n \in \mathbb{N}} B_n$. ⊠

Remark 12.4. The same argument shows that if $N \in \mathbb{N}$ and the cardinality of every E-class is N, then there is a sequence $(B_n)_{n<N}$ of pairwise disjoint Borel transversals of E for which $X = \bigcup_{n<N} B_n$.

A *homomorphism* from an equivalence relation E on X to an equivalence relation F on Y is a function $\pi \colon X \to Y$ sending E-related points to F-related points.

Proposition 12.5. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X, F is smooth, and there is a countable-to-one Borel homomorphism $\pi \colon X \to Y$ from E to F. Then E is smooth.*

Proof. Define $x \; E' \; y \iff \pi(x) \; F \; \pi(y)$. If F is smooth, then E' is smooth, so Proposition 12.2 yields a sequence $(B_n)_{n \in \mathbb{N}}$ of Borel partial transversals of E' for which $X = \bigcup_{n \in \mathbb{N}} B_n$. As $E \subseteq E'$, it follows that every partial transversal of E' is a partial transversal of E, so one more application of Proposition 12.2 ensures that E is smooth. ⊠

PROPOSITION 12.6. *Suppose that X is a standard Borel space, E is a countable smooth Borel equivalence relation on X, and F is a finite-index Borel superequivalence relation of E. Then F is smooth.*

PROOF. Simply note that the restriction of F to every partial transversal of E is finite, and appeal to Propositions 12.1 and 12.2. ⊠

We say that E is *generically smooth* if there is a comeager Borel set on which E is smooth, and *generically nowhere smooth* if the only Borel sets on which E is smooth are meager. Let $\mathbb{E}_0$ denote the equivalence relation on $2^{\mathbb{N}}$ given by $c \mathbin{\mathbb{E}_0} d \iff \exists n \in \mathbb{N} \forall m \geq n \; c(m) = d(m)$.

PROPOSITION 12.7. *The relation $\mathbb{E}_0$ is generically nowhere smooth.*

PROOF. By Proposition 12.2, it is enough to show that if $B \subseteq 2^{\mathbb{N}}$ is a non-meager Borel set, then it is not a partial transversal of $\mathbb{E}_0$. Towards this end, appeal to localization to obtain $t \in 2^{<\mathbb{N}}$ with the property that B is comeager in $\mathcal{N}_t$. As the function $\phi\colon \mathcal{N}_t \to \mathcal{N}_t$ given by $\phi(t \frown (i) \frown c) = t \frown (1-i) \frown c$ is category preserving, it follows that $B \cap \phi^{-1}(B)$ is comeager in $\mathcal{N}_t$. But if $x \in B \cap \phi^{-1}(B)$, then x and $\phi(x)$ are distinct $\mathbb{E}_0$-related points of B. ⊠

We say that E is *category smooth* if it is generically smooth with respect to every Polish topology on X generating its Borel structure.

THEOREM 12.8 (Harrington-Kechris-Louveau). *Suppose that X is a standard Borel space and E is a countable Borel equivalence relation on X. Then E is smooth if and only if E is category smooth.*

PROOF. If E is not smooth, then the Glimm-Effros dichotomy for countable Borel equivalence relations yields a Borel embedding $\pi\colon 2^{\mathbb{N}} \to X$ of $\mathbb{E}_0$ into E. But Proposition 12.7 ensures that E is not generically smooth with respect to any topology on X agreeing on $\pi(2^{\mathbb{N}})$ with the push-forward of the topology on $2^{\mathbb{N}}$ through π. ⊠

We say that E is *μ-smooth* if there is a μ-conull Borel set on which E is smooth, and *μ-nowhere smooth* if the only Borel sets on which E is smooth are μ-null. Let $\mathbb{\mu}_0$ denote the Borel measure on $2^{\mathbb{N}}$ given by $\mathbb{\mu}_0(\mathcal{N}_t) = 1/2^{|t|}$, for all $t \in 2^{<\mathbb{N}}$.

PROPOSITION 12.9. *The relation $\mathbb{E}_0$ is $\mathbb{\mu}_0$-nowhere smooth.*

PROOF. By Proposition 12.2, it is enough to show that if $B \subseteq 2^{\mathbb{N}}$ is a $\mathbb{\mu}_0$-positive Borel set, then it is not a partial transversal of $\mathbb{E}_0$.

Towards this end, appeal to Proposition 7.1 to obtain $t \in 2^{<\mathbb{N}}$ with the property that $\mu_0(B \cap \mathcal{N}_t)/\mu_0(\mathcal{N}_t) > 1/2$. As the function $\phi \colon \mathcal{N}_t \to \mathcal{N}_t$ given by $\phi(t \frown (i) \frown c) = t \frown (1-i) \frown c$ is $(\mu_0 \upharpoonright \mathcal{N}_t)$-preserving, it follows that $\mu_0(B \cap \phi^{-1}(B)) > 0$. But if $x \in B \cap \phi^{-1}(B)$, then x and $\phi(x)$ are distinct $\mathbb{E}_0$-related points of B. ⊠

We say that E is *measure smooth* if it is μ-smooth for all Borel probability measures μ on X.

THEOREM 12.10 (Harrington-Kechris-Louveau). *Suppose that X is a standard Borel space and E is a countable Borel equivalence relation on X. Then E is smooth if and only if E is measure smooth.*

PROOF. If E is not smooth, then the Glimm-Effros dichotomy for countable Borel equivalence relations yields a Borel embedding $\pi \colon 2^{\mathbb{N}} \to X$ of $\mathbb{E}_0$ into E. But Proposition 12.9 ensures that E is $(\pi_* \mu_0)$-nowhere smooth, thus E is not measure smooth. ⊠

13. Combinatorics

The results of the last section allow one to build in a Borel fashion any structure on the classes of a countable smooth Borel equivalence relation that one can build on a countable set. While the analogous statement is false for non-smooth countable Borel equivalence relations, one can still carry out such constructions that depend only upon names for sets in a countable separating family, rather than upon names for points themselves. Here we describe several particularly useful ways of leveraging this fact.

Given a binary relation R on X, we say that a set $Y \subseteq X$ is *R-complete* if it intersects every vertical section of R.

PROPOSITION 13.1 (Slaman-Steel). *Suppose that X is a standard Borel space and R is a transitive Borel binary relation on X whose vertical sections are all countably infinite. Then there is a decreasing sequence $(B_n)_{n \in \mathbb{N}}$ of R-complete Borel sets with empty intersection.*

PROOF. By the isomorphism theorem for standard Borel spaces, we can assume that $X = 2^{\mathbb{N}}$. For all $s \in 2^{<\mathbb{N}}$, set

$$D_s = \{c \in 2^{\mathbb{N}} \mid |\mathcal{N}_s \cap R_c| = \aleph_0 \implies \forall d \in R_c \; |\mathcal{N}_s \cap R_d| = \aleph_0\}.$$

For all $n \in \mathbb{N}$, put $D_n = \bigcap_{s \in 2^n} D_s$, define $s_n \colon D_n \to 2^n$ by

$$s_n(c) = \min_{\mathrm{lex}} \{s \in 2^n \mid |\mathcal{N}_s \cap R_c| = \aleph_0\},$$

and set $A_n = \{c \in D_n \mid s_n(c) = c \upharpoonright n\}$. The Lusin-Novikov uniformization theorem ensures that these functions and sets are Borel.

Lemma 13.2. *Suppose that $n \in \mathbb{N}$ and $c \in A_{n+1}$. Then $c \in A_n$.*

Proof. Note first that if $s \in 2^n$ and $|\mathcal{N}_s \cap R_c| = \aleph_0$, then there exists $i < 2$ with $|\mathcal{N}_{s \frown (i)} \cap R_c| = \aleph_0$, so the fact that $c \in D_{s \frown (i)}$ ensures that $|\mathcal{N}_{s \frown (i)} \cap R_d| = \aleph_0$ for all $d \in R_c$, thus $|\mathcal{N}_s \cap R_d| = \aleph_0$ for all $d \in R_c$. It follows that $c \in D_n$, and the fact that $|\mathcal{N}_s \cap R_c| = \aleph_0$ if and only if there exists $i < 2$ with $|\mathcal{N}_{s \frown (i)} \cap R_c| = \aleph_0$ also ensures that $s_n(c) \sqsubseteq s_{n+1}(c)$. As $s_{n+1}(c) = c \upharpoonright (n+1)$, this implies that $s_n(c) = c \upharpoonright n$, thus $c \in A_n$. ⊠

Lemma 13.3. *Suppose that $c \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$. Then $|A_n \cap R_c| = \aleph_0$.*

Proof. By a straightforward induction of length 2^n, there exists $d \in D_n \cap R_c$. Set $s = s_n(d)$, and observe that $|\mathcal{N}_s \cap R_d| = \aleph_0$ and $\mathcal{N}_s \cap R_d \subseteq A_n \cap R_d \subseteq A_n \cap R_c$. ⊠

Lemma 13.4. *The set $A = \bigcap_{n \in \mathbb{N}} A_n$ is an R-antichain.*

Proof. Suppose that $c \in A$ and $d \in R_c$ are distinct, and fix $n \in \mathbb{N}$ sufficiently large that $c \upharpoonright n \neq d \upharpoonright n$. As $c \in A_n$, it follows that $c \in D_n$ and $s_n(c) = c \upharpoonright n$. As $c \; R \; d$, it follows that $d \in D_n$ and $s_n(d) = c \upharpoonright n$, so $d \notin A_n$. ⊠

It remains to show that the sets $B_n = A_n \setminus A$ are R-complete. Given $c \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, appeal to Lemma 13.3 to obtain distinct points $d \in A_n \cap R_c$ and $e \in A_n \cap R_d$. As Lemma 13.4 ensures that at most one of these points is in A, it follows that at least one is in B_n. ⊠

A *graph* on X is an irreflexive symmetric set $G \subseteq X \times X$. A map $c \colon X \to Y$ is a *coloring* of G if it sends G-related points to distinct points. A graph is *locally finite* if its vertical sections are all finite.

Proposition 13.5 (Kechris-Solecki-Todorcevic). *Suppose that X is a standard Borel space and G is a locally finite Borel graph on X. Then there is a Borel coloring $c \colon X \to \mathbb{N}$ of G.*

Proof. Fix an enumeration $(B_n)_{n \in \mathbb{N}}$ of a Borel separating family for X that is closed under intersections. Then the Lusin-Novikov uniformization theorem ensures that the coloring $c \colon X \to \mathbb{N}$ of G given by $c(x) = \min\{n \in \mathbb{N} \mid x \in B_n \text{ and } B_n \cap G_x = \emptyset\}$ is Borel. ⊠

A set $B \subseteq X$ is *G-independent* if $G \upharpoonright B = \emptyset$. A graph is *locally countable* if its vertical sections are all countable.

Proposition 13.6 (Kechris-Solecki-Todorcevic). *Suppose that X is a standard Borel space and G is a locally countable Borel graph on X for which there is a Borel coloring $c \colon X \to \mathbb{N}$. Then there is a maximal G-independent Borel set $B \subseteq X$.*

PROOF. Recursively set $B_n = \{x \in c^{-1}(n) \mid \bigcup_{m<n} B_m \cap G_x = \emptyset\}$ for all $n \in \mathbb{N}$. As the Lusin-Novikov uniformization theorem ensures that these sets are Borel, so too is the maximal G-independent set $B = \bigcup_{n \in \mathbb{N}} B_n$. ⊠

We say that a graph has *degree at most* k if all of its vertical sections have cardinality at most k.

PROPOSITION 13.7 (Kechris-Solecki-Todorcevic). *Suppose that X is a standard Borel space and G is a Borel graph on X of degree at most some $k \in \mathbb{N}$. Then there is a Borel coloring $c \colon X \to k+1$ of G.*

PROOF. As G is locally finite, Proposition 13.5 yields a Borel $\mathbb{N}$-coloring of G, so Proposition 13.6 yields maximal $(G \restriction \sim\bigcup_{i<j} B_i)$-independent Borel sets $B_j \subseteq \sim\bigcup_{i<j} B_i$ for $j \leq k$. Define $c \colon X \to k+1$ by $c(x) = j \iff x \in B_j$. ⊠

For each $n \in \mathbb{N}$, let $[X]^n$ denote the set of subsets of X of cardinality n, equipped with the standard Borel structure it inherits from X^n. Let $[X]^{<\aleph_0}$ denote the disjoint union of these spaces. The *intersection graph* on a set $\mathcal{S} \subseteq [X]^{<\aleph_0}$ is the graph on $\mathcal{S}$ with respect to which two distinct sets are related if they intersect.

PROPOSITION 13.8 (Kechris-Miller). *Suppose that X is a standard Borel space and $\mathcal{S} \subseteq [X]^{<\aleph_0}$ is Borel. Then the intersection graph on $\mathcal{S}$ has a Borel $\mathbb{N}$-coloring if and only if it is locally countable.*

PROOF. Note first that if the intersection graph on $\mathcal{S}$ is not locally countable, then there exists $x \in X$ appearing in uncountably many $S \in \mathcal{S}$, in which case the set of such S forms an uncountable clique in the intersection graph on $\mathcal{S}$. As the existence of such cliques rules out the existence of $\mathbb{N}$-colorings (let alone Borel $\mathbb{N}$-colorings), it only remains to show that if the intersection graph on $\mathcal{S}$ is locally countable, then it has a Borel $\mathbb{N}$-coloring.

Towards this end, note that the vertical sections of the Borel set

$$R = \{((x,y),S) \in (X \times X) \times \mathcal{S} \mid x,y \in S\}$$

are all countable, so the Lusin-Novikov uniformization theorem ensures that $\mathrm{proj}_{X \times X}(R)$ is Borel. As this projection also has countable vertical sections, another application of the Lusin-Novikov uniformization theorem ensures that $\bigcup \mathcal{S}$ is Borel and yields Borel functions $\phi_n \colon \bigcup \mathcal{S} \to \bigcup \mathcal{S}$ such that $\mathrm{proj}_{X \times X}(R) = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(\phi_n)$. Fix an enumeration $(B_n)_{n \in \mathbb{N}}$ of a Borel separating family for X that is closed under finite intersections.

For each set $S \in \mathcal{S}$, let $a(S)$ be the lexicographically minimal sequence $(n_i^S)_{i<|S|}$ of natural numbers for which there is an injection $\phi\colon |S| \to S$ such that $\forall i, j < |S|\ (\phi(i) \in B_{n_j^S} \iff i = j)$, and let $b(S)$ be the lexicographically minimal sequence $(n_{i,j}^S)_{i,j<|S|}$ of natural numbers with the property that

$$\forall i, j < |S|\ \phi_{n_{i,j}^S}(B_{n_i^S} \cap S) = B_{n_j^S} \cap S.$$

Then $a \times b$ is a Borel coloring of the intersection graph on $\mathcal{S}$. ⊠

Proposition 13.9 (Kechris-Miller). *Suppose that X is a standard Borel space and $\mathcal{S} \subseteq [X]^{<\aleph_0}$ is a Borel set on which the intersection graph is locally countable. Then there is a maximal Borel set $\mathcal{R} \subseteq \mathcal{S}$ of pairwise disjoint sets.*

Proof. By Propositions 13.6 and 13.8. ⊠

A permutation σ is an *involution* if $\sigma^2 = \mathrm{id}$.

Theorem 13.10 (Feldman-Moore). *Suppose that X is a standard Borel space and $R \subseteq X \times X$ is a reflexive symmetric Borel set whose vertical sections are all countable. Then there are Borel involutions $I_n\colon X \to X$ with the property that $R = \bigcup_{n\in\mathbb{N}} \mathrm{graph}(I_n)$.*

Proof. Set $\mathcal{S} = \{\{x, y\} \in [X]^2 \mid x\ R\ y\}$, and appeal to Proposition 13.8 to obtain a Borel coloring $c\colon \mathcal{S} \to \mathbb{N}$ of the intersection graph on $\mathcal{S}$. For each $n \in \mathbb{N}$, let I_n denote the involution of X, with support $\bigcup c^{-1}(\{n\})$, given by $I_n(x) = y \iff c(\{x, y\}) = n$. As the graphs of these involutions are Borel, so too are the involutions themselves, thus the family $\{\mathrm{id}\} \cup \{I_n \mid n \in \mathbb{N}\}$ is as desired. ⊠

The *orbit equivalence relation* induced by a group action $\Gamma \curvearrowright X$ is given by $x\ E_\Gamma^X\ y \iff \exists \gamma \in \Gamma\ \gamma \cdot x = y$.

Theorem 13.11 (Feldman-Moore). *Suppose that X is a standard Borel space and E is a countable Borel equivalence relation on X. Then there is a countable group Γ of Borel automorphisms of X whose induced orbit equivalence relation is E.*

Proof. Appeal to Theorem 13.10 to obtain Borel automorphisms $T_n\colon X \to X$ for which $E = \bigcup_{n\in\mathbb{N}} \mathrm{graph}(T_n)$, and let Γ be the group generated by these automorphisms. ⊠

14. Hyperfiniteness

A Borel equivalence relation E is *hyperfinite* if it is the union of an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of finite Borel subequivalence relations.

PROPOSITION 14.1. *Suppose that X is a standard Borel space and E is a countable smooth Borel equivalence relation on X. Then E is hyperfinite.*

PROOF. By Proposition 12.2, there are Borel partial transversals $B_n \subseteq X$ of E for which $X = \bigcup_{n \in \mathbb{N}} B_n$. For all $n \in \mathbb{N}$, let F_n be the equivalence relation on X generated by $E \upharpoonright \bigcup_{m \leq n} B_m$. ⊠

PROPOSITION 14.2. *The equivalence relation $\mathbb{E}_0$ is hyperfinite.*

PROOF. For all $n \in \mathbb{N}$, let F_n denote the equivalence relation on $2^{\mathbb{N}}$ given by $c \mathrel{F_n} d \iff \forall m \geq n \; c(m) = d(m)$. ⊠

A Borel equivalence relation E is *hypersmooth* if it is the union of an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of smooth Borel subequivalence relations.

PROPOSITION 14.3 (Dougherty-Jackson-Kechris). *Suppose that X is a standard Borel space and E is a countable hypersmooth Borel equivalence relation on X. Then E is hyperfinite.*

PROOF. Fix an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of smooth Borel equivalence relations on X whose union is E. By Proposition 12.2, there are Borel partial transversals $B_{m,n}$ of E_m such that $X = \bigcup_{n \in \mathbb{N}} B_{m,n}$ for all $m \in \mathbb{N}$. For all $n \in \mathbb{N}$, a straightforward induction reveals that for all $m \in \mathbb{N}$, the equivalence relations $F_{m,n}$ generated by $\bigcup_{i \leq m} E_i \upharpoonright \bigcup_{j \leq n} B_{i,j}$ are finite, and the Lusin-Novikov uniformization theorem ensures that they are all Borel. Set $F_n = F_{n,n}$ for all $n \in \mathbb{N}$. ⊠

PROPOSITION 14.4 (Dougherty-Jackson-Kechris). *Suppose that X and Y are standard Borel spaces, E is a countable Borel equivalence relation on X, F is a hyperfinite Borel equivalence relation on Y, and there is a countable-to-one Borel homomorphism $\phi \colon X \to Y$ from E to F. Then E is hyperfinite.*

PROOF. Fix an increasing sequence $(F_n)_{n \in \mathbb{N}}$ of finite Borel equivalence relations on Y whose union is F. Proposition 12.5 then ensures that the equivalence relations $E_n = E \cap (\phi \times \phi)^{-1}(F_n)$ are smooth, so E is hypersmooth, thus Proposition 14.3 implies that E is hyperfinite. ⊠

PROPOSITION 14.5 (Dougherty-Jackson-Kechris). *Suppose that X is a standard Borel space and E is a countable Borel equivalence relation on X. Then the family of Borel sets on which E is hyperfinite is closed under countable unions and saturations.*

PROOF. Suppose first that $B \subseteq X$ is a Borel set on which E is hyperfinite. The Lusin-Novikov uniformization theorem then ensures that $[B]_E$ is Borel and that there is a Borel reduction of $E \upharpoonright [B]_E$ to $E \upharpoonright B$, so Proposition 14.4 implies that E is hyperfinite on $[B]_E$.

Suppose now that $(B_n)_{n \in \mathbb{N}}$ is a sequence of Borel sets on which E is hyperfinite. As the Lusin-Novikov uniformization theorem ensures that the sets $[B_n]_E \setminus \bigcup_{m<n} [B_m]_E$ are Borel, it follows that E is hyperfinite on their union. ⊠

We say that an equivalence relation F has *finite index* over an equivalence relation E if every F-class is the union of finitely-many E-classes.

Proposition 14.6 (Jackson-Kechris-Louveau). *Suppose that X is a standard Borel space, E is a hyperfinite Borel equivalence relation on X, and F is a finite-index Borel superequivalence relation of E. Then F is hyperfinite.*

Proof. By Proposition 13.9, there is a Borel set $\mathcal{S} \subseteq [X]^{<\aleph_0}$ of transversals of restrictions of E to F-classes whose union is F-complete. Fix an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of finite Borel equivalence relations on X whose union is E. For all $n \in \mathbb{N}$, let F_n be the equivalence relation on $\mathcal{S}$ given by $S \; F_n \; T \iff \forall x \in S \exists y \in T \; x \; E_n \; y$. Then the equivalence relation $F_\infty = \bigcup_{n \in \mathbb{N}} F_n$ is hyperfinite. Observe that $S \; F_\infty \; T \iff S \times T \subseteq F$ for all $S, T \in \mathcal{S}$, appeal to the Lusin-Novikov uniformization theorem to obtain a Borel function $\phi \colon X \to \mathcal{S}$ such that $\forall x \in X \; \phi(x) \subseteq [x]_F$, and note that ϕ is a reduction of F to F_∞, thus Proposition 14.4 ensures that F is hyperfinite. ⊠

The *orbit equivalence relation* induced by a bijection $T \colon X \to X$ is given by $x \; E_T^X \; y \iff \exists n \in \mathbb{Z} \; T^n(x) = y$.

Proposition 14.7 (Slaman-Steel, Weiss). *Suppose that X is a standard Borel space and E is a hyperfinite Borel equivalence relation on X. Then E is the orbit equivalence relation induced by a Borel automorphism $T \colon X \to X$.*

Proof. Fix a Borel linear ordering $\leq$ of X, an increasing sequence $(F_n)_{n \in \mathbb{N}}$ of finite Borel equivalence relations on X such that F_0 is equality and $E = \bigcup_{n \in \mathbb{N}} F_n$, and Borel selectors $s_n \colon X \to X$ for each F_n. Given distinct E-related points $x, y \in X$, let $n(x, y)$ denote the maximal natural number n for which $s_n(x) \neq s_n(y)$, and put $x \preceq y \iff s_n(x) < s_n(y)$. Then $\preceq \upharpoonright C$ is isomorphic to the usual ordering of $\mathbb{N}$, $-\mathbb{N}$, or $\mathbb{Z}$ for every infinite E-class C. As E is smooth on the union B of the E-classes C for which $\preceq \upharpoonright C$ is not isomorphic to $\mathbb{Z}$, it is easy to find a Borel automorphism $T \colon B \to B$ generating $E \upharpoonright B$. But the $\preceq$-successor generates E on $\sim B$. ⊠

The *tail equivalence relation* induced by a function $T\colon X \to X$ is the equivalence relation on X given by

$$x \; E_t(T) \; y \iff \exists m, n \in \mathbb{N} \; T^m(x) = T^n(y).$$

Proposition 14.8 (Dougherty-Jackson-Kechris). *Suppose that X is a standard Borel space and $T\colon X \to X$ is a Borel function. Then the tail equivalence relation induced by T is hypersmooth.*

Proof. The *aperiodic part* of T is the set of all $x \in X$ with the property that $T^m(x) \neq T^n(x)$ for all distinct $m, n \in \mathbb{N}$. As this set is Borel and Proposition 12.1 ensures that the restriction of $E_t(T)$ to its complement is smooth, we can assume that T is aperiodic. Let R be the Borel partial order on X given by $x \; R \; y \iff \exists n \in \mathbb{N} \; T^n(x) = y$. By Proposition 13.1, there is a decreasing sequence $(B_n)_{n \in \mathbb{N}}$ of R-complete Borel sets with empty intersection. For all $n \in \mathbb{N}$, define $i_n\colon X \to \mathbb{N}$ by $i_n(x) = \min\{i \in \mathbb{N} \mid T^i(x) \in B_n\}$, as well as $s_n\colon X \to B_n$ by $s_n(x) = T^{i_n(x)}(x)$, and F_n on X by $x \; F_n \; y \iff s_n(x) = s_n(y)$. ⊠

Proposition 14.9 (Dougherty-Jackson-Kechris). *Suppose that X is a standard Borel space and E is a hyperfinite Borel equivalence relation on X. Then E is Borel reducible to $\mathbb{E}_0$.*

Proof. By the isomorphism theorem for standard Borel spaces, we can assume that $X = 2^{\mathbb{N}}$. As the disjoint union of two copies of $\mathbb{E}_0$ is Borel reducible to $\mathbb{E}_0$, and smooth Borel equivalence relations are trivially Borel reducible to $\mathbb{E}_0$, Proposition 12.1 allows us to assume that E is aperiodic. Fix a Borel automorphism $T\colon X \to X$ generating E. Set $B_0 = X$, and given $n \in \mathbb{N}$ and a Borel set $B_n \subseteq X$, let G_n be the graph on B_n in which two distinct points $x, y \in B_n$ are related if there exist $k \in \mathbb{N}$ and $z \in \{x, y\}$ such that $\{x, y\} = \{z, T^k(z)\}$ and $\forall 0 < j < k \; T^j(x) \notin B_n$, and let B_{n+1} be a maximal G_n-independent Borel subset of B_n. By again throwing out an E-invariant Borel set on which E is smooth, we can assume that $\bigcap_{n \in \mathbb{N}} B_n = \emptyset$. For each $n \in \mathbb{N}$ and $x \in X$, fix $i_n(x) \in \mathbb{N}$ least for which $T^{-i_n(x)}(x) \in B_n$, let $b_n(x) \in 3^n$ denote the base two representation of $i_{n+1}(x) - i_n(x)$, and define $\phi_n\colon X \to 2^{n \cdot 3^n}$ by $\phi_n(x) = \bigoplus_{k < 3^n} T^{k - i_n(x)}(x) \upharpoonright n$. Then the function $\phi\colon X \to 2^{\mathbb{N}}$ given by $\phi(x) = \bigoplus_{n \in \mathbb{N}} \phi_n(x) \frown b_n(x)$ is a Borel reduction of E to $\mathbb{E}_0$. ⊠

Remark 14.10. Along with the Glimm-Effros dichotomy for countable Borel equivalence relations, Proposition 14.9 implies that every hyperfinite Borel equivalence relation is Borel reducible to every non-smooth hyperfinite Borel equivalence relation.

We say that E is *generically hyperfinite* if there is a comeager Borel set on which E is hyperfinite.

THEOREM 14.11 (Sullivan-Weiss-Wright, Woodin, Hjorth-Kechris). *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then E is generically hyperfinite.*

PROOF. Let $F_\emptyset$ denote equality on X. Given $s \in \mathbb{N}^{<\mathbb{N}}$ and a finite Borel equivalence relation F_s on X, appeal to Theorem 13.10 to obtain a sequence $(I_{n,s})_{n \in \mathbb{N}}$ of Borel involutions of X/F_s with the property that $E/F_s = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(I_{n,s})$, and for each $n \in \mathbb{N}$, let $F_{s \frown (n)}$ be the extension of F_s with respect to which two F_s-inequivalent points x and y are related if and only if $I_n([x]_{F_s}) = [y]_{F_s}$.

For each $b \in \mathbb{N}^\mathbb{N}$, set $F_b = \bigcup_{n \in \mathbb{N}} F_{b \restriction n}$. Note that if $s \in \mathbb{N}^{<\mathbb{N}}$ and $x \; E \; y$, then there exists $n \in \mathbb{N}$ such that $x \; F_{s \frown (n)} \; y$. It follows that for all $x \in X$, the set of $b \in \mathbb{N}^\mathbb{N}$ with $[x]_E \subseteq [x]_{F_b}$ is dense G_δ, so the Kuratowski-Ulam Theorem yields that $\{x \in X \mid [x]_E = [x]_{F_b}\}$ is comeager for comeagerly many $b \in \mathbb{N}^\mathbb{N}$, thus E is generically hyperfinite. ⊠

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