MAD FAMILY INEXTENSIBLE TO F_σ IDEAL

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ABSTRACT. Under CH, Laflamme constructs a mad family which is not extendable to every F_σ ideal [4]. He also pointed out a construction of $\neg$CH and there exists mad family which is not extendable to F_σ ideal with cardinality smaller than continuum by using Cohen-indestructible mad family.

We will construct a model which has a mad family inextensible to F_σ-ideal by using finite support iteration of Mathias-Prikry type forcings.

1. Introduction

When we analyze ideals on countable set, Katětov order play very important role. Suppose $\mathcal{I}$ and $\mathcal{J}$ are ideals on countable set X and Y, respectively. Then $\mathcal{I} \leq_K \mathcal{J}$ if there exists $\phi : Y \to X$ such that for each $I \in \mathcal{I}$, $\phi^{-1}[I] \in \mathcal{J}$. We call this ordering Katětov order. If the function is finite-to-one, then we call this ordering Katětov-Blass order, denoted by $\mathcal{I} \leq_{KB} \mathcal{J}$. For example, if $\mathcal{I} \subset \mathcal{J}$, then $\mathcal{I} \leq_K \mathcal{J}$.

When we investigate Katětov order, the uniformity number and the covering number are significant

Let $\mathcal{I}$ be on ω. Then the uniformity number of $\mathcal{I}$, denoted by $\mathsf{non}^*(\mathcal{I})$ and the covering number of $\mathcal{I}$, denoted by $\mathsf{cov}^*(\mathcal{I})$ are defined by

$$\begin{aligned} \mathsf{non}^*(\mathcal{I}) &= \min\{|\mathcal{H}| : \mathcal{H} \subset [\omega]^\omega \wedge \forall I \in \mathcal{I} \exists H \in \mathcal{H}(|A \cap H| < \aleph_0)\}. \\ \mathsf{cov}^*(\mathcal{I}) &= \min\{|G| : \mathcal{G} \subset \mathcal{I} \wedge \forall X \in [\omega]^\omega \exists G \in \mathcal{G}(|X \cap G| = \aleph_0)\}. \end{aligned}$$

Observation 1.1. *Let $\mathcal{A}$ be a MAD family on ω and let $\mathcal{I}(\mathcal{A})$ be an ideal generated by $\mathcal{A}$.* $\mathsf{cov}^*(\mathcal{I}(\mathcal{A})) = |\mathcal{A}|$.

These cardinal invariants has following relation with the Katětov order.

Proposition 1.2. [1, 3]. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on countable set.*

(1) $\mathcal{I} \leq_K \mathcal{J}$ implies $\mathsf{cov}^*(\mathcal{I}) \geq \mathsf{cov}^*(\mathcal{J})$.

(2) $\mathcal{I} \leq_{KB} \mathcal{J}$ implies $\mathsf{non}^*(\mathcal{I}) \leq \mathsf{non}^*(\mathcal{J})$.

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Concerning to the uniformity number of F_σ-ideal, we have some interesting results. Let $\mathcal{I}$ be an ideal on ω and let X be an $\mathcal{I}$-positive set. Then define ideal $\mathcal{I} \restriction X$ on X by $\mathcal{I} \restriction X = \{I \cap X : I \in \mathcal{I}\}$.

Proposition 1.3. [5] *If $\mathcal{I}$ is tall F_σ-ideal, then there exists $\mathcal{I}$-positive set X such that $\mathcal{I} \restriction X$ is countably tall, that is, for every countable set $\{X_n : n \in \omega\}$ of infinite subsets of X, there exists $I \in \mathcal{I}$ such that $|X_n \cap J| = \aleph_0$ for every $n \in \omega$. So $\mathsf{non}^*(\mathcal{I} \restriction X) > \omega$.*

Observation 1.4. *(1) For every $X \in \mathcal{I}^+$, $\mathcal{I} \leq_K \mathcal{I} \restriction X$.*

(2) If $\mathcal{J}$ is F_σ-ideal including $\mathcal{I}(\mathcal{A})$, then there exists $\mathcal{J}$-positive set X such that $\mathcal{I}(\mathcal{A}) \leq_K \mathcal{J} \restriction X$ and $\mathcal{J} \restriction X$ is countably tall.

(3) If $\mathcal{I}$ is F_σ-ideal, then $\mathcal{I} \restriction X$ is F_σ-ideal for every $\mathcal{I}$-positive set X.

These observation says that if we construct a model which satisfies that $\mathfrak{a} < \mathsf{cov}^*(\mathcal{I})$ for every countably tall F_σ ideal, then there exists a mad family which is not extendable to every F_σ ideal in the model.

Question 1.5. *Is it consistent $\mathfrak{a} < \mathsf{cov}^*(\mathcal{I})$ for every countable tall F_σ ideal?*

In [4], Laflamme introduce a forcing notion which is ω^ω-bounding and diagonalize an F_σ filter, in other word, enlarge the covering number of the dual ideal of the filter. By countable support iteration of this forcing and bookkeeping argument, we can construct a model which enlarge $\mathsf{cov}^*(\mathcal{I})$ for every countably tall F_σ ideal. However we can not say whether almost disjoint number $\mathfrak{a}$ become large or not. Instead of using this forcing notion, we shall use the finite support iteration of Mathias-Prikry type forcings.

2. finite support iteration of Mathias-Prikry forcing with F_σ-ideals

When we construct a model satisfying $\mathfrak{a} = \omega_1$, we sometime use combinatorial principle like $\diamondsuit_{\mathfrak{d}}$ [2] or "parametrized diamond principle" [7]. Let define parametrized diamond principles.

Definition 3. We call triple (A, B, E) is Borel invariants if

(1) A, B and E are Borel subset in some Polish space.
(2) A and B have cardinality at most $\mathfrak{c}$.
(3) $E \subset A \times B$.
(4) $\forall a \in A \exists b \in B((a, b) \in E)$.
(5) $\forall b \in B \exists a \in A((a, b) \notin E)$.

Definition 4. Let (A, B, E) be a Borel invariants. We call $F : 2^{<\omega} \to A$ Borel function if for every $\alpha < \omega_1$, $F \restriction 2^\alpha$ is Borel function.

Then $\diamondsuit(A, B, E)$ holds if for every Borel function $F : 2^{<\omega_1} \to A$ there exists $g : \omega_1 \to B$ such that for every $f : \omega_1 \to 2$

$$\{\alpha < \omega_1 : (F(f \restriction \alpha), g(\alpha)) \in E\} \text{ is stationary.}$$

We denote $\diamondsuit(\omega^\omega, \omega, \leq^*)$ by $\diamondsuit(\mathfrak{d})$ and denote $\diamondsuit(\omega^\omega, \omega^\omega, ^* \not\geq)$ by $\diamondsuit(\mathfrak{b})$. Then it is known that $\diamondsuit(\mathfrak{d})$ implies $\diamondsuit(\mathfrak{b})$.

Some parametrized diamonds effect almost disjoint number.

Theorem 4.1. [7] *$\diamondsuit(\mathfrak{b})$ implies $\mathfrak{a} = \omega_1$.*

So if the iteration of the forcing introduced by Laflamme forces $\diamondsuit(\mathfrak{b})$, we have desired result. When we use countable support iteration of proper forcings, there is nice criterion whether $\diamondsuit(A, B, E)$ holds or not.

Theorem 4.2. [7] *Suppose that $\langle \mathcal{Q}_\alpha : \alpha < \omega_2 \rangle$ is a sequence of Borel partial orders such that for each $\alpha < \omega_2$ $\mathcal{Q}_\alpha$ is equivalent to $\mathcal{P}(2)^+ \times \mathcal{Q}_\alpha$ as a forcing notion and let $\mathcal{P}_{\omega_2}$ be the countable support iteration of this sequence. If $\mathcal{P}_{\omega_2}$ is proper and (A, B, E) is a Borel invariant then $\mathcal{P}(\omega_2)$ forces $\langle A, B, E \rangle \leq_{\omega_1}$ iff $\mathcal{P}_{\omega_2}$ forces $\diamondsuit(A, B, E)$.*

In this note, by using finite support iteration of Mathias-Prikry type forcings $\mathbb{M}(\mathcal{I}^*)$ for F_σ-ideal, we shall show that it is consistent that $\mathfrak{a} < \mathsf{cov}^*(\mathcal{I})$ for all F_σ-ideal $\mathcal{I}$.

Let $\mathcal{I}$ be an ideal on ω. Then Mathias-Prikry forcing $\mathbb{M}(\mathcal{I}^*)$ is the forcing notion such that $\langle s, A \rangle \in \mathbb{M}(\mathcal{I}^*)$ if $s \in [\omega]^{<\omega}$, $A \in \mathcal{I}^*$ and $s \cap A = \emptyset$ ordered by $\langle s, A \rangle \leq \langle t, G \rangle$ if $s \supset t$, $A \subset B$ and $s \setminus t \subset B$.

For each $p \in \mathbb{M}(\mathcal{I}^*)$, $s_p \in [\omega]^{<\omega}$ and $A_p \in \mathcal{I}^*$ denote $p = \langle s_p, A_p \rangle$.

Theorem 4.3. *Let κ be an ordinal with $\mathsf{cf}(\kappa) > \omega_1$. Let $\mathbb{P}_\kappa$ be the κ-stage finite support iteration of Mathias-Prikry type forcing associated with F_σ-ideals. Then $V^{\mathbb{P}_\kappa} \models \diamondsuit(\mathfrak{b})$.*

This theorem shows desired statement.

Corollary 4.4. *It is consistent that for every F_σ-ideal $\mathcal{I}$, $\mathfrak{a} < \mathsf{cov}^*(\mathcal{I})$.*

Proof from Theorem 4.3. Let $\kappa > \omega_1$ be a cardinal with $\mathsf{cf}(\kappa) > \omega_1$ and $\kappa^\omega = \kappa$. By bookkeeping argument, we can construct κ-stage finite support iteration of Mathias-Prikry type forcings so that $\mathsf{cov}^*(\mathcal{I}) = \mathfrak{c}$ for every F_σ-ideal. By Theorem 4.3, $\diamondsuit(\mathfrak{b})$ holds in $V^{\mathbb{P}_\kappa}$. So $\mathfrak{a} = \omega_1 < \mathsf{cov}^*(\mathcal{I})$ for every F_σ-ideal $\mathcal{I}$.

□

To prove Theorem 4.3, we use the following preservation statement.

Theorem 4.5. *Suppose γ is an ordinal. Let $\mathbb{Q}_\gamma$ be a finite support iteration of $\langle \mathbb{Q}_\beta, \dot{\mathbb{M}}(\dot{\mathcal{I}}_\beta) : \beta < \gamma \rangle$, where $\dot{\mathcal{I}}_\beta$ is a $\mathbb{Q}_\beta$-name for an F_σ-ideal. $\mathbb{P}$ is a forcing notion which has a $\mathbb{P}$-name $\dot{c}$ such that $\Vdash_{\mathbb{P}}$ "$\exists^\infty n(x(n) < \dot{c}(n))$" for $x \in \omega^\omega \cap V$. Let $\dot{x}$ be a $\mathbb{Q}_\gamma$-name such that $\Vdash_{\mathbb{Q}_\gamma} \dot{x} \in \omega^\omega$. Then $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}_\gamma}$ "$\exists^\infty n(\dot{x}(n) < \dot{c}(n)))$".*

We will interpret a Mathias-Prikry forcing and a F_σ-ideal in forcing extensions by using its code rather than by taking the ground model forcing notion and the ground model ideal.

More precisely we should write $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}_\gamma}$ " $\exists^\infty n\,(\dot{c}(n) = i_*(\dot{x})(n))$" where i_* is the canonical map from $\mathbb{Q}_\gamma$-names to $\mathbb{P}*\dot{\mathbb{Q}}_\gamma$-names induced by the complete embedding $i : \mathbb{Q}_\gamma \to \mathbb{P}*\dot{\mathbb{Q}}_\gamma$.

In [6], there are similar argument to force parametrized diamond by using the finite support iteration of Suslin c.c.c forcings. But we will give a proof for the completion.

Proof. **First Step** Let $\mathcal{I}_0$ be an F_σ ideal and $\mathcal{I}_0^n$ be closed subsets of 2^ω for $n < \omega$ such that $\mathcal{I}_0 = \bigcup_{n<\omega} \mathcal{I}_0^n$.

Let $\dot{x}$ be a $\mathbb{M}(\mathcal{I}^*)$-name such that $\Vdash_{\mathbb{M}(\mathcal{I}^*)} \dot{x} \in \omega^\omega$. Let $\dot{c}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\exists n \in \omega(x(n) < \dot{c}(n))$" for $x \in \omega^\omega \cap V$. Let $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{M}}(\mathcal{I}_0)$ and $m \in \omega$.

It suffices to show that there exists $(p_1, \dot{q}_1) \leq (p_0, \dot{q}_0)$ and $l \geq m$ such that $(p_1, \dot{q}_1) \Vdash_{\mathbb{P}*\dot{\mathbb{M}}(\mathcal{I})}$ "$\dot{x}(l) < \dot{c}(l)$".

Without loss of generality, we can assume that for some $s \in [\omega]^{<\omega}$ and $n \in \omega$ $p_0 \Vdash_{\mathbb{P}}$ "$s_{\dot{q}} = s$ and $\omega \setminus A_{\dot{q}} \in \mathcal{I}_0^n$ ".

Claim 4.6. *Let $\mathcal{I}$ be an F_σ-ideal and $\{\mathcal{I}^n : n \in \omega\}$ be closed subsets for $n \in \omega$ such that $\mathcal{I} = \bigcup_{n\in\omega} \mathcal{I}^n$. Put $\mathcal{I}^n$ Let $\dot{x}$ be a $\mathbb{M}(\mathcal{I})$-name such that $\Vdash_{\dot{x}} \in \omega^\omega$.*

For each $s \in [\omega]^{<\omega}$, $n \in \omega$ and $i \in \omega$, put

$$x_{s,n}(i) = \min\{j \in \omega; \forall I \in \mathcal{I}^n(\neg\langle s, \omega \setminus I\rangle \Vdash_{\mathbb{M}(\mathcal{I}^*)} \text{“}\dot{x}(i) > j\text{”})\}.$$

Then $x_{s,n} \in \omega^\omega$.

Proof of Claim. Assume to the contrary that there exists $i \in \omega$ and $I_l \in \mathcal{I}^n$ such that $\langle s, \omega \setminus I_l\rangle \Vdash \dot{x}(i) \geq l$. Since $\mathcal{I}^n$ is compact, there exists $I \in \mathcal{I}^n$ and subsequence I_{l_k} such that I_{l_k} converges to I.

Then there exists $\langle t, B\rangle \leq \langle s, \omega \setminus I\rangle$ such that $\langle t, B\rangle \Vdash \dot{x}(i) = l$ for some $l \in \omega$. Since I_{l_k} converges to I, there exists k_0 such that $l < l_{k_0}$ and $(\omega \setminus I) \cap \max(t \setminus s) = (\omega \setminus I_{l_{k_0}}) \cap \max(t \setminus s)$. Then $t \setminus s \subset \omega \setminus I_{l_{k_0}}$. So $\langle t, B\rangle$ is compatible with $\langle s, \omega \setminus I_{l_{k_0}}\rangle$. It is contradiction. □

Let $x_{s,n} \in \omega^\omega \cap V$ such that

$$x_{s,n}(i) = \min\{j \in \omega; \forall I \in \mathcal{I}_0^n(\neg\langle s, \omega \setminus I\rangle \Vdash_{\mathbb{M}(\mathcal{I}^*)} \text{“}\dot{x}(i) > j\text{”})\}.$$

Let $r \leq_{\mathbb{P}} p_0$ such that $r \Vdash_{\mathbb{P}} x_{s,n}(l) < \dot{c}(l)$ for some $l \geq m$. Then fix decreasing sequence $\langle r_k : k < \omega \rangle$ of $\mathbb{P}$ and $A^* \subset \omega$ so that $r_k \leq_{\mathbb{P}} r$ and $r_k \Vdash_{\mathbb{P}} (\omega \setminus A_{\dot{q}}) \cap k = (\omega \setminus A^*) \cap k$.

Here $A^* \in \mathcal{I}_0^n$. Since $r_k \Vdash_{\mathbb{P}}$ "$A_{\dot{q}} \cap k = A^* \cap k$ and $[(\omega \setminus A_{\dot{q}}) \cap k] \cap \mathcal{I}_0^n \neq \emptyset$" and absoluteness, for every $k \in \omega$ $[(\omega \setminus A^*) \cap k] \cap \mathcal{I}_0^n \neq \emptyset$. So $\omega \setminus A^* \in \mathcal{I}_0^n$.

By definition of $x_{s,n}$, there exists $\langle t, B \rangle \leq \langle s, A^* \rangle$ such that $\langle t, B \rangle \Vdash_{\mathbb{M}(\mathcal{I}^*)}$ "$\dot{x}(l) = x_{s,n}(l)$". Then by definition of order, $t \setminus s \subset A^* \cap (|t| + 1)$. Since $r_{|t|+1} \Vdash A^* \cap (|t| + 1) = A_{\dot{q}} \cap (|t| + 1)$, $r_{|t|+1} \Vdash \langle t, B \rangle$ is compatible with $\langle s, A_{\dot{q}} \rangle$. Let $\dot{q}_1$ be a $\mathbb{M}(\mathcal{I}^*)$-name such that $r_{|t|+1} \Vdash \dot{q}_1 \leq_{\mathbb{M}(\mathcal{I})} \langle t, B \rangle, \langle s, A_{\dot{q}} \rangle$. Put $p_1 = r_{|t|+1}$. Then $\langle p_1, \dot{q}_1 \rangle \Vdash \dot{c}(l) > x_{s,n}(l) = \dot{x}(l)$.

Successor Step

Suppose the lemma holds for γ. Let $\dot{x}$ be a $\mathbb{Q}_{\gamma+1}$-name such that $\Vdash_{\mathbb{Q}_{\gamma+1}}$ " $\dot{x} \in \omega^\omega$". Let $(p_0, \dot{q}_0) \in \mathbb{P} * \dot{\mathbb{Q}}_{\gamma+1}$ and $m \in \omega$. Without loss of generality we can assume $(p_0, \dot{q}_0 \restriction \gamma) \Vdash_{\mathbb{P} * \dot{\mathbb{Q}}_\gamma}$ "$\dot{q}_0(\gamma) = \langle s, \dot{A} \rangle$ and $\dot{A} \in \mathcal{I}_{\gamma+1}^n$" for some $n \in \omega$ and $s \in \omega^{<\omega}$.

Apply Claim 4.6 in $V^{\mathbb{Q}_\gamma}$ for $\dot{x}$ and put $\dot{x}_{s,n}$ a $\mathbb{Q}_\gamma$-name such that

$$\Vdash_{\mathbb{Q}_\gamma} \text{" } \dot{x}_{s,n}(i) = \min\{j : \forall \dot{I} \in \mathcal{I}_{\gamma+1}^n \left(\neg \langle s, \omega \setminus \dot{I} \rangle \Vdash \dot{x}(i) > j \right)\}\text{".}$$

By induction hypothesis there are $(p', \dot{q}') \in \mathbb{P} * \dot{\mathbb{Q}}_\gamma$ and $l \geq m$ such that $(p', \dot{q}') \leq_{\mathbb{P} * \dot{\mathbb{Q}}_\gamma} (p_0, \dot{q}_0 \restriction \gamma)$ and $(p', \dot{q}') \Vdash_{\mathbb{P} * \dot{\mathbb{Q}}_\gamma}$ " $\dot{c}(l) > \dot{x}_{s,n}(l)$". Since $\mathbb{Q}_\gamma \lessdot \mathbb{P} * \dot{\mathbb{Q}}_\gamma$, there is a $\mathbb{Q}_\gamma$-name $\dot{Q}$ for a partial order such that $\mathbb{P} * \dot{\mathbb{Q}}_\gamma \cong \mathbb{Q}_\gamma * \dot{Q}$. Let q^* be a projection of $(p', \dot{q}')$ to $\mathbb{Q}_\gamma$. Find $\mathbb{Q}_\gamma$-names $\langle \dot{r}_k : k \in \omega \rangle$ and $\dot{A}^*$ such that

(i) $\Vdash_{\mathbb{Q}_\gamma}$ " $\omega \setminus \dot{A}^* \in \mathcal{I}_{\gamma+1}^n$ and $\dot{r}_k \in \dot{Q}$" for $k \in \omega$,
(ii) $(q^*, \dot{r}_0) \leq (p', \dot{q}')$,
(iii) $\Vdash_{\mathbb{Q}_\gamma}$ " $\dot{r}_{k+1} \leq_{\dot{Q}} \dot{r}_k$" for $k \in \omega$ and,
(iv) $(q^*, \dot{r}_k) \Vdash_{\mathbb{Q}_\gamma * \dot{Q}}$ " $\dot{A} \cap k = \dot{A}^* \cap k$" for $k \in \omega$.

Then there are $q_1^* \leq_{\mathbb{Q}_\gamma} q^*$, $t \in [\omega]^{<\omega}$ and a $\mathbb{Q}_\gamma$-name $\dot{B}$ such that $q_1^* \Vdash_{\mathbb{Q}_\gamma}$ "$\langle t, \dot{B} \rangle \leq_{\mathbb{Q}} \langle s, \dot{A}^* \rangle$ and $\langle t, \dot{B} \rangle \Vdash_{\dot{\mathbb{Q}}}$ " $\dot{x}(l) \leq \dot{x}_{s,n}(l)$"".

Since $(q^*, \dot{r}_{|t|+1}) \Vdash_{\mathbb{Q}_\gamma * \dot{Q}}$ "$\dot{A} \cap (|t| + 1) = \dot{A}^* \cap (|t| + 1)$" and $q_1^* \Vdash_{\mathbb{Q}_\gamma}$ " $t \setminus s \subset \dot{A}^* \cap (|t| + 1)$", $(q_1^*, \dot{r}_{|t|+1}) \Vdash_{\mathbb{Q}_\gamma * \dot{Q}}$ "$\langle t, \dot{B} \rangle$ is compatible with $\langle s, \dot{A} \rangle$".

Choose $(p_1, \dot{q}_1) \in \mathbb{P} * \dot{\mathbb{Q}}_{\gamma+1}$ so that $(p_1, \dot{q}_1 \restriction \gamma) = (q_1^*, \dot{r}_{|t|+1})$ and $(p_1, \dot{q}_1 \restriction \gamma) \Vdash_{\mathbb{P} * \dot{\mathbb{Q}}_\gamma}$ " $\dot{q}_1(\gamma) \leq_{\mathbb{Q}} \langle s, \dot{A} \rangle, \langle t, \dot{B} \rangle$". Then $(p_1, \dot{q}_1 \restriction \gamma) \Vdash_{\mathbb{P} * \dot{\mathbb{Q}}_\gamma}$ " $\dot{x}_{s,n}(l) < \dot{c}(l)$ and $\dot{q}_1(\gamma) \Vdash_{\dot{\mathbb{Q}}}$ " $\dot{x}(l) \leq \dot{x}_{s,n}(l)$"". Therefore $(p_1, \dot{q}_1) \Vdash_{\mathbb{P} * \dot{\mathbb{Q}}_{\gamma+1}}$ " $\dot{x}(l) < \dot{c}(l)$".

Limit Step

Suppose γ is a limit ordinal and for $\beta < \gamma$ the lemma holds. Without loss of generality we can assume the cofinality of γ is ω. Let $\langle \gamma_i : i \in \omega \rangle$ be a strictly increasing sequence converging to γ. Let $(p_0, \dot{q}_0) \in \mathbb{P} * \dot{\mathbb{Q}}_\gamma$, $m \in \omega$ and $\dot{x}$ be a $\mathbb{Q}_\gamma$-name such that $\Vdash_{\mathbb{Q}_\gamma}$ " $\dot{x} \in \omega^\omega$ ". Suppose $(p_0, \dot{q}_0) \in \mathbb{P} * \dot{\mathbb{Q}}_{\gamma_j}$.

In $V^{\mathbb{Q}_{\gamma_j}}$ let $\langle r_k : k \in \omega \rangle$ be a decreasing sequence in $\mathbb{Q}_{[\gamma_j, \gamma)}$ such that $r_k \Vdash_{\mathbb{Q}_{[\gamma_j,\gamma)}}$ "$\dot{x} \restriction k = x_j \restriction k$" where $x_j \in \omega^\omega \cap V^{\mathbb{Q}_{\gamma_j}}$.

Back in V let $\dot{r}_k$ and $\dot{x}_j$ be $\mathbb{Q}_{\gamma_j}$-names such that $\Vdash_{\mathbb{Q}_{\gamma_j}}$ "$\langle \dot{r}_k : k \in \omega \rangle$ and $\dot{x}_j$ satisfies the above".

By induction hypothesis there exist $\langle p', \dot{q}' \rangle \leq_{\mathbb{P} * \dot{\mathbb{Q}}_{\gamma_j}} \langle p_0, \dot{q}_0 \rangle$ and $l \geq m$ such that $(p', \dot{q}') \Vdash_{\mathbb{P} * \dot{\mathbb{Q}}_{\gamma_j}}$ " $\dot{c}(l) > \dot{x}_j(l)$". Put $p_1 = p'$ and $\Vdash_{\mathbb{P}}$ " $\dot{q}_1 = \dot{q}'^\frown \dot{r}_{l+1}$"

Then $\langle p_1, \dot{q}_1 \rangle \Vdash_{\mathbb{P} * \dot{\mathbb{Q}}_\gamma}$ " $\dot{c}(l) > \dot{x}_j(l) = \dot{x}(l)$".

□

We shall prove that the finite support iteration of Mathias-Prikry type forcings with F_σ-ideals forces $\diamondsuit(\mathfrak{b})$ by Theorem 4.5.

Proof of Theorem 4.3 from Theorem 4.5. We will show $V^{\mathbb{Q}_\kappa} \models \diamondsuit(\mathfrak{b})$.

Let $\dot{F}$ be a $\mathbb{Q}_\kappa$-name such that $\Vdash_{\mathbb{Q}_\kappa}$ " $\dot{F} : 2^{<\omega_1} \to \omega^\omega$". Since $\mathbb{Q}_\kappa$ has the c.c.c, a real $\dot{r}_\alpha$ coding the Borel function $\dot{F} \restriction 2^\alpha$ appears at an intermediate stage. By $cf(\kappa) > \omega_1$ we can assume $\dot{F}$ is a $\mathbb{Q}_\beta$-name for some $\beta < \kappa$. Since the cofinality of the order type of $[\beta, \kappa)$ is $cf(\kappa) > \omega_1$ for $\beta < \kappa$ and $\mathbb{Q}_\kappa = \mathbb{Q}_\beta * \dot{\mathbb{Q}}_{[\beta,\kappa)}$, we can assume $\dot{F}$ is a Borel function in the ground model. Let F be a Borel function in the ground model. Let $\dot{c}_\alpha$ be a $\mathbb{Q}_{\omega_1}$-name such that $\Vdash_{\mathbb{Q}_{\omega_1}}$ " $\exists^\infty n\, (\dot{c}_\alpha(n) > \dot{x}(n))$" for $\dot{x} \in \omega^\omega \cap V^{\mathbb{Q}_\alpha}$. We can obtain such $\dot{c}_\alpha$. For example let $\dot{c}_\alpha$ be a $\mathbb{Q}_{\omega_1}$-name for a Cohen real over $V^{\mathbb{Q}_\alpha}$.

Let $\dot{f}$ be a $\mathbb{Q}_\kappa$-name such that $\Vdash_{\mathbb{Q}_\kappa}$ " $\dot{f} : \omega_1 \to 2$". Then the following claim holds.

Claim 4.7. *Define* $C_{\dot{f}} \subset \omega_1$ *by*

$$C_{\dot{f}} = \{\alpha < \omega_1 : \dot{f} \restriction \alpha \text{ is a } \mathbb{Q}_{\alpha \cup [\omega_1,\kappa)}\text{-name}\}.$$

Then $C_{\dot{f}}$ *contains a club.*

Remark 4.8. *More precisely we should write*

$$C_{\dot{f}} = \{\alpha < \omega_1 : \text{there exists a } \mathbb{Q}_{\alpha \cup [\omega_1,\kappa)}\text{-name } \dot{x}_\alpha \text{ such that } \Vdash_{\mathbb{Q}_\kappa} \text{“ } \dot{f} \restriction \alpha = i_*(\dot{x}_\alpha)\text{”}\}$$

where i_* *is the class function from* $\mathbb{Q}_{\alpha \cup [\omega_1,\kappa)}$*-names to* $\mathbb{Q}_\kappa$*-names induced by the complete embedding* $i : \mathbb{Q}_{\alpha \cup [\omega_1,\kappa)} \lessdot \mathbb{Q}_\kappa$*. For convenience we will think of a* $\mathbb{Q}_\kappa$*-name* $\dot{x}$ *as* $\mathbb{Q}_I$*-name if there exists a* $\mathbb{Q}_I$*-name* $\dot{y}$

such that $\Vdash_{\mathbb{D}_\kappa}$ *"* $\dot{x} = i_{I*}(\dot{y})$ *" where* i_I *is the complete embedding from* $\mathbb{Q}_I$ *to* $\mathbb{Q}_\kappa$*.*

For $\alpha \in C_{\dot{f}}$, $F(\dot{f} \restriction \alpha)$ is a $\mathbb{Q}_{\alpha\cup[\omega_1,\kappa)}$-name such that $\Vdash_{\mathbb{Q}_{\alpha\cup[\omega_1,\kappa)}}$ " $F(\dot{f} \restriction \alpha) \in \omega^\omega$". By Theorem 4.5, $\alpha \in C_{\dot{f}}$ implies $\Vdash_{\mathbb{Q}_\kappa}$ " $\exists^\infty n \in \omega(F(\dot{f} \restriction \alpha)(n) < \dot{c}_\alpha(n))$". So $\Vdash_{\mathbb{Q}_\kappa}$ " $\langle \dot{c}_\alpha : \alpha < \omega_1 \rangle$ is a $\diamondsuit(\omega^\omega, \omega^\omega, \not\geq^*$)-sequence for F".

□

Question 4.9. *In ZFC, is there a mad family which is inextensible to every* F_σ*-ideal?*

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