

# MAD FAMILY INEXTENSIBLE TO $F_\sigma$ IDEAL

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ABSTRACT. Under CH, Laflamme constructs a mad family which is not extendable to every  $F_\sigma$  ideal [4]. He also pointed out a construction of  $\neg$ CH and there exists mad family which is not extendable to  $F_\sigma$  ideal with cardinality smaller than continuum by using Cohen-indestructible mad family.

We will construct a model which has a mad family inextendible to  $F_\sigma$ -ideal by using finite support iteration of Mathias-Prikry type forcings.

## 1. INTRODUCTION

When we analyze ideals on countable set, Katětov order play very important role. Suppose  $\mathcal{I}$  and  $\mathcal{J}$  are ideals on countable set  $X$  and  $Y$ , respectively. Then  $\mathcal{I} \leq_K \mathcal{J}$  if there exists  $\phi : Y \rightarrow X$  such that for each  $I \in \mathcal{I}$ ,  $\phi^{-1}[I] \in \mathcal{J}$ . We call this ordering Katětov order. If the function is finite-to-one, then we call this ordering Katětov-Blass order, denoted by  $\mathcal{I} \leq_{KB} \mathcal{J}$ . For example, if  $\mathcal{I} \subset \mathcal{J}$ , then  $\mathcal{I} \leq_K \mathcal{J}$ .

When we investigate Katětov order, the uniformity number and the covering number are significant

Let  $\mathcal{I}$  be on  $\omega$ . Then the uniformity number of  $\mathcal{I}$ , denoted by  $\text{non}^*(\mathcal{I})$  and the covering number of  $\mathcal{I}$ , denoted by  $\text{cov}^*(\mathcal{I})$  are defined by

$$\begin{aligned} \text{non}^*(\mathcal{I}) &= \min\{|\mathcal{H}| : \mathcal{H} \subset [\omega]^\omega \wedge \forall I \in \mathcal{I} \exists H \in \mathcal{H} (|A \cap H| < \aleph_0)\}. \\ \text{cov}^*(\mathcal{I}) &= \min\{|\mathcal{G}| : \mathcal{G} \subset \mathcal{I} \wedge \forall X \in [\omega]^\omega \exists G \in \mathcal{G} (|X \cap G| = \aleph_0)\}. \end{aligned}$$

**Observation 1.1.** *Let  $\mathcal{A}$  be a MAD family on  $\omega$  and let  $\mathcal{I}(\mathcal{A})$  be an ideal generated by  $\mathcal{A}$ .  $\text{cov}^*(\mathcal{I}(\mathcal{A})) = |\mathcal{A}|$ .*

These cardinal invariants has following relation with the Katětov order.

**Proposition 1.2.** [1, 3]. *Let  $\mathcal{I}$  and  $\mathcal{J}$  be ideals on countable set.*

- (1)  $\mathcal{I} \leq_K \mathcal{J}$  implies  $\text{cov}^*(\mathcal{I}) \geq \text{cov}^*(\mathcal{J})$ .
- (2)  $\mathcal{I} \leq_{KB} \mathcal{J}$  implies  $\text{non}^*(\mathcal{I}) \leq \text{non}^*(\mathcal{J})$ .

Concerning to the uniformity number of  $F_\sigma$ -ideal, we have some interesting results. Let  $\mathcal{I}$  be an ideal on  $\omega$  and let  $X$  be an  $\mathcal{I}$ -positive set. Then define ideal  $\mathcal{I} \upharpoonright X$  on  $X$  by  $\mathcal{I} \upharpoonright X = \{I \cap X : I \in \mathcal{I}\}$ .

**Proposition 1.3.** [5] *If  $\mathcal{I}$  is tall  $F_\sigma$ -ideal, then there exists  $\mathcal{I}$ -positive set  $X$  such that  $\mathcal{I} \upharpoonright X$  is countably tall, that is, for every countable set  $\{X_n : n \in \omega\}$  of infinite subsets of  $X$ , there exists  $I \in \mathcal{I}$  such that  $|X_n \cap I| = \aleph_0$  for every  $n \in \omega$ . So  $\text{non}^*(\mathcal{I} \upharpoonright X) > \omega$ .*

**Observation 1.4.** (1) *For every  $X \in \mathcal{I}^+$ ,  $\mathcal{I} \leq_K \mathcal{I} \upharpoonright X$ .*

(2) *If  $\mathcal{J}$  is  $F_\sigma$ -ideal including  $\mathcal{I}(\mathcal{A})$ , then there exists  $\mathcal{J}$ -positive set  $X$  such that  $\mathcal{I}(\mathcal{A}) \leq_K \mathcal{J} \upharpoonright X$  and  $\mathcal{J} \upharpoonright X$  is countably tall.*

(3) *If  $\mathcal{I}$  is  $F_\sigma$ -ideal, then  $\mathcal{I} \upharpoonright X$  is  $F_\sigma$ -ideal for every  $\mathcal{I}$ -positive set  $X$ .*

These observation says that if we construct a model which satisfies that  $\mathfrak{a} < \text{cov}^*(\mathcal{I})$  for every countably tall  $F_\sigma$  ideal, then there exists a mad family which is not extendable to every  $F_\sigma$  ideal in the model.

**Question 1.5.** *Is it consistent  $\mathfrak{a} < \text{cov}^*(\mathcal{I})$  for every countable tall  $F_\sigma$  ideal?*

In [4], Laflamme introduce a forcing notion which is  $\omega^\omega$ -bounding and diagonalize an  $F_\sigma$  filter, in other word, enlarge the covering number of the dual ideal of the filter. By countable support iteration of this forcing and bookkeeping argument, we can construct a model which enlarge  $\text{cov}^*(\mathcal{I})$  for every countably tall  $F_\sigma$  ideal. However we can not say whether almost disjoint number  $\mathfrak{a}$  become large or not. Instead of using this forcing notion, we shall use the finite support iteration of Mathias-Prikry type forcings.

## 2. FINITE SUPPORT ITERATION OF MATHIAS-PRIKRY FORCING WITH $F_\sigma$ -IDEALS

When we construct a model satisfying  $\mathfrak{a} = \omega_1$ , we sometime use combinatorial principle like  $\diamond_\delta$  [2] or “parametrized diamond principle” [7]. Let define parametrized diamond principles.

**Definition 3.** We call triple  $(A, B, E)$  is Borel invariants if

- (1)  $A, B$  and  $E$  are Borel subset in some Polish space.
- (2)  $A$  and  $B$  have cardinality at most  $\mathfrak{c}$ .
- (3)  $E \subset A \times B$ .
- (4)  $\forall a \in A \exists b \in B ((a, b) \in E)$ .
- (5)  $\forall b \in B \exists a \in A ((a, b) \notin E)$ .

**Definition 4.** Let  $(A, B, E)$  be a Borel invariants. We call  $F : 2^{<\omega} \rightarrow A$  Borel function if for every  $\alpha < \omega_1$ ,  $F \upharpoonright 2^\alpha$  is Borel function.

Then  $\diamond(A, B, E)$  holds if for every Borel function  $F : 2^{<\omega_1} \rightarrow A$  there exists  $g : \omega_1 \rightarrow B$  such that for every  $f : \omega_1 \rightarrow 2$

$$\{\alpha < \omega_1 : (F(f \upharpoonright \alpha), g(\alpha)) \in E\} \text{ is stationary.}$$

We denote  $\diamond(\omega^\omega, \omega, \leq^*)$  by  $\diamond(\mathfrak{d})$  and denote  $\diamond(\omega^\omega, \omega^\omega, * \not\leq)$  by  $\diamond(\mathfrak{b})$ . Then it is known that  $\diamond(\mathfrak{d})$  implies  $\diamond(\mathfrak{b})$ .

Some parametrized diamonds effect almost disjoint number.

**Theorem 4.1.** [7]  $\diamond(\mathfrak{b})$  implies  $\mathfrak{a} = \omega_1$ .

So if the iteration of the forcing introduced by Laflamme forces  $\diamond(\mathfrak{b})$ , we have desired result. When we use countable support iteration of proper forcings, there is nice criterion whether  $\diamond(A, B, E)$  holds or not.

**Theorem 4.2.** [7] Suppose that  $\langle \mathcal{Q}_\alpha : \alpha < \omega_2 \rangle$  is a sequence of Borel partial orders such that for each  $\alpha < \omega_2$   $\mathcal{Q}_\alpha$  is equivalent to  $\mathcal{P}(2)^+ \times \mathcal{Q}_\alpha$  as a forcing notion and let  $\mathcal{P}_{\omega_2}$  be the countable support iteration of this sequence. If  $\mathcal{P}_{\omega_2}$  is proper and  $(A, B, E)$  is a Borel invariant then  $\mathcal{P}(\omega_2)$  forces  $\langle A, B, E \rangle \leq_{\omega_1}$  iff  $\mathcal{P}_{\omega_2}$  forces  $\diamond(A, B, E)$ .

In this note, by using finite support iteration of Mathias-Prikry type forcings  $\mathbb{M}(\mathcal{I}^*)$  for  $F_\sigma$ -ideal, we shall show that it is consistent that  $\mathfrak{a} < \text{cov}^*(\mathcal{I})$  for all  $F_\sigma$ -ideal  $\mathcal{I}$ .

Let  $\mathcal{I}$  be an ideal on  $\omega$ . Then Mathias-Prikry forcing  $\mathbb{M}(\mathcal{I}^*)$  is the forcing notion such that  $\langle s, A \rangle \in \mathbb{M}(\mathcal{I}^*)$  if  $s \in [\omega]^{<\omega}$ ,  $A \in \mathcal{I}^*$  and  $s \cap A = \emptyset$  ordered by  $\langle s, A \rangle \leq \langle t, G \rangle$  if  $s \supset t$ ,  $A \subset B$  and  $s \setminus t \subset B$ .

For each  $p \in \mathbb{M}(\mathcal{I}^*)$ ,  $s_p \in [\omega]^{<\omega}$  and  $A_p \in \mathcal{I}^*$  denote  $p = \langle s_p, A_p \rangle$ .

**Theorem 4.3.** Let  $\kappa$  be an ordinal with  $\text{cf}(\kappa) > \omega_1$ . Let  $\mathbb{P}_\kappa$  be the  $\kappa$ -stage finite support iteration of Mathias-Prikry type forcing associated with  $F_\sigma$ -ideals. Then  $V^{\mathbb{P}_\kappa} \models \diamond(\mathfrak{b})$ .

This theorem shows desired statement.

**Corollary 4.4.** It is consistent that for every  $F_\sigma$ -ideal  $\mathcal{I}$ ,  $\mathfrak{a} < \text{cov}^*(\mathcal{I})$ .

*Proof from Theorem 4.3.* Let  $\kappa > \omega_1$  be a cardinal with  $\text{cf}(\kappa) > \omega_1$  and  $\kappa^\omega = \kappa$ . By bookkeeping argument, we can construct  $\kappa$ -stage finite support iteration of Mathias-Prikry type forcings so that  $\text{cov}^*(\mathcal{I}) = \mathfrak{c}$  for every  $F_\sigma$ -ideal. By Theorem 4.3,  $\diamond(\mathfrak{b})$  holds in  $V^{\mathbb{P}_\kappa}$ . So  $\mathfrak{a} = \omega_1 < \text{cov}^*(\mathcal{I})$  for every  $F_\sigma$ -ideal  $\mathcal{I}$ . □

To prove Theorem 4.3, we use the following preservation statement.

**Theorem 4.5.** *Suppose  $\gamma$  is an ordinal. Let  $\mathbb{Q}_\gamma$  be a finite support iteration of  $\langle \mathbb{Q}_\beta, \dot{\mathbb{M}}(\dot{\mathcal{I}}_\beta) : \beta < \gamma \rangle$ , where  $\dot{\mathcal{I}}_\beta$  is a  $\mathbb{Q}_\beta$ -name for an  $F_\sigma$ -ideal.  $\mathbb{P}$  is a forcing notion which has a  $\mathbb{P}$ -name  $\dot{c}$  such that  $\Vdash_{\mathbb{P}} \text{“}\exists^\infty n(x(n) < \dot{c}(n))\text{”}$  for  $x \in \omega^\omega \cap V$ . Let  $\dot{x}$  be a  $\mathbb{Q}_\gamma$ -name such that  $\Vdash_{\mathbb{Q}_\gamma} \dot{x} \in \omega^\omega$ . Then  $\Vdash_{\mathbb{P} * \dot{\mathbb{Q}}_\gamma} \text{“}\exists^\infty n(\dot{x}(n) < \dot{c}(n))\text{”}$ .*

We will interpret a Mathias-Prikry forcing and a  $F_\sigma$ -ideal in forcing extensions by using its code rather than by taking the ground model forcing notion and the ground model ideal.

More precisely we should write  $\Vdash_{\mathbb{P} * \dot{\mathbb{Q}}_\gamma} \text{“}\exists^\infty n(\dot{c}(n) = i_*(\dot{x})(n))\text{”}$  where  $i_*$  is the canonical map from  $\mathbb{Q}_\gamma$ -names to  $\mathbb{P} * \dot{\mathbb{Q}}_\gamma$ -names induced by the complete embedding  $i : \mathbb{Q}_\gamma \rightarrow \mathbb{P} * \dot{\mathbb{Q}}_\gamma$ .

In [6], there are similar argument to force parametrized diamond by using the finite support iteration of Suslin c.c.c forcings. But we will give a proof for the completion.

*Proof. First Step* Let  $\mathcal{I}_0$  be an  $F_\sigma$  ideal and  $\mathcal{I}_0^n$  be closed subsets of  $2^\omega$  for  $n < \omega$  such that  $\mathcal{I}_0 = \bigcup_{n < \omega} \mathcal{I}_0^n$ .

Let  $\dot{x}$  be a  $\mathbb{M}(\mathcal{I}^*)$ -name such that  $\Vdash_{\mathbb{M}(\mathcal{I}^*)} \dot{x} \in \omega^\omega$ . Let  $\dot{c}$  be a  $\mathbb{P}$ -name such that  $\Vdash_{\mathbb{P}} \text{“}\exists n \in \omega(x(n) < \dot{c}(n))\text{”}$  for  $x \in \omega^\omega \cap V$ . Let  $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{M}}(\mathcal{I}_0)$  and  $m \in \omega$ .

It suffices to show that there exists  $(p_1, \dot{q}_1) \leq (p, \dot{q})$  and  $l \geq m$  such that  $(p_1, \dot{q}_1) \Vdash_{\mathbb{P} * \dot{\mathbb{M}}(\mathcal{I})} \text{“}\dot{x}(l) < \dot{c}(l)\text{”}$ .

Without loss of generality, we can assume that for some  $s \in [\omega]^{<\omega}$  and  $n \in \omega$   $p_0 \Vdash_{\mathbb{P}} \text{“}s_{\dot{q}} = s \text{ and } \omega \setminus A_{\dot{q}} \in \mathcal{I}_0^n\text{”}$ .

**Claim 4.6.** *Let  $\mathcal{I}$  be an  $F_\sigma$ -ideal and  $\{\mathcal{I}^n : n \in \omega\}$  be closed subsets for  $n \in \omega$  such that  $\mathcal{I} = \bigcup_{n \in \omega} \mathcal{I}^n$ . Put  $\mathcal{I}^n$ . Let  $\dot{x}$  be a  $\mathbb{M}(\mathcal{I})$ -name such that  $\Vdash_{\dot{x} \in \omega^\omega}$ .*

*For each  $s \in [\omega]^{<\omega}$ ,  $n \in \omega$  and  $i \in \omega$ , put*

$$x_{s,n}(i) = \min\{j \in \omega; \forall I \in \mathcal{I}^n(\neg \langle s, \omega \setminus I \rangle \Vdash_{\mathbb{M}(\mathcal{I}^*)} \text{“}\dot{x}(i) > j\text{”})\}.$$

*Then  $x_{s,n} \in \omega^\omega$ .*

*Proof of Claim.* Assume to the contrary that there exists  $i \in \omega$  and  $I_l \in \mathcal{I}^n$  such that  $\langle s, \omega \setminus I_l \rangle \Vdash \dot{x}(i) \geq l$ . Since  $\mathcal{I}^n$  is compact, there exists  $I \in \mathcal{I}^n$  and subsequence  $I_{l_k}$  such that  $I_{l_k}$  converges to  $I$ .

Then there exists  $\langle t, B \rangle \leq \langle s, \omega \setminus I \rangle$  such that  $\langle t, B \rangle \Vdash \dot{x}(i) = l$  for some  $l \in \omega$ . Since  $I_{l_k}$  converges to  $I$ , there exists  $k_0$  such that  $l < l_{k_0}$  and  $(\omega \setminus I) \cap \max(t \setminus s) = (\omega \setminus I_{l_{k_0}}) \cap \max(t \setminus s)$ . Then  $t \setminus s \subset \omega \setminus I_{l_{k_0}}$ . So  $\langle t, B \rangle$  is compatible with  $\langle s, \omega \setminus I_{l_{k_0}} \rangle$ . It is contradiction.  $\square$

Let  $x_{s,n} \in \omega^\omega \cap V$  such that

$$x_{s,n}(i) = \min\{j \in \omega; \forall I \in \mathcal{I}_0^n(\neg \langle s, \omega \setminus I \rangle \Vdash_{\mathbb{M}(\mathcal{I}^*)} \text{“}\dot{x}(i) > j\text{”})\}.$$

Let  $r \leq_{\mathbb{P}} p_0$  such that  $r \Vdash_{\mathbb{P}} x_{s,n}(l) < \dot{c}(l)$  for some  $l \geq m$ . Then fix decreasing sequence  $\langle r_k : k < \omega \rangle$  of  $\mathbb{P}$  and  $A^* \subset \omega$  so that  $r_k \leq_{\mathbb{P}} r$  and  $r_k \Vdash_{\mathbb{P}} (\omega \setminus A_{\dot{q}}) \cap k = (\omega \setminus A^*) \cap k$ .

Here  $A^* \in \mathcal{I}_0^n$ . Since  $r_k \Vdash_{\mathbb{P}}$  “ $A_{\dot{q}} \cap k = A^* \cap k$  and  $[(\omega \setminus A_{\dot{q}}) \cap k] \cap \mathcal{I}_0^n \neq \emptyset$ ” and absoluteness, for every  $k \in \omega$   $[(\omega \setminus A^*) \cap k] \cap \mathcal{I}_0^n \neq \emptyset$ . So  $\omega \setminus A^* \in \mathcal{I}_0^n$ .

By definition of  $x_{s,n}$ , there exists  $\langle t, B \rangle \leq \langle s, A^* \rangle$  such that  $\langle t, B \rangle \Vdash_{\mathbb{M}(\mathcal{I}^*)}$  “ $\dot{x}(l) = x_{s,n}(l)$ ”. Then by definition of order,  $t \setminus s \subset A^* \cap (|t| + 1)$ . Since  $r_{|t|+1} \Vdash A^* \cap (|t| + 1) = A_{\dot{q}} \cap (|t| + 1)$ ,  $r_{|t|+1} \Vdash \langle t, B \rangle$  is compatible with  $\langle s, A_{\dot{q}} \rangle$ . Let  $\dot{q}_1$  be a  $\mathbb{M}(\mathcal{I}^*)$ -name such that  $r_{|t|+1} \Vdash \dot{q}_1 \leq_{\mathbb{M}(\mathcal{I}^*)} \langle t, B \rangle, \langle s, A_{\dot{q}} \rangle$ . Put  $p_1 = r_{|t|+1}$ . Then  $\langle p_1, \dot{q}_1 \rangle \Vdash \dot{c}(l) > x_{s,n}(l) = \dot{x}(l)$ .

### Successor Step

Suppose the lemma holds for  $\gamma$ . Let  $\dot{x}$  be a  $\mathbb{Q}_{\gamma+1}$ -name such that  $\Vdash_{\mathbb{Q}_{\gamma+1}}$  “ $\dot{x} \in \omega^\omega$ ”. Let  $(p_0, \dot{q}_0) \in \mathbb{P} * \dot{\mathbb{Q}}_{\gamma+1}$  and  $m \in \omega$ . Without loss of generality we can assume  $(p_0, \dot{q}_0 \upharpoonright \gamma) \Vdash_{\mathbb{P} * \dot{\mathbb{Q}}_\gamma}$  “ $\dot{q}_0(\gamma) = \langle s, \dot{A} \rangle$  and  $\dot{A} \in \mathcal{I}_{\gamma+1}^n$ ” for some  $n \in \omega$  and  $s \in \omega^{<\omega}$ .

Apply Claim 4.6 in  $V^{\mathbb{Q}_\gamma}$  for  $\dot{x}$  and put  $\dot{x}_{s,n}$  a  $\mathbb{Q}_\gamma$ -name such that

$$\Vdash_{\mathbb{Q}_\gamma} \text{ “ } \dot{x}_{s,n}(i) = \min\{j : \forall \dot{I} \in \mathcal{I}_{\gamma+1}^n \left( \neg \langle s, \omega \setminus \dot{I} \rangle \Vdash \dot{x}(i) > j \right) \text{ ”.}$$

By induction hypothesis there are  $(p', \dot{q}') \in \mathbb{P} * \dot{\mathbb{Q}}_\gamma$  and  $l \geq m$  such that  $(p', \dot{q}') \leq_{\mathbb{P} * \dot{\mathbb{Q}}_\gamma} (p_0, \dot{q}_0 \upharpoonright \gamma)$  and  $(p', \dot{q}') \Vdash_{\mathbb{P} * \dot{\mathbb{Q}}_\gamma}$  “ $\dot{c}(l) > \dot{x}_{s,n}(l)$ ”. Since  $\mathbb{Q}_\gamma \triangleleft \mathbb{P} * \dot{\mathbb{Q}}_\gamma$ , there is a  $\mathbb{Q}_\gamma$ -name  $\dot{Q}$  for a partial order such that  $\mathbb{P} * \dot{\mathbb{Q}}_\gamma \cong \mathbb{Q}_\gamma * \dot{Q}$ . Let  $q^*$  be a projection of  $(p', \dot{q}')$  to  $\mathbb{Q}_\gamma$ . Find  $\mathbb{Q}_\gamma$ -names  $\langle \dot{r}_k : k \in \omega \rangle$  and  $\dot{A}^*$  such that

- (i)  $\Vdash_{\mathbb{Q}_\gamma}$  “ $\omega \setminus \dot{A}^* \in \mathcal{I}_{\gamma+1}^n$  and  $\dot{r}_k \in \dot{Q}$ ” for  $k \in \omega$ ,
- (ii)  $(q^*, \dot{r}_0) \leq (p', \dot{q}')$ ,
- (iii)  $\Vdash_{\mathbb{Q}_\gamma}$  “ $\dot{r}_{k+1} \leq_{\dot{Q}} \dot{r}_k$ ” for  $k \in \omega$  and,
- (iv)  $(q^*, \dot{r}_k) \Vdash_{\mathbb{Q}_\gamma * \dot{Q}}$  “ $\dot{A} \cap k = \dot{A}^* \cap k$ ” for  $k \in \omega$ .

Then there are  $q_1^* \leq_{\mathbb{Q}_\gamma} q^*$ ,  $t \in [\omega]^{<\omega}$  and a  $\mathbb{Q}_\gamma$ -name  $\dot{B}$  such that  $q_1^* \Vdash_{\mathbb{Q}_\gamma}$  “ $\langle t, \dot{B} \rangle \leq_{\mathbb{Q}} \langle s, \dot{A}^* \rangle$  and  $\langle t, \dot{B} \rangle \Vdash_{\dot{Q}}$  “ $\dot{x}(l) \leq \dot{x}_{s,n}(l)$ ””.

Since  $(q^*, \dot{r}_{|t|+1}) \Vdash_{\mathbb{Q}_\gamma * \dot{Q}}$  “ $\dot{A} \cap (|t| + 1) = \dot{A}^* \cap (|t| + 1)$ ” and  $q_1^* \Vdash_{\mathbb{Q}_\gamma}$  “ $t \setminus s \subset \dot{A}^* \cap (|t| + 1)$ ”,  $(q_1^*, \dot{r}_{|t|+1}) \Vdash_{\mathbb{Q}_\gamma * \dot{Q}}$  “ $\langle t, \dot{B} \rangle$  is compatible with  $\langle s, \dot{A}^* \rangle$ ”.

Choose  $(p_1, \dot{q}_1) \in \mathbb{P} * \dot{\mathbb{Q}}_{\gamma+1}$  so that  $(p_1, \dot{q}_1 \upharpoonright \gamma) = (q_1^*, \dot{r}_{|t|+1})$  and  $(p_1, \dot{q}_1 \upharpoonright \gamma) \Vdash_{\mathbb{P} * \dot{\mathbb{Q}}_\gamma}$  “ $\dot{q}_1(\gamma) \leq_{\mathbb{Q}} \langle s, \dot{A} \rangle, \langle t, \dot{B} \rangle$ ”. Then  $(p_1, \dot{q}_1 \upharpoonright \gamma) \Vdash_{\mathbb{P} * \dot{\mathbb{Q}}_\gamma}$  “ $\dot{x}_{s,n}(l) < \dot{c}(l)$  and  $\dot{q}_1(\gamma) \Vdash_{\dot{Q}}$  “ $\dot{x}(l) \leq \dot{x}_{s,n}(l)$ ”””. Therefore  $(p_1, \dot{q}_1) \Vdash_{\mathbb{P} * \dot{\mathbb{Q}}_{\gamma+1}}$  “ $\dot{x}(l) < \dot{c}(l)$ ”.

### Limit Step

Suppose  $\gamma$  is a limit ordinal and for  $\beta < \gamma$  the lemma holds. Without loss of generality we can assume the cofinality of  $\gamma$  is  $\omega$ . Let  $\langle \gamma_i : i \in \omega \rangle$  be a strictly increasing sequence converging to  $\gamma$ . Let  $(p_0, \dot{q}_0) \in \mathbb{P} * \dot{\mathbb{Q}}_\gamma$ ,  $m \in \omega$  and  $\dot{x}$  be a  $\mathbb{Q}_\gamma$ -name such that  $\Vdash_{\mathbb{Q}_\gamma} \text{“} \dot{x} \in \omega^\omega \text{”}$ . Suppose  $(p_0, \dot{q}_0) \in \mathbb{P} * \dot{\mathbb{Q}}_{\gamma_j}$ .

In  $V^{\mathbb{Q}_{\gamma_j}}$  let  $\langle r_k : k \in \omega \rangle$  be a decreasing sequence in  $\mathbb{Q}_{[\gamma_j, \gamma]}$  such that  $r_k \Vdash_{\mathbb{Q}_{[\gamma_j, \gamma]}} \text{“} \dot{x} \upharpoonright k = x_j \upharpoonright k \text{”}$  where  $x_j \in \omega^\omega \cap V^{\mathbb{Q}_{\gamma_j}}$ .

Back in  $V$  let  $\dot{r}_k$  and  $\dot{x}_j$  be  $\mathbb{Q}_{\gamma_j}$ -names such that  $\Vdash_{\mathbb{Q}_{\gamma_j}} \text{“} \langle \dot{r}_k : k \in \omega \rangle$  and  $\dot{x}_j$  satisfies the above”.

By induction hypothesis there exist  $\langle p', \dot{q}' \rangle \leq_{\mathbb{P} * \dot{\mathbb{Q}}_{\gamma_j}} \langle p_0, \dot{q}_0 \rangle$  and  $l \geq m$  such that  $\langle p', \dot{q}' \rangle \Vdash_{\mathbb{P} * \dot{\mathbb{Q}}_{\gamma_j}} \text{“} \dot{c}(l) > \dot{x}_j(l) \text{”}$ . Put  $p_1 = p'$  and  $\Vdash_{\mathbb{P}} \text{“} \dot{q}_1 = \dot{q}' \hat{\wedge} \dot{r}_{l+1} \text{”}$

Then  $\langle p_1, \dot{q}_1 \rangle \Vdash_{\mathbb{P} * \dot{\mathbb{Q}}_\gamma} \text{“} \dot{c}(l) > \dot{x}_j(l) = \dot{x}(l) \text{”}$ .

□

We shall prove that the finite support iteration of Mathias-Prikry type forcings with  $F_\sigma$ -ideals forces  $\diamond(\mathfrak{b})$  by Theorem 4.5.

*Proof of Theorem 4.3 from Theorem 4.5.* We will show  $V^{\mathbb{Q}_\kappa} \models \diamond(\mathfrak{b})$ .

Let  $\dot{F}$  be a  $\mathbb{Q}_\kappa$ -name such that  $\Vdash_{\mathbb{Q}_\kappa} \text{“} \dot{F} : 2^{<\omega_1} \rightarrow \omega^\omega \text{”}$ . Since  $\mathbb{Q}_\kappa$  has the c.c.c, a real  $\dot{r}_\alpha$  coding the Borel function  $\dot{F} \upharpoonright 2^\alpha$  appears at an intermediate stage. By  $cf(\kappa) > \omega_1$  we can assume  $\dot{F}$  is a  $\mathbb{Q}_\beta$ -name for some  $\beta < \kappa$ . Since the cofinality of the order type of  $[\beta, \kappa)$  is  $cf(\kappa) > \omega_1$  for  $\beta < \kappa$  and  $\mathbb{Q}_\kappa = \mathbb{Q}_\beta * \dot{\mathbb{Q}}_{[\beta, \kappa)}$ , we can assume  $\dot{F}$  is a Borel function in the ground model. Let  $F$  be a Borel function in the ground model. Let  $\dot{c}_\alpha$  be a  $\mathbb{Q}_{\omega_1}$ -name such that  $\Vdash_{\mathbb{Q}_{\omega_1}} \text{“} \exists^\infty n (\dot{c}_\alpha(n) > \dot{x}(n)) \text{”}$  for  $\dot{x} \in \omega^\omega \cap V^{\mathbb{Q}_\alpha}$ . We can obtain such  $\dot{c}_\alpha$ . For example let  $\dot{c}_\alpha$  be a  $\mathbb{Q}_{\omega_1}$ -name for a Cohen real over  $V^{\mathbb{Q}_\alpha}$ .

Let  $\dot{f}$  be a  $\mathbb{Q}_\kappa$ -name such that  $\Vdash_{\mathbb{Q}_\kappa} \text{“} \dot{f} : \omega_1 \rightarrow 2 \text{”}$ . Then the following claim holds.

**Claim 4.7.** Define  $C_{\dot{f}} \subset \omega_1$  by

$$C_{\dot{f}} = \{\alpha < \omega_1 : \dot{f} \upharpoonright \alpha \text{ is a } \mathbb{Q}_{\alpha \cup [\omega_1, \kappa)}\text{-name}\}.$$

Then  $C_{\dot{f}}$  contains a club.

**Remark 4.8.** More precisely we should write

$$C_{\dot{f}} = \{\alpha < \omega_1 : \text{there exists a } \mathbb{Q}_{\alpha \cup [\omega_1, \kappa)}\text{-name } \dot{x}_\alpha \text{ such that } \Vdash_{\mathbb{Q}_\kappa} \text{“} \dot{f} \upharpoonright \alpha = i_*(\dot{x}_\alpha) \text{”}\}$$

where  $i_*$  is the class function from  $\mathbb{Q}_{\alpha \cup [\omega_1, \kappa)}$ -names to  $\mathbb{Q}_\kappa$ -names induced by the complete embedding  $i : \mathbb{Q}_{\alpha \cup [\omega_1, \kappa)} \leq \mathbb{Q}_\kappa$ . For convenience we will think of a  $\mathbb{Q}_\kappa$ -name  $\dot{x}$  as  $\mathbb{Q}_I$ -name if there exists a  $\mathbb{Q}_I$ -name  $\dot{y}$

such that  $\Vdash_{\mathbb{D}_\kappa} \dot{x} = i_{I^*}(\dot{y})$  where  $i_I$  is the complete embedding from  $\mathbb{Q}_I$  to  $\mathbb{Q}_\kappa$ .

For  $\alpha \in C_{\dot{j}}$ ,  $F(\dot{f} \upharpoonright \alpha)$  is a  $\mathbb{Q}_{\alpha \cup \{\omega_1, \kappa\}}$ -name such that  $\Vdash_{\mathbb{Q}_{\alpha \cup \{\omega_1, \kappa\}}} F(\dot{f} \upharpoonright \alpha) \in \omega^\omega$ . By Theorem 4.5,  $\alpha \in C_{\dot{j}}$  implies  $\Vdash_{\mathbb{Q}_\kappa} \exists^\infty n \in \omega (F(\dot{f} \upharpoonright \alpha)(n) < \dot{c}_\alpha(n))$ . So  $\Vdash_{\mathbb{Q}_\kappa} \langle \dot{c}_\alpha : \alpha < \omega_1 \rangle$  is a  $\diamond(\omega^\omega, \omega^\omega, \not\leq^*)$ -sequence for  $F$ . □

**Question 4.9.** *In ZFC, is there a mad family which is inextensible to every  $F_\sigma$ -ideal?*

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