Independence number for partitions of ω

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Abstract

In this paper we will define a cardinal invariant corresponding to the independence number for partitions of ω. By using Cohen forcing we will prove that this cardinal invariant is consistently smaller than the continuum.

1 Introduction

The structure $([\omega]^\omega, \subset^*)$ of the set of all infinite subsets of ω ordered by "almost inclusion" is well studied in set theory. To describe much of the combinatorial structure of $([\omega]^\omega, \subset^*)$ cardinal invariants of the continuum are introduced like, for example, the reaping number $\mathfrak{r}$ or the independence number $\mathfrak{i}$.

In recent years partial orders similar to $([\omega]^\omega, \subset^*)$ have been focused on and analogous cardinal invariants have been defined and investigated. For example $((\omega)^\omega, \leq^*)$, the set of all infinite partitions of ω ordered by "almost coarser", and the cardinal invariants $\mathfrak{p}_d$, $\mathfrak{t}_d$, $\mathfrak{s}_d$, $\mathfrak{r}_d$, $\mathfrak{a}_d$ and $\mathfrak{h}_d$ have been defined and investigated in [2], [3] and [4].

In this work we will define the dual-independence number $\mathfrak{i}_d$ analogous to the independence number $\mathfrak{i}$ and get a consistency result.

Once we define dual-independence number $\mathfrak{i}_d$, we can prove the following proposition similar to the proof of $\mathfrak{r} \leq \mathfrak{i}$.

Proposition 1.1 (Brendle). $\mathfrak{r}_d \leq \mathfrak{i}_d$.

And $\mathfrak{r}_d$ has the following property.

Theorem 1.2. *[3] MA implies* $\mathfrak{r}_d = \mathfrak{c}$.

So it is consistent that $\mathfrak{i}_d = \mathfrak{c}$. And it is natural to ask the following question.

Question 1.3. *Is it consistent that* $\mathfrak{i}_d < \mathfrak{c}$*?*

In section 2 we will define the dual-independence number and study its properties. In section 3 we will prove that $\mathfrak{i}_d < \mathfrak{c}$ is consistent by using Cohen forcing.

2 $(\omega)^\omega$ and dual-independent family

We start with the definition of "partition of ω".

Definition 2.1. *X is a partition of ω if X is a subset of $\wp(\omega)$, $\bigcup X = \omega$ and for each $a, b \in X$ if $a \neq b$, then $a \cap b = \emptyset$. By (ω) we denote all partitions of ω. Also by $(\omega)^\omega$ we denote all infinite partitions of ω and by $(\omega)^{<\omega}$ we denote all finite partitions of ω.*

For partitions of ω we give the ordering "coarser".

Definition 2.2. *For $X, Y \in (\omega)$ X is coarser than Y (Y is finer than X) if for each $x \in X$ there exists a subset Y' of Y such that $x = \bigcup Y'$.*

For $X, Y \in (\omega)^\omega$ X is almost coarser than Y (Y is almost finer than Y) if for all but finitely many $x \in X$ there exists $Y' \subset Y$ such that $x = \bigcup Y'$.

We can easily check that $((\omega), \leq)$ is a lattice. For each X, $Y \in (\omega)$ by $X \wedge Y$ we denote the infimum of X and Y. For $X, Y \in (\omega)^\omega$ by $X \perp Y$ we mean that $X \wedge Y \in (\omega)^{<\omega}$.

As $([\omega]^\omega, \subset^*)$, $((\omega)^\omega, \leq^*)$ has the following properties:

Lemma 2.3. *[3] Suppose that $X_0 \geq X_1 \geq X_2 \geq \dots$ is a decreasing sequence of $(\omega)^\omega$. Then there exists $Y \in (\omega)^\omega$ such that $Y \leq^* X_n$ for $n \in \omega$.*

Lemma 2.4. *[3] For $X, Y \in (\omega)^\omega$ if $\neg(X \leq^* Y)$, then there exists $Z \in (\omega)^\omega$ such that $Z \leq^* X$ and $Z \perp Y$.*

So $((\omega)^\omega, \leq^*)$ is similar to $([\omega]^\omega, \subset^*)$. On the other hand there is a serious difference: $([\omega]^\omega, \subset^*)$ is a Boolean algebra but $((\omega)^\omega, \leq^*)$ is just a lattice and not a Boolean algebra.

In general when we define independence, we use complementation. But $((\omega)^\omega, \leq^*)$ doesn't have any natural complementation. So we will define independence for $((\omega)^\omega, \leq^*)$ without mentioning complementation.

Definition 2.5. *Let $\mathcal{I}$ be a subset of $(\omega)^\omega$. $\mathcal{I}$ is dual-independent if for all $\mathcal{A}$ and $\mathcal{B}$ finite subsets of $\mathcal{I}$ with $\mathcal{A} \cap \mathcal{B} = \emptyset$ there exists $C \in (\omega)^\omega$ such that*

(i) $C \leq^* A$ for $A \in \mathcal{A}$ and

(ii) $C \perp B$ for $B \in \mathcal{B}$.

Then define dual-independence number $\mathfrak{i}_d$ by

$$\mathfrak{i}_d = \min\{|\mathcal{I}| : \mathcal{I} \text{ is a maximal dual-independent family}\}.$$

Since there is no natural complementation for an element of $((\omega)^\omega, \leq^*)$, it becomes more difficult to handle dual-independent families than to handle independent families for a Boolean algebra. But the following lemmata helps to handle dual-independent families.

Lemma 2.6. *[3] If $X, Y \in (\omega)^\omega$ and $\neg(X \leq^* Y)$, then there exists an infinite sequence $\{a_n\}_{n\in\omega}$ of different elements of X such that*

$$\forall n \in \omega \exists y \in Y\, (y \cap a_{2n} \neq \emptyset \wedge y \cap a_{2n+1} \neq \emptyset)$$

or there exists a finite subset A of X such that the set

$$\{x \in X \setminus A : \exists y \in Y (x \cap y \neq \emptyset \wedge \bigcup A \cap y \neq \emptyset)\}$$

is infinite.

Proof. Suppose that we have defined a sequence $\{a_n\}_{n<2k}$ but for any two $a, b \in X \setminus \{a_0, \ldots, a_{2k-1}\}$ and $y \in Y$ we have $a \cap y = \emptyset$ or $b \cap y = \emptyset$. Let A denote the finite family $\{a_0, \ldots, a_{2k-1}\}$ and let

$$\mathcal{F} = \{x \in X \setminus A : \exists y \in Y \left(x \cap y \neq \emptyset \wedge \bigcup A \cap y \neq \emptyset\right)\}.$$

If $\mathcal{F}$ is finite, then the partition

$$X_* = \{\bigcup A \cup \bigcup \mathcal{F}\} \cup (X \setminus A \cup \mathcal{F})$$

is a finite modification of X which is coarser than Y. It is a contradiction to $\neg(X \leq^* Y)$.

□

By this lemma we can prove the following useful lemma.

Lemma 2.7. *If $X \in (\omega)^\omega$ and $\mathcal{B}$ is a finite subset of $(\omega)^\omega$ such that $\neg(X \leq^* B)$ for $B \in \mathcal{B}$, then there exists $Z \leq X$ such that $Z \perp B$ for $B \in \mathcal{B}$.*

Proof. Let $\mathcal{B} = \{B_i : i < n\}$. By the above lemma for each $i < n$ there exists an infinite sequence $\{a^i_k\}_{k\in\omega}$ of different elements of X such that

$$\forall k \in \omega \exists b \in B_i (b \cap a^i_{2k} \neq \emptyset \wedge b \cap a^i_{2k+1} \neq \emptyset)$$

or there exists a finite subset A_i of X and an infinite sequence $\{a^i_k\}_{k\in\omega}$ of different elements of $X \setminus A_i$ such that

$$\forall k \in \omega \exists b \in B_i (b \cap a^i_k \neq \emptyset \wedge \bigcup A_i \cap b \neq \emptyset).$$

In the first case we define $A_i = \emptyset$.

Recursively we shall construct a subsequence $\{b^i_k\}_{k\in\omega}$ of $\{a^i_k\}_{k\in\omega}$ for $i < n$. Given $\{b^i_l\}_{l<2k}$ for $i < n$ and b^i_{2k}, b^i_{2k+1} for $i < j$ for some $j < n$.

$\underline{A_j = \emptyset}$ Choose $k_0 \in \omega$ such that

$$\{a^j_{2k_0}, a^j_{2k_0+1}\} \cap \left(\bigcup_{i<n} A_i \cup \{b^i_l : i < n \wedge l < 2k\} \cup \{b^i_{2k}, b^i_{2k+1} : i < j\} \right) = \emptyset.$$

Put $b^j_{2k} = a^j_{2k_0}$ and $b^j_{2k+1} = a^j_{2k_0+1}$.

$\underline{A_j \neq \emptyset}$ Choose $k_0 < k_1 \in \omega$ such that

$$\{a^j_{k_0}, a^j_{k_1}\} \cap \left(\bigcup_{i<n} A_i \cup \{b^i_l : i < n \wedge l < 2k\} \cup \{b^i_{2k}, b^i_{2k+1} : i < j\} \right) = \emptyset.$$

Put $b^j_{2k} = a^j_{k_0}$ and $b^j_{2k+1} = a^j_{k_1}$.

Define $Z = \{\bigcup_{i<n} b^i_{2k} : k \in \omega\} \cup \{\omega \setminus \bigcup_{k\in\omega} \bigcup_{i<n} b^i_{2k}\}$. Then $Z \leq X$ and for each $z \in Z$ and $i < n$ there exists $b \in B_i$ such that

$$b \cap z \neq \emptyset \wedge (\omega \setminus \bigcup_{k\in\omega} \bigcup_{i<n} b^i_{2k}) \cap b \neq \emptyset.$$

Hence $Z \perp B_i$ for $i < n$.

□

So it becomes easier to check dual-independence.

Corollary 2.8. *$\mathcal{I}$ is dual-independent if and only if for each finite subset $\mathcal{A}$ of $\mathcal{I}$ and $B \in \mathcal{I} \setminus \mathcal{A}$*

$$\bigwedge \mathcal{A} \not\leq^* B.$$

3 Cohen forcing and dual-independence number

By using Cohen forcing we will prove it is consistent that $\mathfrak{i}_d < \mathfrak{c}$.

Theorem 3.1. *Suppose $V \models CH$. Then $V^{\mathbb{C}(\omega_2)} \models \mathfrak{i}_d = \omega_1$.*

To prove Theorem 3.1 we use the following lemma.

Lemma 3.2. *Assume $p \in \mathbb{C}$, $\mathcal{I}$ is a countable dual-independent family and $\dot{X}$ is a $\mathbb{C}$-name such that $p \Vdash$ "$\dot{X}$ is a non-trivial infinite partition of ω and $\{\dot{X}\} \cup \mathcal{I}$ is dual-independent". Then there exists $X^* \in (\omega)^\omega \cap V$ such that $\{X^*\} \cup \mathcal{I}$ is dual-independent and $p \Vdash \dot{X} \perp X^*$.*

Proof of 3.1 from 3.2 Within the ground model we shall define a maximal dual-independent family $\mathcal{I}$ of size ω_1. It suffices to verify maximality of $\mathcal{I}$ in the extension via $\mathbb{C}$ (see [5] pp256).

By CH, let $\langle p_\xi, \tau_\xi \rangle$ $\xi < \omega_1$ enumerate all pairs $\langle p, \tau \rangle$ such that $p \in \mathbb{C}$ and τ is a nice name for an infinite partition of ω. By recursion, pick an infinite partition of ω as follows. Given $\{X_\eta : \eta < \xi\}$ for some $\xi < \omega_1$. Choose X_ξ so that

(1) $\{X_\xi\} \cup \{X_\eta : \eta < \xi\}$ is dual-independent.

(2) If $p_\xi \Vdash$ "$\{\tau_\xi\} \cup \{X_\eta : \eta < \xi\}$ is dual-independent", then $p_\xi \Vdash X_\xi \perp \tau_\xi$.

(2) is possible by Lemma 3.2. Let $\mathcal{I} = \{X_\eta : \eta < \omega_1\}$. We shall prove $\mathcal{I}$ is maximal. If $\mathcal{I}$ is not maximal in $V[G]$ for some $\mathbb{C}$-generic G, then there exists $p_\xi \in G$ and τ_ξ such that $p_\xi \Vdash \{\tau_\xi\} \cup \mathcal{I}$ is dual-independent. By construction there exists $X_\xi \in \mathcal{I}$ and $p_\xi \Vdash \tau_\xi \perp X_\xi$. It is a contradiction.

□

Proof of 3.2. Let $\mathbb{P}(\mathcal{I})$ be a partial order such that $\langle\sigma,\mathcal{H}\rangle \in \mathbb{P}(\mathcal{I})$ if σ is a partition of a finite subset of ω and $\mathcal{H}$ is a finite subset of $\mathcal{I}$. It is ordered by $\langle\sigma,\mathcal{H}\rangle \leq \langle\tau,\mathcal{G}\rangle$ if

(i) $\forall x \in \tau \exists x' \in \sigma (x \subset x')$,

(ii) $\mathcal{H} \supset \mathcal{G}$,

(iii) $\forall x_0 \neq x_1 \in \tau \forall x_0' \in \sigma\, (x_0 \subset x_0' \to x_1 \cap x_0' = \emptyset)$,

(iv) $\forall Y \in \mathcal{G} \forall y_0, y_1 \in (Y \wedge \tau) \forall y_0', y_1' \in (Y \wedge \sigma)$

$$\left(y_0 \cap y_1 = \emptyset \wedge \bigcup \tau \cap y_0 \neq \emptyset \wedge \bigcup \tau \cap y_1 \neq \emptyset \wedge y_0 \subset y_0' \wedge y_1 \subset y_1' \to y_0' \cap y_1' = \emptyset\right).$$

Claim 3.2.1. *The following sets are dense.*

(i) $D_n = \{\langle\sigma,\mathcal{H}\rangle : n \in \bigcup \sigma\}$ *for* $n \in \omega$.

(ii) $D^l_{\mathcal{A}} = \{\langle\sigma,\mathcal{H}\rangle : \mathcal{A} \subset \mathcal{H} \wedge |\{h \in (\bigwedge \mathcal{H} \wedge \sigma) : h \cap \bigcup \sigma \neq \emptyset\}| \geq l\}$ *for finite subsets* $\mathcal{A}$ *of* $\mathcal{I}$ *and* $l \in \omega$.

(iii) $D_{\mathcal{A},l} = \{\langle\sigma,\mathcal{H}\rangle : \mathcal{A} \subset \mathcal{H} \wedge \exists x \in \sigma\, (|\{h \in \bigwedge \mathcal{H} : x \cap h \neq \emptyset\}| \geq l)\}$ *for finite subsets* $\mathcal{A}$ *of* $\mathcal{I}$ *and* $l \in \omega$.

(iv) Let $\mathcal{A}$ *be a finite subset of* $\mathcal{I}$, $B \in \mathcal{I} \setminus \mathcal{A}$ *and* $A = \bigwedge \mathcal{A}$. *Since* $\neg(A \leq^* B)$ *and by Lemma 2.6, there exists* $\{a_n\}_{n\in\omega}$ *such that*

$$\forall n \in \omega \exists b \in B\, (a_{2n} \cap b \neq \emptyset \wedge a_{2n+1} \cap b \neq \emptyset) \tag{1}$$

or there exists a finite subset A_0 *of* A *such that the set*

$$\mathcal{F}_{A_0} = \{a \in A \setminus A_0 : \exists y \in Y \left(y \cap a \neq \emptyset \wedge y \cap \bigcup A_0 \neq \emptyset\right)\} \tag{2}$$

is infinite. If (1) holds, fix $\{a_n\}_{n\in\omega}$. *If (2) holds, fix* A_0 *and* $\mathcal{F}_{A_0}$

(1) Let $D_{\mathcal{A},B,l} = \{\langle\sigma,\mathcal{H}\rangle : \exists \{a^i : i < 2l\} \subset (A \wedge \sigma)\, (\forall i < 2l (\bigcup \sigma \cap a^i \neq \emptyset) \wedge$ $\wedge \{a^i : i < 2l\}$ *is pairwise disjoint* $\wedge \forall i < l \exists b \in B (a^{2i} \cap b \neq \emptyset \wedge a^{2i+1} \cap b \neq \emptyset))\}$.

(2) Let $D_{\mathcal{A},B,l} = \{\langle\sigma,\mathcal{H}\rangle : \exists \{a^i : i < l\} \subset (A \wedge \sigma)\, (\forall i < l (\bigcup \sigma \cap a^i \neq \emptyset) \wedge$ $\{a^i : i < l\}$ *is pairwise disjoint* $\wedge \forall i < l (\bigcup A_0 \cap a^i = \emptyset) \wedge$ $\forall a \in A_0 (a \cap \bigcup \sigma \neq \emptyset) \wedge \forall i < l \exists b \in B (b \cap a^i \neq \emptyset \wedge b \cap \bigcup A_0 \neq \emptyset))\}$.

(v) *Let* $\{\dot{x}_i : i \in \omega\}$ *be* $\mathbb{C}$*-names such that* $\Vdash \dot{X} = \{\dot{x}_i : i \in \omega\}$ *and* $\min \dot{x}_i < \min \dot{x}_{i+1}$*. Put* $D_{\dot{X},l,q} = \{\langle \sigma, \mathcal{H}\rangle : \exists r \leq q \left(r \Vdash \exists x \in (\dot{X} \wedge \sigma)(\bigcup_{i<l} \dot{x}_i \subset x)\right)\}$ *for* $q \leq p$ *and* $l \in \omega$*.*

Proof of Claim.

(i) Clear.

(ii) Let $\langle \tau, \mathcal{H}\rangle \in \mathbb{P}(\mathcal{I})$. Without loss of generality, we can assume $\mathcal{A} \subset \mathcal{H}$. Let $H = \wedge\mathcal{H}$. Choose $h_i \in H$ for $i < l$ such that $h_i \cap \bigcup \tau = \emptyset$. Choose $n_i \in h_i$. Put $\sigma = \tau \cup \{\{n_i\} : i < l\}$. Then $\{h_i : i < l\} \subset \{h \in (H \wedge \sigma) : h \cap \bigcup \sigma \neq \emptyset\}$. So $\langle \sigma, \mathcal{H}\rangle \in D^l_{\mathcal{A}}$.

We shall prove $\langle \sigma, \mathcal{H}\rangle \leq \langle \tau, \mathcal{H}\rangle$. Let $Y \in \mathcal{H}$. Since $h_i \cap \bigcup \tau = \emptyset$ and $n_i \in h_i$ for $i < l$, $\{y \in (Y \wedge \sigma) : y \cap \bigcup \sigma \neq \emptyset\} = \{y \in (Y \wedge \tau) : y \cap \bigcup \sigma \neq \emptyset\} \dot{\cup} \{y \in Y : \exists i < l (n_i \in y)\}$. Hence $\langle \sigma, \mathcal{H}\rangle \leq \langle \tau, \mathcal{H}\rangle$.

(iii) Let $\langle \tau, \mathcal{H}\rangle \in \mathbb{P}(\mathcal{I})$. Without loss of generality, we can assume $\mathcal{A} \subset \mathcal{H}$. Let $H = \bigwedge \mathcal{H}$. Choose $\{h_i : i < l\}$ distinct elements of H such that $h_i \cap \bigcup \tau = \emptyset$ for $i < l$. Choose $n_i \in h_i$ for $i < l$. Put $\sigma = \tau \cup \{\{n_i : i < l\}\}$. Then $\{h \in H : \{n_i : i < l\} \cap h \neq \emptyset\} = \{h_i : i < l\}$. So $\langle \sigma, \mathcal{H}\rangle \in D_{\mathcal{A},l}$.

We shall prove $\langle \sigma, \mathcal{H}\rangle \leq \langle \tau, \mathcal{H}\rangle$.

Since $h_i \cap \bigcup \tau = \emptyset$ and $n_i \in h_i$ for $i < l$, $\{y \in (Y \wedge \sigma) : y \cap \bigcup \sigma \neq \emptyset\} = \{y \in (Y \wedge \tau) : y \cap \bigcup \tau \neq \emptyset\} \dot{\cup} \{\bigcup \{y \in Y : \exists i < l (n_i \in y)\}\}$. Hence $\langle \sigma, \mathcal{H}\rangle \leq \langle \tau, \mathcal{H}\rangle$.

(iv) (1) Let $\langle \tau, \mathcal{H}\rangle \in \mathbb{P}(\mathcal{I})$. Choose distinct $i_j \in \omega$ for $j \leq l$ so that $\bigcup \tau \cap a_{2i_j} = \emptyset$ and $\bigcup \tau \cap a_{2i_j+1} = \emptyset$ for $j < l$. Let $k_n = \min a_n$ for $n \in \omega$. Put $\sigma = \tau \cup \{\{k_{2i_j}\}, \{k_{2i_j+1}\} : j < l\}$. Since $\bigcup \tau \cap a_{2i_j} = \bigcup \tau \cap a_{2i_j+1} = \emptyset$ and $k_n \in a_n$, $\{a_{2i_j}, a_{2i_j+1} : j < l\} \subset (A \wedge \sigma)$, $\{a_{2i_j}, a_{2i_j+1} : j < l\}$ is pairwise distinct and for $i < l$ there exists $b \in B$ such that $b \cap a_{2i_j} \neq \emptyset$ and $b \cap a_{2i_j+1} \neq \emptyset$. So $\langle \sigma, \mathcal{H}\rangle \in D_{\mathcal{A},B,l}$.

We shall prove $\langle \sigma, \mathcal{H}\rangle \leq \langle \tau, \mathcal{H}\rangle$ Let $Y \in \mathcal{H}$. Since $\bigcup \tau \cap a_{2i_j} = \bigcup \tau \cap a_{2i_j+1} = \emptyset$, $\{y \in (Y \wedge \sigma) : y \cap \bigcup \sigma \neq \emptyset\} = \{y \in (Y \wedge \tau) : y \cap \bigcup \sigma \neq \emptyset\} \cup \{y \in (Y \wedge \tau) : \exists j < l (k_{2i_j} \in y \vee k_{2i_j+1} \in y)\}$. Hence $\langle \sigma, \mathcal{H}\rangle \leq \langle \tau, \mathcal{H}\rangle$.

(2) Let $\langle \tau, \mathcal{H}\rangle \in \mathbb{P}(\mathcal{I})$. Without loss of generality we can assume $\bigcup \tau \cap a \neq \emptyset$ for $a \in A_0$. Choose distinct a^i for $i < l$ so that $a^i \cap \bigcup \tau = \emptyset$

and $a^i \in \mathcal{F}_{A_0}$. Let $k_i = \min a^i$ and $\sigma = \tau \cup \{\{k_i\} : i < l\}$. Since $\bigcup \tau \cap a^i = \emptyset$, $a^i \in \mathcal{F}_{A_0}$ and $k_i \in a^i$, $\{a^i : j < l\} \subset (A \wedge \sigma)$, $\{a^i : i < l\}$ is pairwise distinct, $\bigcup A_0 \cap a^i = \emptyset$ and for each $i < l$ there exists $b \in B$ such that $b \cap a^i \neq \emptyset$ and $b \cap \bigcup A_0 \neq \emptyset$. So $\langle \sigma, \mathcal{H} \rangle \in D_{\mathcal{A},B,l}$.

We shall prove $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{H} \rangle$. Let $Y \in \mathcal{H}$. Then $\{y \in (Y \wedge \sigma) : y \cap \bigcup \sigma \neq \emptyset\} = \{y \in (Y \wedge \tau) : y \cap \bigcup \tau \neq \emptyset\} \cup \{y \in (Y \wedge \tau) : \exists i < l (k_i \in y)\}$. Hence $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{H} \rangle$.

(v) Let $\langle \tau, \mathcal{H} \rangle \in \mathbb{P}(\mathcal{I})$ and $q \in \mathbb{C}$. Let $H = \bigwedge \mathcal{H}$. Let $q' \leq q$ and $n_i \in \omega$ such that $q' \Vdash n_i \in \dot{x}_i$ for $i < l$. Without loss of generality we can assume $n_i \in \bigcup \tau$. Since $p \Vdash \{\dot{X}\} \cup \mathcal{I}$ is dual-independent, $p \Vdash \neg(H \leq^* \dot{X})$. So $p \Vdash$ "$\exists \langle h_n : n \in \omega \rangle \subset H \left(\forall n \in \omega \exists x \in \dot{X} (h_{2n} \cap x \neq \emptyset \wedge h_{2n+1} \cap x \neq \emptyset) \right)$ or $\exists H_0 \subset H$ finite $\left(\left| \{h \in H \setminus H_0 : \exists x \in \dot{X} (x \cap h \neq \emptyset \wedge x \cap \bigcup H_0 \neq \emptyset)\} \right| = \omega \right)$". Without loss of generality we can assume

$$q' \Vdash \text{“}\exists \langle h_n : n \in \omega \rangle \subset H \left(\forall n \in \omega \exists x \in \dot{X} (h_{2n} \cap x \neq \emptyset \wedge h_{2n+1} \cap x \neq \emptyset) \right)\text{”} \tag{3}$$

or

$$q' \Vdash \text{“}\exists \text{finite } H_0 \subset H \left(\left| \{h \in H \setminus H_0 : \exists x \in \dot{X} (x \cap h \neq \emptyset \wedge x \cap \bigcup H_0 \neq \emptyset)\} \right| = \omega \right)\text{”}. \tag{4}$$

case(3) Let $r \leq q'$, $\langle h_i : i < 2l \rangle \subset H$ and $\langle k_i : i < 2l \rangle$ such that $\bigcup \sigma \cap h_i = \emptyset$, h_i are pairwise disjoint and

$$r \Vdash \forall i < l \exists x \in \dot{X} \left(k_{2i} \in x \cap h_{2i} \wedge k_{2i+1} \in x \cap h_{2i+1} \right).$$

Put $k_{-1} = k_0$. Then put $\sigma = \{s' : s' = s \cup \{k_{2i}, k_{2i-1} : n_i \in s\} \text{ for } s \in \tau\}$.

We shall prove $\langle \sigma, \mathcal{H} \rangle \in D_{\dot{X},l,q}$. Let $\dot{x}$ be a $\mathbb{C}$-name such that $r \Vdash$ "$\dot{x} \in (\dot{X} \wedge \sigma) \wedge \dot{x}_i \subset \dot{x}$" for some $i < l$. Since $r \Vdash n_i \in \dot{x}_i$, $r \Vdash n_i \in \dot{x}$. Since there exists $s' \in \sigma$ such that $\{n_i, k_{2i}, k_{2i-1}\} \subset s'$, $r \Vdash k_{2i} \in \dot{x}$. Since $r \Vdash$ "$\exists x \in \dot{X}(\{k_{2i}, k_{2i+1}\} \subset x)$" and there exists $s' \in \sigma$ such that $\{k_{2i+1}, k_{2i+2}, n_{i+1}\} \subset s'$, $r \Vdash n_{i+1} \in \dot{x}$. So $r \Vdash \bigcup_{i<l} \dot{x}_i \subset \dot{x}$. Hence $\langle \sigma, \mathcal{H} \rangle \in D_{\dot{X},l,q}$.

Finally we shall prove $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{H} \rangle$. Let $Y \in \mathcal{H}$ and $y_i \in Y$ such that $k_i \in y_i$ for $i < 2l$. Then $\{y \in (Y \wedge \sigma) : y \cap \bigcup \sigma \neq \emptyset\} = \{y \cup$

$\bigcup\{y_{2i}, y_{2i-1} : \exists i < l(n_i \in y)\} : y \in (Y \wedge \tau) \wedge y \cap \bigcup \tau \neq \emptyset\}$. Since $H \leq Y$, $\{h_i : i < 2l\}$ is pairwise disjoint and $\bigcup \tau \cap h_i = \emptyset$ for $i < 2l$, $\{y_i : i < 2l\}$ is pairwise disjoint and $\bigcup \tau \cap y_i = \emptyset$ for $i < l$. So if $y \neq y' \in (Y \wedge \tau)$ with $y \cap \bigcup \tau \neq \emptyset \wedge y' \cap \bigcup \tau \neq \emptyset$, then $(y \cup \bigcup\{y_{2i}, y_{2i-1} : n_i \in y\}) \cap (y' \cup \bigcup\{y_{2i}, y_{2i-1} : n_i \in y'\}) = \emptyset$. Hence $\langle \sigma, \mathcal{H}\rangle \leq \langle \tau, \mathcal{H}\rangle$.

case(4) Let G be $\mathbb{C}$-generic over V with $q' \in G$. We will work in $V[G]$. Let H_0 be a finite subset of H such that the set

$$\{h \in H \setminus H_0 : \exists x \in \dot{X}[G] : h \cap x \neq \emptyset \wedge x \cap \bigcup H_0 \neq \emptyset\}$$

is infinite where $\dot{X}[G]$ is the interpretation of $\dot{X}$ in $V[G]$. Since H_0 is finite, there exists $h' \in H_0$ such that the set

$$\{h \in H \setminus \{h'\} : \exists x \in \dot{X}[G]\,(h \cap x \neq \emptyset \wedge x \cap h' \neq \emptyset)\}$$

is infinite.

Let $\langle h_j : j \in \omega\rangle$ be an enumeration of the set

$$\{h \in H \setminus \{h'\} : \exists x \in \dot{X}[G]\,\Big(h \cap x \neq \emptyset \wedge x \cap h' \neq \emptyset \wedge h \cap \bigcup \tau = \emptyset\Big)\}$$

and $\langle k_j : j \in \omega\rangle$ be natural numbers such that

$$\exists x \in \dot{X}[G](k_{2j} \in x \cap h_j \wedge k_{2j+1} \in x \cap h').$$

Let $\{Y_i : i < m\}$ be an enumeration of $\mathcal{H}$. By induction we shall construct decreasing sequence $\{A_j : j < m\}$ of infinite sets of natural numbers. Put $A_{-1} = \{k_{2i+1} : i \in \omega\} \setminus \bigcup \tau$.

Suppose we already have A_j. Let $A_j \restriction Y_{j+1} = \{A_j \cap y : y \in Y_{j+1}\} \setminus \{\emptyset\}$. If $A_j \restriction Y_{j+1}$ is infinite, put

$$A_{j+1} = \bigcup\{A_j \cap y : y \cap \bigcup \tau = \emptyset \wedge y \in Y_{j+1}\}.$$

If $A_j \restriction Y_{j+1}$ is finite, then choose $y \in Y_{j+1}$ so that $A_j \cap y$ is infinite and put

$$A_{j+1} = y \cap A_j.$$

In both cases A_{j+1} is infinite. Choose j_i for $i < l$ so that $k_{2j_i+1} \in A_{m-1}$ for $i < l$. Then define $\sigma = \{s' : s' = s \cup \{k_{2j_i} : n_i \in s\}$ for $s \in \tau\} \cup \{\{k_{2j_i+1} : i < l\}\}$.

From now on we will work in V and prove $\langle \sigma, \mathcal{H} \rangle \in D_{\dot{X},q,l}$. Let $r \leq q'$ such that

$$r \Vdash \forall i < l \exists x \in \dot{X} \left(k_{2j_i} \in x \cap h_{j_i} \wedge k_{2j_i+1} \in x \cap h' \right).$$

Suppose $r \Vdash$ "$\dot{x} \in (X \wedge \sigma) \wedge \dot{x}_i \subset \dot{x}$" for some $i < l$ and a $\mathbb{C}$-name $\dot{x}$. Since $r \Vdash \dot{x}_i \subset \dot{x}$, $r \Vdash n_i \in \dot{x}$. Since there exists $s' \in \sigma$ such that $\{k_{2j_i}, n_i\} \subset s'$, $r \Vdash \{k_{2j_i}, n_i\} \subset \dot{x}$. Since $r \Vdash \exists x \in \dot{X}(k_{2j_i} \in x \cap h_{j_i} \wedge k_{2j_i+1} \in x \cap h')$, $r \Vdash \{k_{2j_i}, k_{2j_i+1}\} \subset \dot{x}$. Since $\{k_{2j_i+1} : i < l\} \in \sigma$, $r \Vdash k_{2j_{i+1}+1} \in \dot{x}$. By similar argument, we have $r \Vdash \dot{x}_{i+1} \subset \dot{x}$. Therefore $r \Vdash \bigcup_{i<l} \dot{x}_i \subset \dot{x}$. Hence $\langle \sigma, \mathcal{H} \rangle \in D_{\dot{X},q,l}$.

Finally we shall prove $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{H} \rangle$. Let $Y \in \mathcal{H}$. By construction of $\{A_j : j < m\}$, there is $y \in Y$ such that $\{k_{2j_i+1} : i < l\} \subset y$ or for $i < l$ and $y \in Y$ if $k_{2j_i+1} \in y$, then $y \cap \bigcup \tau = \emptyset$.

<u>case 1. There is $y \in Y$ such that $\{k_{2j_i+1} : i < l\} \subset y$.</u>

For each $y \in Y$ let $y_\tau \in (Y \wedge \tau)$ such that $y \subset y_\tau$. Let $y' \in Y$ such that $\{k_{2j_i+1} : i < l\} \subset y'$. Then $\{y \in (Y \wedge \sigma) : y \cap \bigcup \sigma \neq \emptyset\} = \{y'_\tau\} \cup \{y_\tau \cup \bigcup\{y^* \in Y : \exists i < l \, (k_{2j_i} \in y^* \wedge n_i \in y_\tau)\} : y \cap \bigcup \tau \neq \emptyset \wedge y \in Y\}$.

Suppose $y'_\tau \neq y_\tau$ for some $y \in Y$ with $y \cap \bigcup \tau \neq \emptyset$. Since $H \leq Y$, $\{h_{j_i} : i < l\} \cup \{h'\}$ is pairwise disjoint, $y' \subset h'$, $k_{2j_i} \in h_{j_i}$ and $\bigcup \sigma \cap h_i = \emptyset$, $y'_\sigma \cap y_\sigma = y'_\tau \cap (y_\tau \cup \{y^* \in Y : \exists i < l \, (k_{2j_i} \in y^* \wedge n_i \in y_\tau)\}) = \emptyset$.

Let $y^0_\tau \neq y^1_\tau$ such that $y^0_\tau \neq y'_\tau$, $y^1_\tau \neq y'_\tau$, $y^0 \cap \bigcup \tau \neq \emptyset$ and $y^1 \cap \bigcup \tau \neq \emptyset$. Since $H \leq Y$, $\{h_{j_i} : i < l\}$ is pairwise disjoint, $y' \subset h'$, $k_{2j_i} \in h_{j_i}$ and $\bigcup \sigma \cap h_i = \emptyset$, $y^0_\sigma \cap y^1_\sigma = (y^0_\tau \cup \bigcup\{y^* \in Y : \exists i < l \, (k_{2j_i} \in y^* \wedge n_i \in y^0_\tau)\}) \cap (y^1_\tau \cup \bigcup\{y^* \in Y : \exists i < l \, (k_{2j_i} \in y^* \wedge n_i \in y^1_\tau)\} = \emptyset$. Hence $\forall y^0, y^1 \in Y$

$$\left(y^0_\tau \cap y^1_\tau = \emptyset \wedge \bigcup \tau \cap y^0 \neq \emptyset \wedge \bigcup \tau \cap y^1 \neq \emptyset \rightarrow y^0_\sigma \cap y^1_\sigma = \emptyset \right).$$

<u>case 2. for $i < l$ and $y \in Y$ if $k_{2j_i+1} \in y$.</u>

If $\forall i < l \forall y \in Y \, (k_{2j_i} \in y \rightarrow y \cap \bigcup \tau = \emptyset)$, $\{y \in (Y \wedge \sigma) : y \cap \bigcup \sigma \neq \emptyset\} = \{\bigcup\{y \in Y : \exists i < l(k_{2j_i+1} \in y)\}\} \cup \{y_\tau \cup \bigcup\{y^* \in Y : \exists i < l \, (k_{2j_i} \in y^* \wedge n_i \in y_\tau)\} : y \cap \bigcup \tau \neq \emptyset \wedge y \in Y\}$. Since $k_{2j_i+1} \in y$ implies $y \cap \bigcup \tau = \emptyset$, $\bigcup\{y \in Y : \exists i < l(k_{2j_i+1} \in y)\} \cap \bigcup \tau = \emptyset$.

Let $y^0_\tau \neq y^1_\tau$ with $y^0 \cap \bigcup \tau \neq \emptyset$ and $y^1 \cap \bigcup \tau \neq \emptyset$. Since $H \leq Y$ and $\{h_{j_i} : i < l\}$ is pairwise disjoint, $(y^0_\tau \cup \bigcup\{y^* \in Y : \exists i < l(k_{2j_i} \in$

$y^* \wedge n_i \in y^0_\tau)\}) \cap (y^1_\tau \cup \bigcup\{y^* \in Y : \exists i < l(k_{2j_i} \in y^* \wedge n_i \in y^1_\tau)\}) = \emptyset$. Hence $\forall y^0, y^1 \in Y$

$$\left(y^0_\tau \cap y^1_\tau = \emptyset \wedge \bigcup \tau \cap y^0 \neq \emptyset \wedge \bigcup \tau \cap y^1 \neq \emptyset \to y^0_\sigma \cap y^1_\sigma = \emptyset\right).$$

Therefore $\langle \sigma, \mathcal{H}\rangle \leq \langle \tau, \mathcal{H}\rangle$.

Claim ■

Let $\mathcal{D} = \{D_n : n \in \omega\} \cup \{D^l_{\mathcal{A}} : \mathcal{A}$ is a finite subset of $\mathcal{I} \wedge l \in \omega\} \cup \{D_{\mathcal{A},l} : \mathcal{A}$ is a finite subset of $\mathcal{I} \wedge l \in \omega\} \cup \{D_{\mathcal{A},B,l} : \mathcal{A}$ is a finite subset of $\mathcal{I} \wedge B \in \mathcal{I} \setminus \mathcal{A} \wedge l \in \omega\} \cup \{D_{\dot{X},l,q} : q \leq p \wedge l \in \omega\}$ and G is $\mathcal{D}$-generic for $\mathbb{P}(\mathcal{I})$.

Let X_G be a partition generated by $\equiv_G$ where $\equiv_G$ is defined by

$$n \equiv_G m \text{ if } \exists \langle \sigma, \mathcal{H}\rangle \exists x \in \sigma\,(\{n, m\} \subset x)\,.$$

Then by (i) and (ii) $X_G \in (\omega)^\omega$. By (ii) $X_G \wedge \bigwedge \mathcal{A} \in (\omega)^\omega$ for finite $\mathcal{A} \subset \mathcal{I}$. By (iii) $\neg(\bigwedge \mathcal{A} \leq^* X_G)$ for finite $\mathcal{A} \subset \mathcal{I}$. By (iv) $\neg(X_G \wedge \bigwedge \mathcal{A} \leq^* Y)$ for finite $\mathcal{A} \subset \mathcal{I}$ and $Y \in \mathcal{I} \setminus \mathcal{A}$. Therefore $\{X_G\} \cup \mathcal{I}$ is dual-independent by Corollary 2.8. By (v) $p \Vdash \dot{X} \perp X_G$. Hence X_G is a required partition.

□

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