A BIREDUCIBILITY LEMMA

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Abstract. At the request of Adams (via Kechris), we prove a technical lemma involved with reducibility.

If $E \sim_B F$ are aperiodic hyperfinite equivalence relations, then a result of Dougherty-Jackson-Kechris [1] implies that either E embeds onto a complete section of F, or F embeds onto a complete section of E. This is not true, however, if we drop the assumption of hyperfiniteness. Here we show the next best thing:

Proposition 1. *Suppose that E_1 and E_2 are countable Borel equivalence relations on Polish spaces X_1 and X_2. Then the following are equivalent:*

1. $E_1 \sim_B E_2$;
2. *There are partitions of X_i into E_i-invariant Borel sets X_i^1, X_i^2 such that:*
   (a) $E_1|X_1^1$ *Borel embeds onto a complete section of* $E_2|X_2^1$;
   (b) $E_2|X_2^2$ *Borel embeds onto a complete section of* $E_1|X_1^2$.

Proof. As (2) $\Rightarrow$ (1) is a straightforward consequence of the Lusin-Novikov uniformization theorem, we shall only prove (1) $\Rightarrow$ (2). By a standard Schröder-Bernstein argument and the Lusin-Novikov uniformization theorem, there are Borel E_i-complete sections $A_i \subseteq X_i$ such that $E_1|A_1 \cong_B E_2|A_2$. Fix a Borel isomorphism $\pi : A_1 \to A_2$ of $E_1|A_1$ with $E_2|A_2$. Fix countable groups Γ_i of Borel automorphisms of X_i such that $E_i = E_{\Gamma_i}^{X_i}$, as well as an enumeration

$$(\gamma_1^1, \gamma_2^1), (\gamma_1^2, \gamma_2^2), (\gamma_1^3, \gamma_2^3), \ldots$$

of $\Gamma_1 \times \Gamma_2$, and set $B_1^1 = B_2^1 = \emptyset$ and $\varphi_1 = \emptyset$.

Suppose now that we have found pairwise disjoint Borel sets $B_1^1, \ldots, B_1^n \subseteq X_1$, pairwise disjoint Borel sets $B_2^1, \ldots, B_2^n \subseteq X_2$, and Borel isomorphisms $\varphi_i : B_1^i \to B_2^i$ of $E_1|B_1^i$ with $E_2|B_2^i$, for $1 \leq i \leq n$. Set $X_i^n = X \setminus \bigcup_{1 \leq j \leq n} B_i^j$, and define

$$B_1^{n+1} = X_1^n \cap [\gamma_n^2 \circ \pi \circ \gamma_n^1]^{-1}(X_2^n)$$

and

$$B_2^{n+1} = X_2^n \cap [(\gamma_n^1)^{-1} \circ \pi^{-1} \circ (\gamma_n^2)^{-1}]^{-1}(X_1^n).$$

It is clear that B_i^{n+1} is disjoint from $B_i^1, \ldots, B_i^n$, and the map $\varphi_{n+1}(x) = \gamma_n^2 \circ \pi \circ \gamma_n^1(x)$ is a Borel isomorphism of $E_1|B_1^{n+1}$ with $E_2|B_2^{n+1}$.

Set $B_i = \bigcup_{n \in \mathbb{Z}^+} B_i^n$ and $\varphi = \bigcup_{n \in \mathbb{Z}^+} \varphi_n$. It follows from the definition of B_1^n, B_2^n, and φ_n that φ is a Borel isomorphism of $E_1|B_1$ with $E_2|B_2$. To obtain (2), it is therefore enough to show that for all $x \in A_1$, either $[x]_{E_1} \subseteq B_1$ or $[\varphi(x)]_{E_2} \subseteq B_2$. Suppose, towards a contradiction, that this is not the case. Then there exists $x_1 \in [x]_{E_1} \setminus B_1$ and $x_2 \in [\varphi(x)]_{E_2} \setminus B_2$. Fix $\gamma_1 \in \Gamma_1$ such that $\gamma_1(x_1) \in A_1$, fix $\gamma_2 \in \Gamma_2$ such that $x_2 = \gamma_2 \circ \pi \circ \gamma_1(x_1)$, and fix $n \in \mathbb{N}$ such that $(\gamma_1, \gamma_2) = (\gamma_1^n, \gamma_2^n)$. Then $x_i \in B_i^{n+1} \subseteq B_i$, the desired contradiction. □

References

[1] R. Dougherty, S. Jackson, and A. Kechris. The structure of hyperfinite Borel equivalence relations. *Trans. Amer. Math. Soc.*, **341 (1)**, (1994), 193–225