

# Independence number for partitions of $\omega$

Hiroaki Minami

Graduate School of Science and Technology, Kobe University,  
Rokkodai, Nada-ku, Kobe 657-8501, Japan.

January 14, 2007

## Abstract

In this paper we will define a cardinal invariant corresponding to the independence number for partitions of  $\omega$ . By using Cohen forcing we will prove that this cardinal invariant is consistently smaller than the continuum.

## 1 Introduction

The structure  $([\omega]^\omega, \subset^*)$  of the set of all infinite subsets of  $\omega$  ordered by “almost inclusion” is well studied in set theory. To describe much of the combinatorial structure of  $([\omega]^\omega, \subset^*)$  cardinal invariants of the continuum are introduced like, for example, the reaping number  $\mathfrak{r}$  or the independence number  $\mathfrak{i}$ .

In recent years partial orders similar to  $([\omega]^\omega, \subset^*)$  have been focused on and analogous cardinal invariants have been defined and investigated. For example  $((\omega)^\omega, \leq^*)$ , the set of all infinite partitions of  $\omega$  ordered by “almost coarser”, and the cardinal invariants  $\mathfrak{p}_d$ ,  $\mathfrak{t}_d$ ,  $\mathfrak{s}_d$ ,  $\mathfrak{r}_d$ ,  $\mathfrak{a}_d$  and  $\mathfrak{h}_d$  have been defined and investigated in [2], [3] and [4].

In this work we will define the dual-independence number  $\mathfrak{i}_d$  analogous to the independence number  $\mathfrak{i}$  and get a consistency result.

Once we define dual-independence number  $\mathfrak{i}_d$ , we can prove the following proposition similar to the proof of  $\mathfrak{r} \leq \mathfrak{i}$ .

**Proposition 1.1** (Brendle).  $\mathfrak{r}_d \leq \mathfrak{i}_d$ .

And  $\tau_d$  has the following property.

**Theorem 1.2.** [3] *MA implies  $\tau_d = \mathfrak{c}$ .*

So it is consistent that  $\mathfrak{i}_d = \mathfrak{c}$ . And it is natural to ask the following question.

**Question 1.3.** *Is it consistent that  $\mathfrak{i}_d < \mathfrak{c}$ ?*

In section 2 we will define the dual-independence number and study its properties. In section 3 we will prove that  $\mathfrak{i}_d < \mathfrak{c}$  is consistent by using Cohen forcing.

## 2 $(\omega)^\omega$ and dual-independent family

We start with the definition of “partition of  $\omega$ ”.

**Definition 2.1.**  *$X$  is a partition of  $\omega$  if  $X$  is a subset of  $\wp(\omega)$ ,  $\bigcup X = \omega$  and for each  $a, b \in X$  if  $a \neq b$ , then  $a \cap b = \emptyset$ . By  $(\omega)$  we denote all partitions of  $\omega$ . Also by  $(\omega)^\omega$  we denote all infinite partitions of  $\omega$  and by  $(\omega)^{<\omega}$  we denote all finite partitions of  $\omega$ .*

For partitions of  $\omega$  we give the ordering “coarser”.

**Definition 2.2.** *For  $X, Y \in (\omega)$   $X$  is coarser than  $Y$  ( $Y$  is finer than  $X$ ) if for each  $x \in X$  there exists a subset  $Y'$  of  $Y$  such that  $x = \bigcup Y'$ .*

*For  $X, Y \in (\omega)^\omega$   $X$  is almost coarser than  $Y$  ( $Y$  is almost finer than  $X$ ) if for all but finitely many  $x \in X$  there exists  $Y' \subset Y$  such that  $x = \bigcup Y'$ .*

We can easily check that  $((\omega), \leq)$  is a lattice. For each  $X, Y \in (\omega)$  by  $X \wedge Y$  we denote the infimum of  $X$  and  $Y$ . For  $X, Y \in (\omega)^\omega$  by  $X \perp Y$  we mean that  $X \wedge Y \in (\omega)^{<\omega}$ .

As  $([\omega]^\omega, \subset^*)$ ,  $((\omega)^\omega, \leq^*)$  has the following properties:

**Lemma 2.3.** [3] *Suppose that  $X_0 \geq X_1 \geq X_2 \geq \dots$  is a decreasing sequence of  $(\omega)^\omega$ . Then there exists  $Y \in (\omega)^\omega$  such that  $Y \leq^* X_n$  for  $n \in \omega$ .*

**Lemma 2.4.** [3] *For  $X, Y \in (\omega)^\omega$  if  $\neg(X \leq^* Y)$ , then there exists  $Z \in (\omega)^\omega$  such that  $Z \leq^* X$  and  $Z \perp Y$ .*

So  $((\omega)^\omega, \le^*)$  is similar to  $([\omega]^\omega, \subset^*)$ . On the other hand there is a serious difference:  $([\omega]^\omega, \subset^*)$  is a Boolean algebra but  $((\omega)^\omega, \le^*)$  is just a lattice and not a Boolean algebra.

In general when we define independence, we use complementation. But  $((\omega)^\omega, \le^*)$  doesn't have any natural complementation. So we will define independence for  $((\omega)^\omega, \le^*)$  without mentioning complementation.

**Definition 2.5.** Let  $\mathcal{I}$  be a subset of  $(\omega)^\omega$ .  $\mathcal{I}$  is dual-independent if for all  $\mathcal{A}$  and  $\mathcal{B}$  finite subsets of  $\mathcal{I}$  with  $\mathcal{A} \cap \mathcal{B} = \emptyset$  there exists  $C \in (\omega)^\omega$  such that

- (i)  $C \le^* A$  for  $A \in \mathcal{A}$  and
- (ii)  $C \perp B$  for  $B \in \mathcal{B}$ .

Then define dual-independence number  $i_d$  by

$$i_d = \min\{|\mathcal{I}| : \mathcal{I} \text{ is a maximal dual-independent family}\}.$$

Since there is no natural complementation for an element of  $((\omega)^\omega, \le^*)$ , it becomes more difficult to handle dual-independent families than to handle independent families for a Boolean algebra. But the following lemmata helps to handle dual-independent families.

**Lemma 2.6.** [3] If  $X, Y \in (\omega)^\omega$  and  $\neg(X \le^* Y)$ , then there exists an infinite sequence  $\{a_n\}_{n \in \omega}$  of different elements of  $X$  such that

$$\forall n \in \omega \exists y \in Y (y \cap a_{2n} \neq \emptyset \wedge y \cap a_{2n+1} \neq \emptyset)$$

or there exists a finite subset  $A$  of  $X$  such that the set

$$\{x \in X \setminus A : \exists y \in Y (x \cap y \neq \emptyset \wedge \bigcup A \cap y \neq \emptyset)\}$$

is infinite.

**Proof.** Suppose that we have defined a sequence  $\{a_n\}_{n < 2k}$  but for any two  $a, b \in X \setminus \{a_0, \dots, a_{2k-1}\}$  and  $y \in Y$  we have  $a \cap y = \emptyset$  or  $b \cap y = \emptyset$ . Let  $A$  denote the finite family  $\{a_0, \dots, a_{2k-1}\}$  and let

$$\mathcal{F} = \{x \in X \setminus A : \exists y \in Y (x \cap y \neq \emptyset \wedge \bigcup A \cap y \neq \emptyset)\}.$$

If  $\mathcal{F}$  is finite, then the partition

$$X_* = \{\bigcup A \cup \bigcup \mathcal{F}\} \cup (X \setminus A \cup \mathcal{F})$$

is a finite modification of  $X$  which is coarser than  $Y$ . It is a contradiction to  $\neg(X \le^* Y)$ .

□

By this lemma we can prove the following useful lemma.

**Lemma 2.7.** *If  $X \in (\omega)^\omega$  and  $\mathcal{B}$  is a finite subset of  $(\omega)^\omega$  such that  $\neg(X \leq^* B)$  for  $B \in \mathcal{B}$ , then there exists  $Z \leq X$  such that  $Z \perp B$  for  $B \in \mathcal{B}$ .*

**Proof.** Let  $\mathcal{B} = \{B_i : i < n\}$ . By the above lemma for each  $i < n$  there exists an infinite sequence  $\{a_k^i\}_{k \in \omega}$  of different elements of  $X$  such that

$$\forall k \in \omega \exists b \in B_i (b \cap a_{2k}^i \neq \emptyset \wedge b \cap a_{2k+1}^i \neq \emptyset)$$

or there exists a finite subset  $A_i$  of  $X$  and an infinite sequence  $\{a_k^i\}_{k \in \omega}$  of different elements of  $X \setminus A_i$  such that

$$\forall k \in \omega \exists b \in B_i (b \cap a_k^i \neq \emptyset \wedge \bigcup A_i \cap b \neq \emptyset).$$

In the first case we define  $A_i = \emptyset$ .

Recursively we shall construct a subsequence  $\{b_k^i\}_{k \in \omega}$  of  $\{a_k^i\}_{k \in \omega}$  for  $i < n$ .

Given  $\{b_l^i\}_{l < 2k}$  for  $i < n$  and  $b_{2k}^i, b_{2k+1}^i$  for  $i < j$  for some  $j < n$ .

$A_j = \emptyset$  Choose  $k_0 \in \omega$  such that

$$\{a_{2k_0}^j, a_{2k_0+1}^j\} \cap \left( \bigcup_{i < n} A_i \cup \{b_l^i : i < n \wedge l < 2k\} \cup \{b_{2k}^i, b_{2k+1}^i : i < j\} \right) = \emptyset.$$

$$\text{Put } b_{2k}^j = a_{2k_0}^j \text{ and } b_{2k+1}^j = a_{2k_0+1}^j.$$

$A_j \neq \emptyset$  Choose  $k_0 < k_1 \in \omega$  such that

$$\{a_{k_0}^j, a_{k_1}^j\} \cap \left( \bigcup_{i < n} A_i \cup \{b_l^i : i < n \wedge l < 2k\} \cup \{b_{2k}^i, b_{2k+1}^i : i < j\} \right) = \emptyset.$$

$$\text{Put } b_{2k}^j = a_{k_0}^j \text{ and } b_{2k+1}^j = a_{k_1}^j.$$

Define  $Z = \{\bigcup_{i < n} b_{2k}^i : k \in \omega\} \cup \{\omega \setminus \bigcup_{k \in \omega} \bigcup_{i < n} b_{2k}^i\}$ . Then  $Z \leq X$  and for each  $z \in Z$  and  $i < n$  there exists  $b \in B_i$  such that

$$b \cap z \neq \emptyset \wedge (\omega \setminus \bigcup_{k \in \omega} \bigcup_{i < n} b_{2k}^i) \cap b \neq \emptyset.$$

Hence  $Z \perp B_i$  for  $i < n$ .

□

So it becomes easier to check dual-independence.

**Corollary 2.8.**  *$\mathcal{I}$  is dual-independent if and only if for each finite subset  $\mathcal{A}$  of  $\mathcal{I}$  and  $B \in \mathcal{I} \setminus \mathcal{A}$*

$$\bigwedge \mathcal{A} \not\perp^* B.$$

### 3 Cohen forcing and dual-independence number

By using Cohen forcing we will prove it is consistent that  $i_d < \mathfrak{c}$ .

**Theorem 3.1.** *Suppose  $V \models CH$ . Then  $V^{\mathbb{C}(\omega_2)} \models i_d = \omega_1$ .*

To prove Theorem 3.1 we use the following lemma.

**Lemma 3.2.** *Assume  $p \in \mathbb{C}$ ,  $\mathcal{I}$  is a countable dual-independent family and  $\dot{X}$  is a  $\mathbb{C}$ -name such that  $p \Vdash \text{“}\dot{X} \text{ is a non-trivial infinite partition of } \omega \text{ and } \{\dot{X}\} \cup \mathcal{I} \text{ is dual-independent”}$ . Then there exists  $X^* \in (\omega)^\omega \cap V$  such that  $\{X^*\} \cup \mathcal{I}$  is dual-independent and  $p \Vdash \dot{X} \perp X^*$ .*

**Proof of 3.1 from 3.2** Within the ground model we shall define a maximal dual-independent family  $\mathcal{I}$  of size  $\omega_1$ . It suffices to verify maximality of  $\mathcal{I}$  in the extension via  $\mathbb{C}$  (see [5] pp256).

By CH, let  $\langle p_\xi, \tau_\xi \rangle \xi < \omega_1$  enumerate all pairs  $\langle p, \tau \rangle$  such that  $p \in \mathbb{C}$  and  $\tau$  is a nice name for an infinite partition of  $\omega$ . By recursion, pick an infinite partition of  $\omega$  as follows. Given  $\{X_\eta : \eta < \xi\}$  for some  $\xi < \omega_1$ . Choose  $X_\xi$  so that

- (1)  $\{X_\xi\} \cup \{X_\eta : \eta < \xi\}$  is dual-independent.
- (2) If  $p_\xi \Vdash \text{“}\{\tau_\xi\} \cup \{X_\eta : \eta < \xi\} \text{ is dual-independent”}$ , then  $p_\xi \Vdash X_\xi \perp \tau_\xi$ .

(2) is possible by Lemma 3.2. Let  $\mathcal{I} = \{X_\eta : \eta < \omega_1\}$ . We shall prove  $\mathcal{I}$  is maximal. If  $\mathcal{I}$  is not maximal in  $V[G]$  for some  $\mathbb{C}$ -generic  $G$ , then there exists  $p_\xi \in G$  and  $\tau_\xi$  such that  $p_\xi \Vdash \{\tau_\xi\} \cup \mathcal{I}$  is dual-independent. By construction there exists  $X_\xi \in \mathcal{I}$  and  $p_\xi \Vdash \tau_\xi \perp X_\xi$ . It is a contradiction.

□

**Proof of 3.2.** Let  $\mathbb{P}(\mathcal{I})$  be a partial order such that  $\langle \sigma, \mathcal{H} \rangle \in \mathbb{P}(\mathcal{I})$  if  $\sigma$  is a partition of a finite subset of  $\omega$  and  $\mathcal{H}$  is a finite subset of  $\mathcal{I}$ . It is ordered by  $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{G} \rangle$  if

- (i)  $\forall x \in \tau \exists x' \in \sigma (x \subset x')$ ,
- (ii)  $\mathcal{H} \supset \mathcal{G}$ ,
- (iii)  $\forall x_0 \neq x_1 \in \tau \forall x'_0 \in \sigma (x_0 \subset x'_0 \rightarrow x_1 \cap x'_0 = \emptyset)$ ,
- (iv)  $\forall Y \in \mathcal{G} \forall y_0, y_1 \in (Y \wedge \tau) \forall y'_0, y'_1 \in (Y \wedge \sigma)$   
 $(y_0 \cap y_1 = \emptyset \wedge \bigcup_{\tau} \cap y_0 \neq \emptyset \wedge \bigcup_{\tau} \cap y_1 \neq \emptyset \wedge y_0 \subset y'_0 \wedge y_1 \subset y'_1 \rightarrow y'_0 \cap y'_1 = \emptyset)$ .

**Claim 3.2.1.** *The following sets are dense.*

- (i)  $D_n = \{\langle \sigma, \mathcal{H} \rangle : n \in \bigcup \sigma\}$  for  $n \in \omega$ .
- (ii)  $D_{\mathcal{A}}^l = \{\langle \sigma, \mathcal{H} \rangle : \mathcal{A} \subset \mathcal{H} \wedge |\{h \in (\bigwedge \mathcal{H} \wedge \sigma) : h \cap \bigcup \sigma \neq \emptyset\}| \geq l\}$  for finite subsets  $\mathcal{A}$  of  $\mathcal{I}$  and  $l \in \omega$ .
- (iii)  $D_{\mathcal{A}, l} = \{\langle \sigma, \mathcal{H} \rangle : \mathcal{A} \subset \mathcal{H} \wedge \exists x \in \sigma (|\{h \in \bigwedge \mathcal{H} : x \cap h \neq \emptyset\}| \geq l)\}$  for finite subsets  $\mathcal{A}$  of  $\mathcal{I}$  and  $l \in \omega$ .
- (iv) Let  $\mathcal{A}$  be a finite subset of  $\mathcal{I}$ ,  $B \in \mathcal{I} \setminus \mathcal{A}$  and  $A = \bigwedge \mathcal{A}$ . Since  $\neg(A \leq^* B)$  and by Lemma 2.6, there exists  $\{a_n\}_{n \in \omega}$  such that

$$\forall n \in \omega \exists b \in B (a_{2n} \cap b \neq \emptyset \wedge a_{2n+1} \cap b \neq \emptyset) \quad (1)$$

or there exists a finite subset  $A_0$  of  $A$  such that the set

$$\mathcal{F}_{A_0} = \{a \in A \setminus A_0 : \exists y \in Y (y \cap a \neq \emptyset \wedge y \cap \bigcup A_0 \neq \emptyset)\} \quad (2)$$

is infinite. If (1) holds, fix  $\{a_n\}_{n \in \omega}$ . If (2) holds, fix  $A_0$  and  $\mathcal{F}_{A_0}$

- (1) Let  $D_{\mathcal{A}, B, l} = \{\langle \sigma, \mathcal{H} \rangle : \exists \{a^i : i < 2l\} \subset (A \wedge \sigma) (\forall i < 2l (\bigcup \sigma \cap a^i \neq \emptyset) \wedge \bigwedge \{a^i : i < 2l\} \text{ is pairwise disjoint} \wedge \forall i < l \exists b \in B (a^{2i} \cap b \neq \emptyset \wedge a^{2i+1} \cap b \neq \emptyset))\}$ .
- (2) Let  $D_{\mathcal{A}, B, l} = \{\langle \sigma, \mathcal{H} \rangle : \exists \{a^i : i < l\} \subset (A \wedge \sigma) (\forall i < l (\bigcup \sigma \cap a^i \neq \emptyset) \wedge \bigwedge \{a^i : i < l\} \text{ is pairwise disjoint} \wedge \forall i < l (\bigcup A_0 \cap a^i = \emptyset) \wedge \forall a \in A_0 (a \cap \bigcup \sigma \neq \emptyset) \wedge \forall i < l \exists b \in B (b \cap a^i \neq \emptyset \wedge b \cap \bigcup A_0 \neq \emptyset))\}$ .

- (v) Let  $\{\dot{x}_i : i \in \omega\}$  be  $\mathbb{C}$ -names such that  $\Vdash \dot{X} = \{\dot{x}_i : i \in \omega\}$  and  $\min \dot{x}_i < \min \dot{x}_{i+1}$ . Put  $D_{\dot{X}, l, q} = \{\langle \sigma, \mathcal{H} \rangle : \exists r \leq q \left( r \Vdash \exists x \in (\dot{X} \wedge \sigma) (\bigcup_{i < l} \dot{x}_i \subset x) \right)\}$  for  $q \leq p$  and  $l \in \omega$ .

**Proof of Claim.**

- (i) Clear.
- (ii) Let  $\langle \tau, \mathcal{H} \rangle \in \mathbb{P}(\mathcal{I})$ . Without loss of generality, we can assume  $\mathcal{A} \subset \mathcal{H}$ . Let  $H = \bigwedge \mathcal{H}$ . Choose  $h_i \in H$  for  $i < l$  such that  $h_i \cap \bigcup \tau = \emptyset$ . Choose  $n_i \in h_i$ . Put  $\sigma = \tau \cup \{\{n_i\} : i < l\}$ . Then  $\{h_i : i < l\} \subset \{h \in (H \wedge \sigma) : h \cap \bigcup \sigma \neq \emptyset\}$ . So  $\langle \sigma, \mathcal{H} \rangle \in D_{\mathcal{A}}^l$ .

We shall prove  $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{H} \rangle$ . Let  $Y \in \mathcal{H}$ . Since  $h_i \cap \bigcup \tau = \emptyset$  and  $n_i \in h_i$  for  $i < l$ ,  $\{y \in (Y \wedge \sigma) : y \cap \bigcup \sigma \neq \emptyset\} = \{y \in (Y \wedge \tau) : y \cap \bigcup \sigma \neq \emptyset\} \dot{\cup} \{y \in Y : \exists i < l (n_i \in y)\}$ . Hence  $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{H} \rangle$ .

- (iii) Let  $\langle \tau, \mathcal{H} \rangle \in \mathbb{P}(\mathcal{I})$ . Without loss of generality, we can assume  $\mathcal{A} \subset \mathcal{H}$ . Let  $H = \bigwedge \mathcal{H}$ . Choose  $\{h_i : i < l\}$  distinct elements of  $H$  such that  $h_i \cap \bigcup \tau = \emptyset$  for  $i < l$ . Choose  $n_i \in h_i$  for  $i < l$ . Put  $\sigma = \tau \cup \{\{n_i : i < l\}\}$ . Then  $\{h \in H : \{n_i : i < l\} \cap h \neq \emptyset\} = \{h_i : i < l\}$ . So  $\langle \sigma, \mathcal{H} \rangle \in D_{\mathcal{A}, l}$ .

We shall prove  $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{H} \rangle$ .

Since  $h_i \cap \bigcup \tau = \emptyset$  and  $n_i \in h_i$  for  $i < l$ ,  $\{y \in (Y \wedge \sigma) : y \cap \bigcup \sigma \neq \emptyset\} = \{y \in (Y \wedge \tau) : y \cap \bigcup \tau \neq \emptyset\} \dot{\cup} \{\bigcup \{y \in Y : \exists i < l (n_i \in y)\}\}$ . Hence  $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{H} \rangle$ .

- (iv) (1) Let  $\langle \tau, \mathcal{H} \rangle \in \mathbb{P}(\mathcal{I})$ . Choose distinct  $i_j \in \omega$  for  $j \leq l$  so that  $\bigcup \tau \cap a_{2i_j} = \emptyset$  and  $\bigcup \tau \cap a_{2i_j+1} = \emptyset$  for  $j < l$ . Let  $k_n = \min a_n$  for  $n \in \omega$ . Put  $\sigma = \tau \cup \{\{k_{2i_j}\}, \{k_{2i_j+1}\} : j < l\}$ . Since  $\bigcup \tau \cap a_{2i_j} = \bigcup \tau \cap a_{2i_j+1} = \emptyset$  and  $k_n \in a_n$ ,  $\{a_{2i_j}, a_{2i_j+1} : j < l\} \subset (A \wedge \sigma)$ ,  $\{a_{2i_j}, a_{2i_j+1} : j < l\}$  is pairwise distinct and for  $i < l$  there exists  $b \in B$  such that  $b \cap a_{2i_j} \neq \emptyset$  and  $b \cap a_{2i_j+1} \neq \emptyset$ . So  $\langle \sigma, \mathcal{H} \rangle \in D_{\mathcal{A}, B, l}$ .

We shall prove  $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{H} \rangle$ . Let  $Y \in \mathcal{H}$ . Since  $\bigcup \tau \cap a_{2i_j} = \bigcup \tau \cap a_{2i_j+1} = \emptyset$ ,  $\{y \in (Y \wedge \sigma) : y \cap \bigcup \sigma \neq \emptyset\} = \{y \in (Y \wedge \tau) : y \cap \bigcup \sigma \neq \emptyset\} \dot{\cup} \{y \in (Y \wedge \tau) : \exists j < l (k_{2i_j} \in y \vee k_{2i_j+1} \in y)\}$ . Hence  $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{H} \rangle$ .

- (2) Let  $\langle \tau, \mathcal{H} \rangle \in \mathbb{P}(\mathcal{I})$ . Without loss of generality we can assume  $\bigcup \tau \cap a \neq \emptyset$  for  $a \in A_0$ . Choose distinct  $a^i$  for  $i < l$  so that  $a^i \cap \bigcup \tau = \emptyset$

and  $a^i \in \mathcal{F}_{A_0}$ . Let  $k_i = \min a^i$  and  $\sigma = \tau \cup \{\{k_i\} : i < l\}$ . Since  $\bigcup \tau \cap a^i = \emptyset$ ,  $a^i \in \mathcal{F}_{A_0}$  and  $k_i \in a^i$ ,  $\{a^i : j < l\} \subset (A \wedge \sigma)$ ,  $\{a^i : i < l\}$  is pairwise distinct,  $\bigcup A_0 \cap a^i = \emptyset$  and for each  $i < l$  there exists  $b \in B$  such that  $b \cap a^i \neq \emptyset$  and  $b \cap \bigcup A_0 \neq \emptyset$ . So  $\langle \sigma, \mathcal{H} \rangle \in D_{A,B,l}$ .

We shall prove  $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{H} \rangle$ . Let  $Y \in \mathcal{H}$ . Then  $\{y \in (Y \wedge \sigma) : y \cap \bigcup \sigma \neq \emptyset\} = \{y \in (Y \wedge \tau) : y \cap \bigcup \tau \neq \emptyset\} \cup \{y \in (Y \wedge \tau) : \exists i < l (k_i \in y)\}$ . Hence  $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{H} \rangle$ .

- (v) Let  $\langle \tau, \mathcal{H} \rangle \in \mathbb{P}(\mathcal{I})$  and  $q \in \mathbb{C}$ . Let  $H = \bigwedge \mathcal{H}$ . Let  $q' \leq q$  and  $n_i \in \omega$  such that  $q' \Vdash n_i \in \dot{x}_i$  for  $i < l$ . Without loss of generality we can assume  $n_i \in \bigcup \tau$ . Since  $p \Vdash \{\dot{X}\} \cup \mathcal{I}$  is dual-independent,  $p \Vdash \neg(H \leq^* \dot{X})$ . So  $p \Vdash \text{“}\exists \langle h_n : n \in \omega \rangle \subset H \left( \forall n \in \omega \exists x \in \dot{X} (h_{2n} \cap x \neq \emptyset \wedge h_{2n+1} \cap x \neq \emptyset) \right)$  or  $\exists H_0 \subset H$  finite  $\left( \left| \{h \in H \setminus H_0 : \exists x \in \dot{X} (x \cap h \neq \emptyset \wedge x \cap \bigcup H_0 \neq \emptyset) \} \right| = \omega \right)$ ”.

Without loss of generality we can assume

$$q' \Vdash \text{“}\exists \langle h_n : n \in \omega \rangle \subset H \left( \forall n \in \omega \exists x \in \dot{X} (h_{2n} \cap x \neq \emptyset \wedge h_{2n+1} \cap x \neq \emptyset) \right)$$
(3)

or

$$q' \Vdash \text{“}\exists \text{finite } H_0 \subset H \left( \left| \{h \in H \setminus H_0 : \exists x \in \dot{X} (x \cap h \neq \emptyset \wedge x \cap \bigcup H_0 \neq \emptyset) \} \right| = \omega \right)$$
(4)

case(3) Let  $r \leq q'$ ,  $\langle h_i : i < 2l \rangle \subset H$  and  $\langle k_i : i < 2l \rangle$  such that  $\bigcup \sigma \cap h_i = \emptyset$ ,  $h_i$  are pairwise disjoint and

$$r \Vdash \forall i < l \exists x \in \dot{X} (k_{2i} \in x \cap h_{2i} \wedge k_{2i+1} \in x \cap h_{2i+1}).$$

Put  $k_{-1} = k_0$ . Then put  $\sigma = \{s' : s' = s \cup \{k_{2i}, k_{2i-1} : n_i \in s\} \text{ for } s \in \tau\}$ .

We shall prove  $\langle \sigma, \mathcal{H} \rangle \in D_{\dot{X},l,q}$ . Let  $\dot{x}$  be a  $\mathbb{C}$ -name such that  $r \Vdash \text{“}\dot{x} \in (\dot{X} \wedge \sigma) \wedge \dot{x}_i \subset \dot{x}$ ” for some  $i < l$ . Since  $r \Vdash n_i \in \dot{x}_i$ ,  $r \Vdash n_i \in \dot{x}$ . Since there exists  $s' \in \sigma$  such that  $\{n_i, k_{2i}, k_{2i-1}\} \subset s'$ ,  $r \Vdash k_{2i} \in \dot{x}$ . Since  $r \Vdash \text{“}\exists x \in \dot{X} (\{k_{2i}, k_{2i+1}\} \subset x)$ ” and there exists  $s' \in \sigma$  such that  $\{k_{2i+1}, k_{2i+2}, n_{i+1}\} \subset s'$ ,  $r \Vdash n_{i+1} \in \dot{x}$ . So  $r \Vdash \bigcup_{i < l} \dot{x}_i \subset \dot{x}$ . Hence  $\langle \sigma, \mathcal{H} \rangle \in D_{\dot{X},l,q}$ .

Finally we shall prove  $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{H} \rangle$ . Let  $Y \in \mathcal{H}$  and  $y_i \in Y$  such that  $k_i \in y_i$  for  $i < 2l$ . Then  $\{y \in (Y \wedge \sigma) : y \cap \bigcup \sigma \neq \emptyset\} = \{y \cup$



$\bigcup\{y_{2i}, y_{2i-1} : \exists i < l (n_i \in y)\} : y \in (Y \wedge \tau) \wedge y \cap \bigcup \tau \neq \emptyset$ . Since  $H \leq Y$ ,  $\{h_i : i < 2l\}$  is pairwise disjoint and  $\bigcup \tau \cap h_i = \emptyset$  for  $i < 2l$ ,  $\{y_i : i < 2l\}$  is pairwise disjoint and  $\bigcup \tau \cap y_i = \emptyset$  for  $i < l$ . So if  $y \neq y' \in (Y \wedge \tau)$  with  $y \cap \bigcup \tau \neq \emptyset \wedge y' \cap \bigcup \tau \neq \emptyset$ , then  $(y \cup \bigcup\{y_{2i}, y_{2i-1} : n_i \in y\}) \cap (y' \cup \bigcup\{y_{2i}, y_{2i-1} : n_i \in y'\}) = \emptyset$ . Hence  $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{H} \rangle$ .

case(4) Let  $G$  be  $\mathbb{C}$ -generic over  $V$  with  $q' \in G$ . We will work in  $V[G]$ . Let  $H_0$  be a finite subset of  $H$  such that the set

$$\{h \in H \setminus H_0 : \exists x \in \dot{X}[G] : h \cap x \neq \emptyset \wedge x \cap \bigcup H_0 \neq \emptyset\}$$

is infinite where  $\dot{X}[G]$  is the interpretation of  $\dot{X}$  in  $V[G]$ . Since  $H_0$  is finite, there exists  $h' \in H_0$  such that the set

$$\{h \in H \setminus \{h'\} : \exists x \in \dot{X}[G] (h \cap x \neq \emptyset \wedge x \cap h' \neq \emptyset)\}$$

is infinite.

Let  $\langle h_j : j \in \omega \rangle$  be an enumeration of the set

$$\{h \in H \setminus \{h'\} : \exists x \in \dot{X}[G] (h \cap x \neq \emptyset \wedge x \cap h' \neq \emptyset \wedge h \cap \bigcup \tau = \emptyset)\}$$

and  $\langle k_j : j \in \omega \rangle$  be natural numbers such that

$$\exists x \in \dot{X}[G] (k_{2j} \in x \cap h_j \wedge k_{2j+1} \in x \cap h').$$

Let  $\{Y_i : i < m\}$  be an enumeration of  $\mathcal{H}$ . By induction we shall construct decreasing sequence  $\{A_j : j < m\}$  of infinite sets of natural numbers. Put  $A_{-1} = \{k_{2i+1} : i \in \omega\} \setminus \bigcup \tau$ .

Suppose we already have  $A_j$ . Let  $A_j \upharpoonright Y_{j+1} = \{A_j \cap y : y \in Y_{j+1}\} \setminus \{\emptyset\}$ . If  $A_j \upharpoonright Y_{j+1}$  is infinite, put

$$A_{j+1} = \bigcup \{A_j \cap y : y \cap \bigcup \tau = \emptyset \wedge y \in Y_{j+1}\}.$$

If  $A_j \upharpoonright Y_{j+1}$  is finite, then choose  $y \in Y_{j+1}$  so that  $A_j \cap y$  is infinite and put

$$A_{j+1} = y \cap A_j.$$

In both cases  $A_{j+1}$  is infinite. Choose  $j_i$  for  $i < l$  so that  $k_{2j_i+1} \in A_{m-1}$  for  $i < l$ . Then define  $\sigma = \{s' : s' = s \cup \{k_{2j_i} : n_i \in s\} \text{ for } s \in \tau\} \cup \{\{k_{2j_i+1} : i < l\}\}$ .

From now on we will work in  $V$  and prove  $\langle \sigma, \mathcal{H} \rangle \in D_{\dot{X}, q, l}$ . Let  $r \leq q'$  such that

$$r \Vdash \forall i < l \exists x \in \dot{X} (k_{2j_i} \in x \cap h_{j_i} \wedge k_{2j_i+1} \in x \cap h').$$

Suppose  $r \Vdash \text{“}\dot{x} \in (X \wedge \sigma) \wedge \dot{x}_i \subset \dot{x}\text{”}$  for some  $i < l$  and a  $\mathbb{C}$ -name  $\dot{x}$ . Since  $r \Vdash \dot{x}_i \subset \dot{x}$ ,  $r \Vdash n_i \in \dot{x}$ . Since there exists  $s' \in \sigma$  such that  $\{k_{2j_i}, n_i\} \subset s'$ ,  $r \Vdash \{k_{2j_i}, n_i\} \subset \dot{x}$ . Since  $r \Vdash \exists x \in \dot{X} (k_{2j_i} \in x \cap h_{j_i} \wedge k_{2j_i+1} \in x \cap h')$ ,  $r \Vdash \{k_{2j_i}, k_{2j_i+1}\} \subset \dot{x}$ . Since  $\{k_{2j_i+1} : i < l\} \in \sigma$ ,  $r \Vdash k_{2j_i+1} \in \dot{x}$ . By similar argument, we have  $r \Vdash \dot{x}_{i+1} \subset \dot{x}$ . Therefore  $r \Vdash \bigcup_{i < l} \dot{x}_i \subset \dot{x}$ . Hence  $\langle \sigma, \mathcal{H} \rangle \in D_{\dot{X}, q, l}$ .

Finally we shall prove  $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{H} \rangle$ . Let  $Y \in \mathcal{H}$ . By construction of  $\{A_j : j < m\}$ , there is  $y \in Y$  such that  $\{k_{2j_i+1} : i < l\} \subset y$  or for  $i < l$  and  $y \in Y$  if  $k_{2j_i+1} \in y$ , then  $y \cap \bigcup \tau = \emptyset$ .

case 1. There is  $y \in Y$  such that  $\{k_{2j_i+1} : i < l\} \subset y$ .

For each  $y \in Y$  let  $y_\tau \in (Y \wedge \tau)$  such that  $y \subset y_\tau$ . Let  $y' \in Y$  such that  $\{k_{2j_i+1} : i < l\} \subset y'$ . Then  $\{y \in (Y \wedge \sigma) : y \cap \bigcup \sigma \neq \emptyset\} = \{y'_\tau\} \cup \{y_\tau \cup \bigcup \{y^* \in Y : \exists i < l (k_{2j_i} \in y^* \wedge n_i \in y_\tau)\} : y \cap \bigcup \tau \neq \emptyset \wedge y \in Y\}$ .

Suppose  $y'_\tau \neq y_\tau$  for some  $y \in Y$  with  $y \cap \bigcup \tau \neq \emptyset$ . Since  $H \leq Y$ ,  $\{h_{j_i} : i < l\} \cup \{h'\}$  is pairwise disjoint,  $y' \subset h'$ ,  $k_{2j_i} \in h_{j_i}$  and  $\bigcup \sigma \cap h_i = \emptyset$ ,  $y'_\sigma \cap y_\sigma = y'_\tau \cap (y_\tau \cup \{y^* \in Y : \exists i < l (k_{2j_i} \in y^* \wedge n_i \in y_\tau)\}) = \emptyset$ .

Let  $y_\tau^0 \neq y_\tau^1$  such that  $y_\tau^0 \neq y'_\tau$ ,  $y_\tau^1 \neq y'_\tau$ ,  $y^0 \cap \bigcup \tau \neq \emptyset$  and  $y^1 \cap \bigcup \tau \neq \emptyset$ . Since  $H \leq Y$ ,  $\{h_{j_i} : i < l\}$  is pairwise disjoint,  $y' \subset h'$ ,  $k_{2j_i} \in h_{j_i}$  and  $\bigcup \sigma \cap h_i = \emptyset$ ,  $y_\sigma^0 \cap y_\sigma^1 = (y_\tau^0 \cup \bigcup \{y^* \in Y : \exists i < l (k_{2j_i} \in y^* \wedge n_i \in y_\tau^0)\}) \cap (y_\tau^1 \cup \bigcup \{y^* \in Y : \exists i < l (k_{2j_i} \in y^* \wedge n_i \in y_\tau^1)\}) = \emptyset$ . Hence  $\forall y^0, y^1 \in Y$

$$\left( y_\tau^0 \cap y_\tau^1 = \emptyset \wedge \bigcup \tau \cap y^0 \neq \emptyset \wedge \bigcup \tau \cap y^1 \neq \emptyset \rightarrow y_\sigma^0 \cap y_\sigma^1 = \emptyset \right).$$

case 2. for  $i < l$  and  $y \in Y$  if  $k_{2j_i+1} \in y$ .

If  $\forall i < l \forall y \in Y (k_{2j_i} \in y \rightarrow y \cap \bigcup \tau = \emptyset)$ ,  $\{y \in (Y \wedge \sigma) : y \cap \bigcup \sigma \neq \emptyset\} = \{\bigcup \{y \in Y : \exists i < l (k_{2j_i+1} \in y)\}\} \cup \{y_\tau \cup \bigcup \{y^* \in Y : \exists i < l (k_{2j_i} \in y^* \wedge n_i \in y_\tau)\} : y \cap \bigcup \tau \neq \emptyset \wedge y \in Y\}$ . Since  $k_{2j_i+1} \in y$  implies  $y \cap \bigcup \tau = \emptyset$ ,  $\bigcup \{y \in Y : \exists i < l (k_{2j_i+1} \in y)\} \cap \bigcup \tau = \emptyset$ .

Let  $y_\tau^0 \neq y_\tau^1$  with  $y^0 \cap \bigcup \tau \neq \emptyset$  and  $y^1 \cap \bigcup \tau \neq \emptyset$ . Since  $H \leq Y$  and  $\{h_{j_i} : i < l\}$  is pairwise disjoint,  $(y_\tau^0 \cup \bigcup \{y^* \in Y : \exists i < l (k_{2j_i} \in$

$y^* \wedge n_i \in y_\tau^0\}) \cap (y_\tau^1 \cup \bigcup\{y^* \in Y : \exists i < l(k_{2j_i} \in y^* \wedge n_i \in y_\tau^1)\}) = \emptyset$ .  
Hence  $\forall y^0, y^1 \in Y$

$$\left( y_\tau^0 \cap y_\tau^1 = \emptyset \wedge \bigcup \tau \cap y^0 \neq \emptyset \wedge \bigcup \tau \cap y^1 \neq \emptyset \rightarrow y_\sigma^0 \cap y_\sigma^1 = \emptyset \right).$$

Therefore  $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{H} \rangle$ .

Claim ■

Let  $\mathcal{D} = \{D_n : n \in \omega\} \cup \{D_{\mathcal{A}}^l : \mathcal{A} \text{ is a finite subset of } \mathcal{I} \wedge l \in \omega\} \cup \{D_{\mathcal{A},l} : \mathcal{A} \text{ is a finite subset of } \mathcal{I} \wedge l \in \omega\} \cup \{D_{\mathcal{A},B,l} : \mathcal{A} \text{ is a finite subset of } \mathcal{I} \wedge B \in \mathcal{I} \setminus \mathcal{A} \wedge l \in \omega\} \cup \{D_{\dot{X},l,q} : q \leq p \wedge l \in \omega\}$  and  $G$  is  $\mathcal{D}$ -generic for  $\mathbb{P}(\mathcal{I})$ .

Let  $X_G$  be a partition generated by  $\equiv_G$  where  $\equiv_G$  is defined by

$$n \equiv_G m \text{ if } \exists \langle \sigma, \mathcal{H} \rangle \exists x \in \sigma (\{n, m\} \subset x).$$

Then by (i) and (ii)  $X_G \in (\omega)^\omega$ . By (ii)  $X_G \wedge \bigwedge \mathcal{A} \in (\omega)^\omega$  for finite  $\mathcal{A} \subset \mathcal{I}$ . By (iii)  $\neg(\bigwedge \mathcal{A} \leq^* X_G)$  for finite  $\mathcal{A} \subset \mathcal{I}$ . By (iv)  $\neg(X_G \wedge \bigwedge \mathcal{A} \leq^* Y)$  for finite  $\mathcal{A} \subset \mathcal{I}$  and  $Y \in \mathcal{I} \setminus \mathcal{A}$ . Therefore  $\{X_G\} \cup \mathcal{I}$  is dual-independent by Corollary 2.8. By (v)  $p \Vdash \dot{X} \perp X_G$ . Hence  $X_G$  is a required partition.

□

## Acknowledgment

I would like to thank Jörg Brendle for helpful suggestions and discussions during this work.

## References

- [1] Andreas Blass, “Combinatorial cardinal characteristics of the continuum”, in Handbook of Set Theory (A.Kanamori et al.,eds.),to appear.
- [2] J.Brendle, “Martin’s axiom and the dual distributivity number”, MLQ Math. Log. Q. 46 (2000), no. 2, 241–248.
- [3] J.Cichoń, A.Krawczyk, B.Majcher-Iwanow, B.Węglorz, “Dualization of the van Douwen diagram”. J. Symbolic Logic 65 (2000), no. 2, 959–968.

- [4] L.Halbeisen, “On shattering, splitting and reaping partitions”. *Math. Logic Quart.* 44 (1998), no. 1, 123–134.
- [5] Kenneth Kunen, “Set Theory”, *Studies in Logic and the Foundations of Mathematics*, 102. North-Holland Publishing Co., Amsterdam, 1983.