On pair-splitting and pair-reaping pairs of \( \omega \)

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Abstract

In this paper we investigate variations of splitting number and reaping number, pair-splitting number \( s_{\text{pair}} \), pair-reaping number \( r_{\text{pair}} \). We prove that it is consistent that \( s_{\text{pair}} < b \). We also prove it is consistent that \( r_{\text{pair}} > b \).

Introduction

The splitting number \( s \) and the reaping number \( r \) are cardinal invariants related to the structure \( \mathcal{P}(\omega)/\text{fin} \).

For \( X, Y \in [\omega]^{\omega} \) we say \( X \) splits \( Y \) if \( X \cap Y \) and \( Y \setminus X \) are infinite. We call \( S \subset [\omega]^{\omega} \) a splitting family if for each \( Y \in [\omega]^{\omega} \), there exists \( X \in [\omega]^{\omega} \) such that \( X \) splits \( Y \). The splitting number \( s \) is the least size of a splitting family.

We call \( \mathcal{R} \) a reaping family if for each \( X \in [\omega] \), there exists \( Y \in [\omega]^{\omega} \) such that \( Y \) is not split by \( X \), that is, \( X \cap Y \) is finite or \( Y \setminus X \) is finite. The reaping number \( r \) is the least size of a reaping family.

We shall study variations of splitting number and reaping number, pair-splitting number \( s_{\text{pair}} \) and pair-reaping number \( r_{\text{pair}} \). They are introduced and investigated in [7] to analyze dual-reaping number \( r_{\text{d}} \) and dual-splitting number \( s_{\text{d}} \) which are reaping number and splitting number for the structure of all infinite partitions of \( \omega \) ordered by “almost coarser” \( ((\omega)^{\omega}, \leq^*) \) respectively.

We call \( A \subset [\omega]^2 \) unbounded if for \( k \in \omega \), there exists \( a \in A \) such that \( a \cap k = \emptyset \). For \( X \in [\omega]^{\omega} \) and unbounded \( A \subset [\omega]^2 \), \( X \) pair-splits \( A \) if there exist infinitely many \( a \in A \) such that \( a \cap X \neq \emptyset \) and \( a \setminus X \neq \emptyset \). We call \( S \subset [\omega]^{\omega} \) a pair-splitting family if for each unbounded \( A \subset [\omega]^2 \), there exists
$X \in S$ such that $X$ pair-splits $A$. The pair-splitting number $s_{\text{pair}}$ is the least size of a pair-splitting family.

We call $\mathcal{R} \subset \mathcal{P}([\omega]^2)$ a pair-reaping family if for each $A \in \mathcal{R}$, $A$ is unbounded and for $X \in [\omega]^{\omega}$, there exists $A \in \mathcal{R}$ such that $X$ doesn’t pair-split $A$. The pair-reaping number $r_{\text{pair}}$ is the least size of a pair-reaping family.

In [7] it is proved that there is the following relationship between $r_{\text{pair}}$, $s_{\text{pair}}$ and other cardinal invariants.

Proposition 0.1

1. $s_{\text{pair}} \leq \text{non}(\mathcal{M}), \text{non}(\mathcal{N})$.
2. $r_{\text{pair}} \geq \text{cov}(\mathcal{M}), \text{cov}(\mathcal{N})$.
3. $s_{\text{pair}} \geq s$.
4. $r_{\text{pair}} \leq r, s_d$.

It is not known that $r_d \leq s_{\text{pair}}$ or not.

Question 0.1 $r_d \leq s_{\text{pair}}$?

$s \leq \diamondsuit$ and $r \geq \textbf{b}$ hold (see in [2]). And Kamo proved the following statement in [7]:

Theorem 0.1 $r_d \leq \diamondsuit$ and $s_d \geq \textbf{b}$.

So we have the following diagram:
An arrow $\kappa \to \lambda$ denotes the inequality $\kappa \geq \lambda$.

In [7] by using finite support iteration of Hechler forcing, the following consistency results are proved.

**Theorem 0.2** It is consistent that $s_{\text{pair}} < \text{add}(\mathcal{M})$. Dually it is consistent that $r_{\text{pair}} > \text{cof}(\mathcal{M})$.

$r_{\text{pair}}$ is a lower bound of $r$ and $s$ and $s_{\text{pair}}$ is an upper bound of $s$ (and maybe of $r_d$). So it is natural to ask the following question.

**Question 0.2** $s_{\text{pair}} \leq d$? Dually $r_{\text{pair}} \geq b$?

In the present paper we shall investigate the relation ship between $r_{\text{pair}}$ and $b$ and the relationship between $s_{\text{pair}}$ and $d$. In section 1 we shall prove the consistency of $s_{\text{pair}} > d$. In section 2 we shall show the consistency of the consistency of $r_{\text{pair}} < b$. In section 3 we mention the development of results in section 1 and 2.
1 pair-splitting number and dominating number

Notation and Definition We present the related notions. We use standard set theoretical conventions and notation. For a set \( X \), \( X^\omega \) denotes the set of all functions from \( \omega \) to \( X \). For \( f, g \in \omega^\omega \), \( f \) dominates \( g \), written \( f \leq^* g \), if for all but finitely many \( n \in \omega \) \( g(n) \leq f(n) \). We call \( \mathcal{F} \) a dominating family if for each \( g \in \omega^\omega \) there exists \( f \in \mathcal{F} \) such that \( g \leq^* f \). The dominating number \( \mathfrak{d} \) is the least size of a dominating family.

We call \( \mathcal{G} \) an unbounded family if for each \( f \in \omega^\omega \) there exists \( g \in \mathcal{G} \) such that \( g \nleq^* f \), i.e., there exist infinitely many \( n \in \omega \) such that \( g(n) > f(n) \). The unbounded number \( \mathfrak{b} \) is the least size of an unbounded family.

For a set \( X \), \( X^{<\omega} \) denote the set of all functions from natural numbers to \( X \).

We call partial ordering \((T, <)\) a tree if the set \{\( s \in T : s < t \)\} is well-ordered by \(<\). We say \( T \) is a tree on \( X \) if \( T \) is a subtree of \((X^{<\omega}, \subset)\). For a tree \( T \) and \( t \in T \), \( \text{succ}_{\text{T}}(t) \) is the set of all immediate successors of \( t \) in \( T \). For a tree \( T \), \( \text{stem}(T) \) is the first element of \( T \) which has at least \( 2 \)-many immediate successors.

Theorem 1.1 It is consistent \( s_{\text{pair}} > \mathfrak{d} \).

To prove theorem 1.1, we shall construct a proper forcing notion which enlarges \( s_{\text{pair}} \) and is \( \omega^\omega \)-bounding to show \( \mathfrak{d} \) is preserved by the forcing notion.

Definition 1.1 [4, pp340] A forcing notion \( \mathbb{P} \) is \( \omega^\omega \)-bounding if

\[ \vDash_{\mathbb{P}} \forall f \in \omega^\omega \cap V[G] \exists g \in \omega^\omega \cap V (f \leq^* g) . \]

The \( \omega^\omega \)-boundingness has the following good property.

Theorem 1.2 [4, pp341] The countable support iteration of proper \( \omega^\omega \)-bounding forcing notions is \( \omega^\omega \)-bounding.

To prove theorem 1.1 we shall construct a forcing notion which consists of finitely branching trees on \([\omega]^2\) such that the set of successors of any node carries a norm as [8].

To present the desired forcing notion, we define “norm” for finite subsets of \([\omega]^2\). Let \( R(n) \) be a natural number such that if \( m \geq R(n) \), then for any
function \( f : [m]^2 \to 2 \) there exists \( H \in [m]^n \) such that \( |f([H]^2)| = 1 \). Then recursively define \( l_1 = 3 \), \( l_{n+1} = \max\{2l_n, R(l_n)\} \). Then for a finite subset \( A \) of \( [\omega]^2 \) \( \text{norm}(A) \geq n \) if \( A \) contains a complete graph with \( l_n \)-many vertices.

This norm has the following properties:

**Proposition 1.1** For a finite subset \( A \) of \( [\omega]^2 \),

1. \( \text{norm}(A) \geq 1 \) implies for any \( X \in [\omega]^\omega \) there exists \( a \in A \) such that \( a \cap X = \emptyset \) or \( a \subset X \).

2. Suppose \( \text{norm}(A) \geq n+1 \). For \( X \in [\omega]^{\omega} \) let \( A_X^0 = \{a \in A : a \cap X = \emptyset\} \) and \( A_X^1 = \{a \in A : a \subset X\} \). Then \( \text{norm}(A_X^0) \geq n \) or \( \text{norm}(A_X^1) \geq n \).

3. Suppose \( \text{norm}(A) \geq n+1 \). If \( A = A_0 \cup A_1 \), then \( \text{norm}(A_0) \geq n \) or \( \text{norm}(A_1) \geq n \).

**Proof of proposition 1.1**

1. Since \( \text{norm}(A) \geq 1 \), \( A \) contains a complete graph \( A' \subset A \) with 3-many vertices. Then for any 2-coloring of the vertices of \( A' \), there exists an edge whose vertices have the same color. So there exists \( a \in A' \subset A \) such that \( a \subset X \) or \( a \cap X = \emptyset \).

2. Since \( \text{norm}(A) \geq n+1 \), \( A \) contain a complete graph \( A' \) with \( l_{n+1} \)-many vertices. So for each \( X \subset \omega \), \( X \) contains \( l_n \)-many vertices of \( A' \) or \( X \) doesn’t meet \( l_n \)-many vertices of \( A' \) because \( l_{n+1} \geq 2l_n \). Anyway \( A_X^0 = \{a \in A : a \cap X = \emptyset\} \) or \( A_X^1 = \{a \in A : a \subset X\} \) contains a complete graph with \( l_n \)-many vertices. Therefore \( \text{norm}(A_X^0) \geq n \) or \( \text{norm}(A_X^1) \geq n \).

3. Since \( \text{norm}(A) \geq n+1 \), \( A \) contain a complete graph \( A' \) with \( l_{n+1} \)-many vertices. Define \( f : A' \to 2 \) by \( f(a) = i \) if \( a \in A_i \) for \( i < 2 \). Since \( l_{n+1} \geq R(l_n) \), there exists a complete graph \( A^* \subset A' \) which has \( l_n \)-many vertices of \( A' \) and \( |f[A^*]| = 1 \). So \( A^* \subset A_0 \) or \( A^* \subset A_1 \). Hence \( \text{norm}(A_0) \geq n \) or \( \text{norm}(A_1) \geq n \).

\[ \square \]

Then let \( \mathbb{P} \) be the set of perfect trees such that

1. \( T \) is a finitely branching tree on \([\omega]^2\),

2. for any branch of \( T \) and \( n \in \omega \) there exist \( m \geq n \) such that whenever \( t \in T \) with \( |t| \geq m \), \( \text{norm}(\text{succ}_{T}(t)) \geq n \).

For \( T \) and \( S \) in \( \mathbb{P} \), \( T \leq S \) if \( T \subset S \).
Lemma 1.1 Let $G$ be a generic filter on $\mathbb{P}$ and $A_G = \bigcap \{ T : T \in G \}$. Then $A_G \subseteq [\omega]^2$ and for any $X \in [\omega]^\omega \cap V$, $X$ doesn’t pair-split $A_G$.

Proof For $X \in [\omega]^\omega$ define a subset $D_X$ of $\mathbb{P}$ by $T \in D_X$ if for all $t \in T \setminus \{ s : s \subset \text{stem}(T) \}$ and $a \in \text{succ}_T(t)$, $a \subset X$ or $a \cap X = \emptyset$. Then for a given $S \in \mathbb{P}$ we can find $T \leq S$ such that for all $t \in T \setminus \{ s : s \subset \text{stem}(T) \}$ and $a \in \text{succ}_T(t)$, $a \subset X$ or $a \cap X = \emptyset$ by 1 and 2 in Proposition 1.1. So $D_X$ is dense. So $X$ doesn’t pair-split $A_G$.

By this lemma, $\mathbb{P}$ adds an infinite subset of $[\omega]^2$ which is not pair-split by any infinite subset of $\omega$ in ground model. Therefore $\omega_2$-stage countable support iteration of $\mathbb{P}$ forces $s_{\text{pair}} = \omega_2$.

From now on we shall prove $\mathbb{P}$ is $\omega_2$-bounding and proper.

For $T \in \mathbb{P}$, let $\text{ess}(T) = \{ t \in T : \text{stem}(T) \subset t \}$. For $T, S \in \mathbb{P}$, $T \leq^* S$ if $T \leq S$ and for all $t \in \text{ess}(T)$, $\text{norm}(\text{succ}_T(t)) \geq \text{norm}(\text{succ}_S(t)) - 1$. $T \leq_m S$ if $T \leq S$ and for all $t \in T$ with $\text{norm}(\text{succ}_S(t)) \leq m$, we have $\text{succ}_S(t) \subset T$.

As [8] we can prove the following lemmata.

Lemma 1.2 If $S \in \mathbb{P}$ and $W \subset S$, then there is some $T \leq^* S$ such that

I. every branch of $T$ meets $W$, or else

II. $T$ is disjoint from $W$.

Proof Let $S^W$ be the set of all $s \in S$ such that there exists $S' \leq^* S_s$ such that every branch of $S'$ meets $W$ where $S_s$ is the set of $t \in S$ comparable to $s$.

If $\text{stem}(S) \in S^W$, then (I) holds. Otherwise we will construct $T \leq^* S$ which satisfies (II).

Suppose $\text{stem}(S) \notin S^W$. Recursively construct $t \in T$ with $|t| = n$. If $n \leq |\text{stem}(T)|$, $t \in T$ with $|t| = n$ if $t \in S$ with $|t| = n$. If $n \geq |\text{stem}(T)|$, assume $t \in T$ with $|t| \leq n$ are given and $t \notin S^W$ for $t \in T$ with $|t| \leq n$. For $t \in T$ with $|t| = n$, let $A^t = \text{succ}_S(t)$, $A_0^t = S^W \cap A^t$ and $A_1^t = A^t \setminus A_0^t$.

By Proposition 1.1 (iii), $\text{norm}(A^t) \geq \text{norm}(A^t) - 1 - 1$. Since $t \notin S^W$, there is no $S' \leq^* S_i$ such that $S'$ holds I. So $\text{norm}(A_0^t) < n$. Hence $\text{norm}(A^t) \geq \text{norm}(A^t) - 1$. Define $t \in T$ with $|t| = n + 1$ if $t \nmid n \in T$ and $n(t) \in A_1^t \setminus A_1^t$. Then for any $t \in T$ with $|t| = n + 1$, $t \notin S^W$.

By construction $T \leq^* S$ and satisfies II. \qed
Lemma 1.3 Let $\dot{\alpha}$ be a $\mathbb{P}$-name for an ordinal. Let $S \in \mathbb{P}$ such that for $t \in S \setminus \{s : s \subseteq \text{stem}(S)\}$, $\text{norm}(\text{succ}_S(t)) > m + 1$. Then there exists $T \leq_m S$ and a finite subset $w$ of ordinals such that $T \models \dot{\alpha} \in w$.

Proof Let $W$ be the set of nodes $s \in S$ such that there exists $S^s \leq_m S_s$ which decides the value $\dot{\alpha}$. We shall prove that there exists $S_1 \leq^* S$ such that every branch of $S_1$ meets $W$. Suppose $S' \leq^* S$ and $S'' \leq S'$ such that $S'' \models \dot{\alpha} = \beta$ for some $\beta$. Then for some $t \in S''$ for each extension $s$ of $t$ in $S''$ satisfies $\text{norm}(\text{succ}_{S''}(s)) > m$. Because $S''_1 \leq_m S_1$ and $S''$ decides $\dot{\alpha}$, $t \in W$. Hence by Lemma 1.2 there exists $S_1 \leq^* S$ which satisfies I in Lemma 1.2.

Let $S_1 \leq^* S$ such that every branch of $S_1$ meets $W$. Let $W_0$ be the set of minimal elements of $W$ in $S_1$. Since $S_1$ is finitely branching, $W_0$ is finite. (Otherwise, by K"{o}nig’s Lemma we can construct infinitely branch which doesn’t meet $W$). For $v \in W_0$ choose $T^v \leq_m S_v$ and $\alpha_v$ such that $T^v \models \dot{\alpha} = \alpha_v$. Put $T = \bigcup_{v \in W_0} T^v$ and $w = \{\alpha_v : v \in W_0\}$. Then $T \leq_m S$ and $T \models \dot{\alpha} \in w$.

Lemma 1.4 If $S \in \mathbb{P}$, $\dot{\alpha}$ be a $\mathbb{P}$-name for an ordinal and $m < \omega$. Then there exists $T \leq_m S$ and a finite set of ordinals $w$ such that $T \models \dot{\alpha} \in w$.

Proof Choose $k \in \omega$ such that for any $s \in S$ with $|s| \geq k, \text{norm}(\text{succ}_S(s)) > m + 1$. For each $s \in S$ with $|s| = k$, apply Lemma 1.3 to $S_s$ pick $T^s \leq_m S_s$ and a finite set of ordinals $w_s$ so that $T_s \models \dot{\alpha} \in w_s$. Put $T = \bigcup_{s \in S, |s| = k} T_s$ and $w = \bigcup_{s \in S, |s| = k} w_s$. Then $T \leq_m S$ and $T \models \dot{\alpha} \in w$. Since $S$ is finitely branching, $w$ is a finite set.

Proof of theorem 1.1 Lemma 1.4 implies that $\mathbb{P}$ is $\omega^\omega$-bounding. Given a $\mathbb{P}$-name for a function $f$ from $\omega$ to $\omega$ and $S \in \mathbb{P}$, we can construct a sequence $\langle T_n : n \in \omega \rangle$ of conditions of $\mathbb{P}$ such that $T_0 = S$, $T_{n+1} \leq_n T_n$ and for each $n \in \omega$, there exists some finite $w_n$ of natural numbers such that $T_n \models f(n) \in w_n$. Then there exists $T \in \mathbb{P}$ such that $T \leq_n T_n$ and $T \models \forall n \in \omega(f(n) \in w_n)$. Put $g(n) = \max\{w_n\}$. Then $T \models \forall n \in \omega(f(n) \leq g(n))$. So $\mathbb{P}$ is $\omega^\omega$-bounding. Also this claim say $\mathbb{P}$ satisfies Baumgartner’s Axiom A. Hence $\mathbb{P}$ is proper.

Hence the $\omega_2$-stage countable support iteration of $\mathbb{P}$ is $\omega^\omega$-bounding by theorem 1.2. Therefore if $V \models CH$, then the $\omega_2$-stage countable support iteration of $\mathbb{P}$ forces $\omega^\omega \cap V$ is a dominating family. So the $\omega_2$-stage countable support iteration of $\mathbb{P}$ forces $\mathfrak{d} = \omega_1$. Hence it is consistent that $\mathfrak{s}_{pair} > \mathfrak{d}$.

7
Since $s \leq \mathfrak{d}$ (see[2]), we have the following corollary.

**Corollary 1.1** It is consistent that $s < s_{pair}$.

## 2 pair-reaping number and unbounded number

To show the consistency of $\tau_{pair} < b$, we shall use the Laver forcing $\mathbb{L}$. $\mathbb{L}$ is defined by $T \in \mathbb{L}$ if $T \subseteq \omega^{<\omega}$ is a tree and for $s \in T$ with $stem(T) \subseteq s$, $|\text{succ}_T(s)| = \aleph_0$. $\mathbb{L}$ is ordered by inclusion. Then $\mathbb{L}$ adds an unbounded real.

**Proposition 2.1** Let $G$ be a $\mathbb{L}$-generic over $V$ and $f_G = \bigcup \{stem(T) : T \in G\}$. Then $f_G \in \omega^\omega$ and $f_G$ dominates for all $g \in \omega^\omega \cap V$.

Therefore if $\mathbb{L}_{\omega_2}$ is $\omega_2$-stage countable support iteration of Laver forcing, then $V^{\mathbb{L}_{\omega_2}} \models b = \mathfrak{c}$.

By using $\omega_2$-stage countable support iteration of Laver forcing, we shall construct ZFC model which satisfies $\tau_{pair} < b$.

**Theorem 2.1** It is consistent $\tau_{pair} < b$.

By proposition 2.1 it is enough $\mathbb{L}$ preserves $\tau_{pair}$. We shall use the Laver property.

**Definition 2.1** [4] A forcing notion $\mathbb{P}$ have the Laver property if for every $H : \omega \rightarrow \omega \in V$

\[ \models \forall f \in (\Pi_{n \in \omega} H(n)) \cap V[\hat{G}] \exists A : \omega \rightarrow \omega^{<\omega} \in V \forall n \in \omega (f(n) \in A(n) \land |A(n)| \leq 2^n) . \]

**Theorem 2.2** [4] The Laver property is preserved under countable support iteration of proper forcing notions.

**Theorem 2.3** [1, pp353] The Laver forcing $\mathbb{L}$ has the Laver property.

So $\mathbb{L}_{\omega_2}$ has the Laver property. If forcing notion $\mathbb{P}$ has the Laver property, then $\mathbb{P}$ has the following good property:

**Lemma 2.1** Let $\mathbb{P}$ be a forcing notion satisfying the Laver property. Then $\models_{\mathbb{P}} \forall X \in V[\hat{G}] \exists A \in V (X$ doesn’t pair-split $A)$.
Corollary 2.1 It is consistent that $\text{trans-add}(\mathcal{N})$, transitive additivity of null ideal (see [1, pp91]). That is, trans-add(\mathcal{N}) is the smallest size of $\leq^*$-bounded family $F \subseteq \omega^\omega$ such that for every $\phi \in \mathcal{S}$ there is $f \in F$ such that for infinitely many $n \in \omega$ such that $f(n) \notin \phi(n)$.

Then the dual inequality to the corollary 2.2 holds.

Proposition 2.2 $s_{\text{pair}} \geq \text{trans-add}(\mathcal{N})$. 

Proof Let $p \in \mathbb{P}$. Let $\Pi = \langle I_n : n \in \omega \rangle$ be an interval partition of $\omega$ such that $|I_n| = 2^{2n} + 1$. Then $\langle X \upharpoonright I_n : n \in \omega \rangle \in \Pi_{n \in \omega} 2^{I_n}$. By the Laver property there exists $q \leq^* p$ such that $\langle A_n : n \in \omega \rangle \in V$ such that $A_n \subseteq 2^{I_n}$, $|A_n| \leq 2^n$ and $q \models \forall n \in \omega (X \upharpoonright I_n \in A_n)$. For each $n \in \omega \{\langle \sigma(k) : \sigma \in A_n \rangle : k \in A_n\}$ is at most $2^{2n}$-many element. But $|I_n| = 2^{2n} + 1$. So there exists $k_0^n$ and $k_1^n$ in $I_n$ such that $k_0^n \neq k_1^n$ and $\langle \sigma(k_0^n) : \sigma \in A_n \rangle = \langle \sigma(k_1^n) : \sigma \in A_n \rangle$. Put $a_n = \{k_0^n, k_1^n\}$ and $A = \{a_n : n \in \omega\} \in V$. Then $q \models X \upharpoonright I_n \cap a_n = \emptyset$ or $a_n \subseteq X \upharpoonright I_n$ for $n \in \omega$. Therefore $q \not\models X$ doesn’t pair-split $A$. 

Proof of Theorem 2.1 Suppose $V \models CH$. By Theorem 2.2 and 2.3 $\mathbb{L}_{\omega_2}$ has the Laver property. By Lemma 2.1 for each $X \in [\omega]^\omega \cap V^{\omega_2}$ there exists an unbounded $A \subseteq [\omega]^2$ such that $V^{\omega_2} \models X$ doesn’t pair-split $A$. So $\{A \subseteq [\omega]^2 : A \text{ unbounded}\} \cap V$ is pair-reaping family. Since $V \models CH$, $\{A \subseteq [\omega]^2 : A \text{ unbounded}\} \cap V$ has the cardinality at most $\omega_1$. Therefore $V^{\omega_2} \models r_{\text{pair}} < b$.

Since $r \geq b$ (see [2]), we have the following corollary.

Corollary 2.1 It is consistent that $r > r_{\text{pair}}$. 


Let $\mathcal{S}$ be the collection of functions $\phi$ from $\omega$ to $[\omega]^{<\omega}$ such that $|\phi(n)| \leq n + 1$. $I$ is the smallest cardinal $\kappa$ such that for every $h \in \omega^{\omega}$ there is a set $\Phi \subseteq \mathcal{S}$ with cardinality $\kappa$ so that, for every $f \in \omega$ with $f(n) < h(n)$ for all $n < \omega$, there is $\phi \in \Phi$ such that for all but finitely many $n \in \omega$ we have $f(n) \notin \phi(n)$.

As the proof of Theorem 2.1 we can prove the following statement.

Corollary 2.2 $r_{\text{pair}} \leq I$. 

Pawlikowski shows that the dual notion to the definition of $I$ is the characterization of $\text{trans-add}(\mathcal{N})$, transitive additivity of null ideal (see [1, pp91]). That is, $\text{trans-add}(\mathcal{N})$ is the smallest size of $\leq^*$-bounded family $F \subseteq \omega^\omega$ such that for every $\phi \in \mathcal{S}$ there is $f \in F$ such that for infinitely many $n \in \omega$ such that $f(n) \notin \phi(n)$.
It is known the following relation between trans-add($\mathcal{N}$) and $\mathfrak{d}$.

**Theorem 2.4** [6] It is consistent that trans-add($\mathcal{N}$) $> \mathfrak{d}$.

By theorem 2.4 and proposition 2.2 it is consistent that $s_{\text{pair}} > \mathfrak{d}$.

## 3 Further results

In this section we mention the development of above results in the paper [3] written by Hrušák, Meza-Alcántara and the author.

Hrušák and Meza-Alcántara study cardinal invariants of ideals on $\omega$ and they define the pair-splitting number and the pair-reaping number independently of the author and they showed the pair-splitting number and the pair-reaping number are described as cardinal invariants of an ideal on $\omega$.

Let $\mathcal{I}$ be an ideal on $\omega$. Define the cardinal invariants associate with $\mathcal{I}$ by

$$
\text{cov}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I} \land \forall I \in \mathcal{I} \exists A \in \mathcal{A} (|A \cap I| = \aleph_0)\}
$$

$$
\text{non}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subset [\omega]^\omega \land \forall I \in \mathcal{I} \exists A \in \mathcal{A} (|A \cap I| < \aleph_0)\}.
$$

**Theorem 3.1** [3] Let $\mathcal{G}_{FC}$ be an ideal on $[\omega]^2$ defined by

$$
\mathcal{G}_{FC} = \{A \subset [\omega]^2 : \chi(\omega, A) < \aleph_0\}
$$

where $\chi(\omega, A) = \min\{k \in \omega : \exists f : \omega \rightarrow k \forall a \in A(|f[a]| = 2)\}$.

Then $\text{non}^*(\mathcal{G}_{FC}) = r_{\text{pair}}$ and $\text{cov}^*(\mathcal{G}_{FC}) = s_{\text{pair}}$.

From now on we assume $2^\omega$ is equipped with product topology and the topology of $\mathcal{P}(\omega)$ is induced by identification of each subset of $\omega$ with its characteristic function.

Then $\mathcal{G}_{FC}$ is an $F_\sigma$-ideal on $[\omega]^2$. As theorem 2.4, 1.1 and theorem 2.1 we can show the following theorem.

**Theorem 3.2** Suppose $\mathcal{I}$ is an $F_\sigma$-ideal on $\omega$.

1. [6] It is consistent that $\mathfrak{d} < \text{cov}^*(\mathcal{I})$.

2. [3] It is consistent that $\mathfrak{b} > \text{non}^*(\mathcal{I})$.

Also the following statement holds as corollary 2.2 and proposition 2.2.
Corollary 3.1 Suppose $\mathcal{I}$ is an $F_\sigma$-ideal.

1. If $\non^*(\mathcal{I}) \neq \omega$, then $\non^*(\mathcal{I}) \leq 1$.

2. If $\non^*(\mathcal{I}) \neq \omega$, then $\cov^*(\mathcal{I}) \geq \transadd(\mathcal{I})$.

So many results in section 1 and 2 follows from theorem 3.2 and corollary 3.1.

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References


