On pair-splitting and pair-reaping pairs of ω

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Abstract

In this paper we investigate variations of splitting number and reaping number, pair-splitting number $\mathfrak{s}_{pair}$, pair-reaping number $\mathfrak{r}_{pair}$. We prove that it is consistent that $\mathfrak{s}_{pair} < \mathfrak{d}$. We also prove it is consistent that $\mathfrak{r}_{pair} > \mathfrak{b}$.

Introduction

The splitting number $\mathfrak{s}$ and the reaping number $\mathfrak{r}$ are cardinal invariants related to the structure $\mathcal{P}(\omega)/fin$.

For $X, Y \in [\omega]^\omega$ we say X splits Y if $X \cap Y$ and $Y \setminus X$ are infinite. We call $\mathcal{S} \subset [\omega]^\omega$ a splitting family if for each $Y \in [\omega]^\omega$, there exists $X \in [\omega]^\omega$ such that X splits Y. The splitting number $\mathfrak{s}$ is the least size of a splitting family.

We call $\mathcal{R}$ a reaping family if for each $X \in [\omega]$, there exists $Y \in [\omega]^\omega$ such that Y is not split by X, that is, $X \cap Y$ is finite or $Y \setminus X$ is finite. The reaping number $\mathfrak{r}$ is the least size of a reaping family.

We shall study variations of splitting number and reaping number, pair-splitting number $\mathfrak{s}_{pair}$ and pair-reaping number $\mathfrak{r}_{pair}$. They are introduced and investigated in [7] to analyze dual-reaping number $\mathfrak{r}_d$ and dual-splitting number $\mathfrak{s}_d$ which are reaping number and splitting number for the structure of all infinite partitions of ω ordered by "almost coarser" $((\omega)^\omega, \leq^*)$ respectively.

We call $A \subset [\omega]^2$ unbounded if for $k \in \omega$, there exists $a \in A$ such that $a \cap k = \emptyset$. For $X \in [\omega]^\omega$ and unbounded $A \subset [\omega]^2$, X pair-splits A if there exist infinitely many $a \in A$ such that $a \cap X \neq \emptyset$ and $a \setminus X \neq \emptyset$. We call $\mathcal{S} \subset [\omega]^\omega$ a pair-splitting family if for each unbounded $A \subset [\omega]^2$, there exists

$X \in \mathcal{S}$ such that X pair-splits A. The pair-splitting number $\mathfrak{s}_{pair}$ is the least size of a pair-splitting family.

We call $\mathcal{R} \subset \mathcal{P}([\omega]^2)$ a pair-reaping family if for each $A \in \mathcal{R}$, A is unbounded and for $X \in [\omega]^\omega$, there exists $A \in \mathcal{R}$ such that X doesn't pair-split A. The pair-reaping number $\mathfrak{r}_{pair}$ is the least size of a pair-reaping family.

In [7] it is proved that there is the following relationship between $\mathfrak{r}_{pair}$, $\mathfrak{s}_{pair}$ and other cardinal invariants.

Proposition 0.1 *1.* $\mathfrak{s}_{pair} \leq non(\mathcal{M}), non(\mathcal{N})$.

2. $\mathfrak{r}_{pair} \geq cov(\mathcal{M}), cov(\mathcal{N})$.

3. $\mathfrak{s}_{pair} \geq \mathfrak{s}$.

4. $\mathfrak{r}_{pair} \leq \mathfrak{r}, \mathfrak{s}_d$.

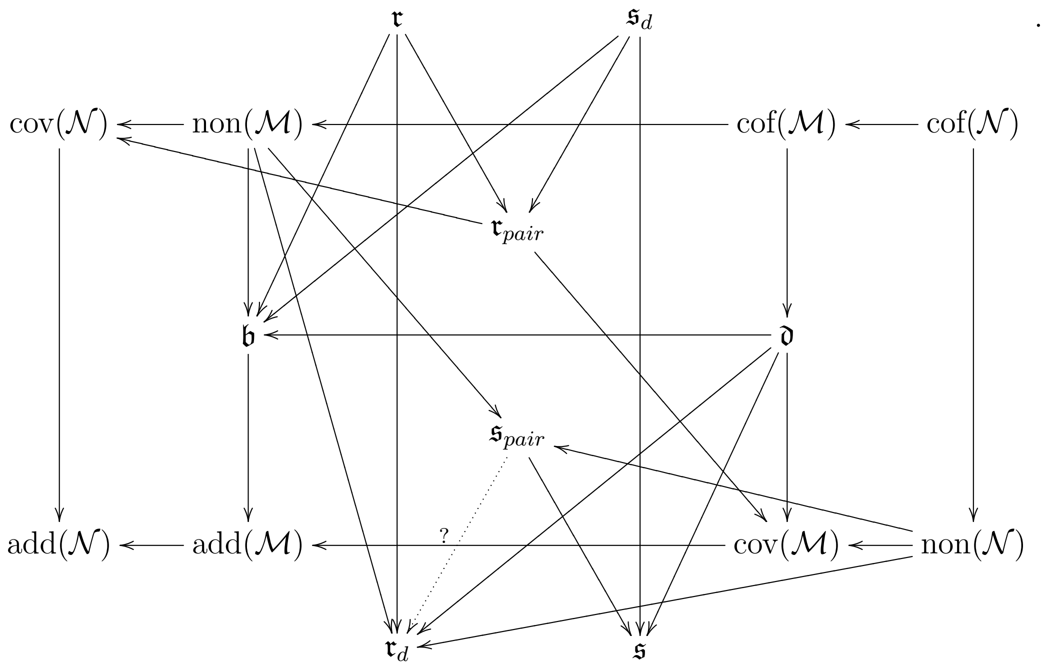
It is not known that $\mathfrak{r}_d \leq \mathfrak{s}_{pair}$ or not.

Question 0.1 $\mathfrak{r}_d \leq \mathfrak{s}_{pair}$*?*

$\mathfrak{s} \leq \mathfrak{d}$ and $\mathfrak{r} \geq \mathfrak{b}$ hold (see in [2]). And Kamo proved the following statement in [7]:

Theorem 0.1 $\mathfrak{r}_d \leq \mathfrak{d}$ *and* $\mathfrak{s}_d \geq \mathfrak{b}$.

So we have the following diagram:



An arrow $\kappa \to \lambda$ denotes the inequality $\kappa \geq \lambda$.

In [7] by using finite support iteration of Hechler forcing, the following consistency results are proved.

Theorem 0.2 *It is consistent that $\mathfrak{s}_{pair} < add(\mathcal{M})$. Dually it is consistent that $\mathfrak{r}_{pair} > cof(\mathcal{M})$.*

$\mathfrak{r}_{pair}$ is a lower bound of $\mathfrak{r}$ and $\mathfrak{s}$ and $\mathfrak{s}_{pair}$ is an upper bound of $\mathfrak{s}$ (and maybe of $\mathfrak{r}_d$). So it is natural to ask the following question.

Question 0.2 $\mathfrak{s}_{pair} \leq \mathfrak{d}$*? Dually* $\mathfrak{r}_{pair} \geq \mathfrak{b}$*?*

In the present paper we shall investigate the relation ship between $\mathfrak{r}_{pair}$ and $\mathfrak{b}$ and the relationship between $\mathfrak{s}_{pair}$ and $\mathfrak{d}$. In section 1 we shall prove the consistency of $\mathfrak{s}_{pair} > \mathfrak{d}$. In section 2 we shall show the consistency of the consistency of $\mathfrak{r}_{pair} < \mathfrak{b}$. In section 3 we mention the development of results in section 1 and 2.

1 pair-splitting number and dominating number

Notation and Definition We present the related notions. We use standard set theoretical conventions and notation. For a set X, X^ω denotes the set of all functions from ω to X. For $f, g \in \omega^\omega$, f dominates g, written $f \leq^* g$, if for all but finitely many $n \in \omega$ $g(n) \leq f(n)$. We call $\mathcal{F}$ a dominating family if for each $g \in \omega^\omega$ there exists $f \in \mathcal{F}$ such that $g \leq^* f$. The dominating number $\mathfrak{d}$ is the least size of a dominating family.

We call $\mathcal{G}$ an unbounded family if for each $f \in \omega^\omega$ there exists $g \in \mathcal{G}$ such that $g \not\leq^* f$, i.e., there exist infinitely many $n \in \omega$ such that $g(n) > f(n)$. The unbounded number $\mathfrak{b}$ is the least size of an unbounded family.

For a set X, $X^{<\omega}$ denote the set of all functions from natural numbers to X.

We call partial ordering $(T, <)$ a tree if the set $\{s \in T : s < t\}$ is well-ordered by $<$. We say T is a tree on X if T is a subtree of $(X^{<\omega}, \subset)$. For a tree T and $t \in T$, $succ_T(t)$ is the set of all immediate successors of t in T. For a tree T, $stem(T)$ is the first element of T which has at least 2-many immediate successors.

Theorem 1.1 *It is consistent $\mathfrak{s}_{pair} > \mathfrak{d}$.*

To prove theorem 1.1, we shall construct a proper forcing notion which enlarges $\mathfrak{s}_{pair}$ and is ω^ω-bounding to show $\mathfrak{d}$ is preserved by the forcing notion.

Definition 1.1 *[4, pp340] A forcing notion $\mathbb{P}$ is ω^ω-bounding if*

$$\Vdash_{\mathbb{P}} \forall f \in \omega^\omega \cap V[G] \exists g \in \omega^\omega \cap V \, (f \leq^* g) \, .$$

The ω^ω-boundingness has the following good property.

Theorem 1.2 *[4, pp341] The countable support iteration of proper ω^ω-bounding forcing notions is ω^ω-bounding.*

To prove theorem 1.1 we shall construct a forcing notion which consists of finitely branching trees on $[\omega]^2$ such that the set of successors of any node carries a norm as [8].

To present the desired forcing notion, we define "norm" for finite subsets of $[\omega]^2$. Let $R(n)$ be a natural number such that if $m \geq R(n)$, then for any function $f : [m]^2 \to 2$ there exists $H \in [m]^n$ such that $|f([H]^2)| = 1$. Then recursively define $l_1 = 3$, $l_{n+1} = \max\{2l_n, R(l_n)\}$. Then for a finite subset A of $[\omega]^2$ norm$(A) \geq n$ if A contains a complete graph with l_n-many vertices.

This norm has the following properties:

Proposition 1.1 *For a finite subset A of $[\omega]^2$,*

1. *$norm(A) \geq 1$ implies for any $X \in [\omega]^\omega$ there exists $a \in A$ such that $a \cap X = \emptyset$ or $a \subset X$.*

2. *Suppose $norm(A) \geq n+1$. For $X \in [\omega]^\omega$ let $A^0_X = \{a \in A : a \cap X = \emptyset\}$ and $A^1_X = \{a \in A : a \subset X\}$. Then $norm(A^0_X) \geq n$ or $norm(A^1_X) \geq n$.*

3. *Suppose $norm(A) \geq n+1$. If $A = A_0 \cup A_1$, then $norm(A_0) \geq n$ or $norm(A_1) \geq n$.*

Proof of proposition 1.1

1. Since $norm(A) \geq 1$, A contains a complete graph $A' \subset A$ with 3-many vertices. Then for any 2-coloring of the vertices of A', there exists an edge whose vertices have the same color. So there exists $a \in A' \subset A$ such that $a \subset X$ or $a \cap X = \emptyset$.

2. Since $norm(A) \geq n+1$, A contain a complete graph A' with l_{n+1}-many vertices. So for each $X \subset \omega$, X contains l_n-many vertices of A' or X doesn't meet l_n-many vertices of A' because $l_{n+1} \geq 2l_n$. Anyway $A^0_X = \{a \in A : a \cap X = \emptyset\}$ or $A^1_X = \{a \in A : a \subset X\}$ contains a complete graph with l_n-many vertices. Therefore $norm(A^0_X) \geq n$ or $norm(A^1_X) \geq n$.

3. Since $norm(A) \geq n+1$, A contain a complete graph A' with l_{n+1}-many vertices. Define $f : A' \to 2$ by $f(a) = i$ if $a \in A_i$ for $i < 2$. Since $l_{n+1} \geq R(l_n)$, there exists a complete graph $A^* \subset A'$ which has l_n-many vertices of A' and $|f[A^*]| = 1$. So $A^* \subset A_0$ or $A^* \subset A_1$. Hence $norm(A_0) \geq n$ or $norm(A_1) \geq n$. $\square$

Then let $\mathbb{P}$ be the set of perfect trees such that

1. T is a finitely branching tree on $[\omega]^2$,

2. for any branch of T and $n \in \omega$ there exist $m \geq n$ such that whenever $t \in T$ with $|t| \geq m$, $norm(succ_T(t)) \geq n$.

For T and S in $\mathbb{P}$, $T \leq S$ if $T \subset S$.

Lemma 1.1 *Let G be a generic filter on $\mathbb{P}$ and $A_G = \bigcap\{T : T \in G\}$. Then $A_G \subset [\omega]^2$ and for any $X \in [\omega]^\omega \cap V$, X doesn't pair-split A_G.*

Proof For $X \in [\omega]^\omega$ define a subset D_X of $\mathbb{P}$ by $T \in D_X$ if for all $t \in T \setminus \{s : s \subset \mathrm{stem}(T)\}$ and $a \in \mathrm{succ}_T(t)$, $a \subset X$ or $a \cap X = \emptyset$. Then for a given $S \in \mathbb{P}$ we can find $T \leq S$ such that for all $t \in T \setminus \{s : s \subset \mathrm{stem}(T)\}$ and $a \in \mathrm{succ}_T(t)$, $a \subset X$ or $a \cap X = \emptyset$ by 1 and 2 in Proposition 1.1. So D_X is dense. So X doesn't pair-split A_G.

$\square$

By this lemma, $\mathbb{P}$ adds an infinite subset of $[\omega]^2$ which is not pair-split by any infinite subset of ω in ground model. Therefore ω_2-stage countable support iteration of $\mathbb{P}$ forces $\mathfrak{s}_{pair} = \omega_2$.

From now on we shall prove $\mathbb{P}$ is ω^ω-bounding and proper.

For $T \in \mathbb{P}$, let $\mathrm{ess}(T) = \{t \in T : \mathrm{stem}(T) \subset t\}$. For $T, S \in \mathbb{P}$, $T \leq^* S$ if $T \leq S$ and for all $t \in \mathrm{ess}(T)$, $\mathrm{norm}(\mathrm{succ}_T(t)) \geq \mathrm{norm}(\mathrm{succ}_S(t)) - 1$. $T \leq_m S$ if $T \leq S$ and for all $t \in T$ with $\mathrm{norm}(\mathrm{succ}_S(t)) \leq m$, we have $\mathrm{succ}_S(t) \subset T$.

As [8] we can prove the following lemmata.

Lemma 1.2 *If $S \in \mathbb{P}$ and $W \subset S$, then there is some $T \leq^* S$ such that*

I. every branch of T meets W, or else

II. T is disjoint from W.

Proof Let S^W be the set of all $s \in S$ such that there exists $S' \leq^* S_s$ such that every branch of S' meets W where S_s is the set of $t \in S$ comparable to s.

If $\mathrm{stem}(S) \in S^W$, then (I) holds. Otherwise we will construct $T \leq^* S$ which satisfies (II).

Suppose $\mathrm{stem}(S) \notin S^W$. Recursively construct $t \in T$ with $|t| = n$. If $n \leq |\mathrm{stem}(T)|$, $t \in T$ with $|t| = n$ if $t \in S$ with $|t| = n$. If $n \geq |\mathrm{stem}(T)|$, assume $t \in T$ with $|t| \leq n$ are given and $t \notin S^W$ for $t \in T$ with $|t| \leq n$. For $t \in T$ with $|t| = n$, let $A^t = \mathrm{succ}_S(t)$, $A^t_0 = S^W \cap A^t$ and $A^t_1 = A^t \setminus A^t_0$. By Proposition 1.1 (iii), $\mathrm{norm}(A^t_i) \geq \mathrm{norm}(A^t) - 1$ for some $i < 2$. Since $t \notin S^W$, there is no $S' \leq^* S_t$ such that S' holds I. So $\mathrm{norm}(A^t_0) < n$. Hence $\mathrm{norm}(A^t_1) \geq \mathrm{norm}(A^t) - 1$. Define $t \in T$ with $|t| = n+1$ if $t \restriction n \in T$ and $t(n) \in A_1^{t\restriction n}$. Then for any $t \in T$ with $|t| = n+1$, $t \notin S^W$.

By construction $T \leq^* S$ and satisfies II. $\square$

Lemma 1.3 *Let $\dot{\alpha}$ be a $\mathbb{P}$-name for an ordinal. Let $S \in \mathbb{P}$ such that for $t \in S \setminus \{s : s \subset stem(S)\}$, $norm(succ_S(t)) > m + 1$. Then there exists $T \leq_m S$ and a finite subset w of ordinal such that $T \Vdash \dot{\alpha} \in w$.*

Proof Let W be the set of nodes $s \in S$ such that there exists $S^s \leq_m S_s$ which decides the value $\dot{\alpha}$.

We shall prove that there exists $S_1 \leq^* S$ such that every branch of S_1 meets W. Suppose $S' \leq^* S$ and $S'' \leq S'$ such that $S'' \Vdash \dot{\alpha} = \beta$ for some β. Then for some $t \in S''$ for each extension s of t in S'' satisfies $\text{norm}(\text{succ}_{S''}(s)) > m$. Because $S''_t \leq_m S_t$ and S'' decides $\dot{\alpha}$, $t \in W$. Hence by Lemma 1.2 there exists $S_1 \leq^* S$ which satisfies I in Lemma 1.2.

Let $S_1 \leq^* S$ such that every branch of S_1 meets W. Let W_0 be the set of minimal elements of W in S_1. Since S_1 is finitely branching, W_0 is finite. (Otherwise, by Köning's Lemma we can construct infinitely branch which doesn't meet W). For $v \in W_0$ choose $T^v \leq_m S_v$ and α_v such that $T^v \Vdash \dot{\alpha} = \alpha_v$. Put $T = \bigcup_{v \in W_0} T^v$ and $w = \{\alpha_v : v \in W_0\}$. Then $T \leq_m S$ and $T \Vdash \dot{\alpha} \in w$.

□

Lemma 1.4 *If $S \in \mathbb{P}$, $\dot{\alpha}$ be a $\mathbb{P}$-name for an ordinal and $m < \omega$. Then there exists $T \leq_m S$ and a finite set of ordinals w such that $T \Vdash \dot{\alpha} \in w$.*

Proof Choose $k \in \omega$ such that for any $s \in S$ with $|s| \geq k$ $\text{norm}(\text{succ}_S(s)) > m + 1$. For each $s \in S$ with $|s| = k$, apply Lemma 1.3 to S_s pick $T^s \leq_m S_s$ and a finite set of ordinals w_s so that $T_s \Vdash \dot{\alpha} \in w_s$. Put $T = \bigcup_{s \in S, |s|=k} T_s$ and $w = \bigcup_{s \in S \cap \omega^k} w_s$. Then $T \leq_m S$ and $T \Vdash \dot{\alpha} \in w$. Since S is finitely branching, w is a finite set. □

Proof of theorem 1.1 Lemma 1.4 implies that $\mathbb{P}$ is ω^ω-bounding. Given a $\mathbb{P}$-name for a function $\dot{f}$ from ω to ω and $S \in \mathbb{P}$, we can construct a sequence $\langle T_n : n \in \omega \rangle$ of conditions of $\mathbb{P}$ such that $T_0 = S$, $T_{n+1} \leq_n T_n$ and for each $n \in \omega$, there exists some finite w_n of natural numbers such that $T_n \Vdash \dot{f}(n) \in w_n$. Then there exists $T \in \mathbb{P}$ such that $T \leq_n T_n$ and $T \Vdash \forall n \in \omega(\dot{f}(n) \in w_n)$. Put $g(n) = \max\{w_n\}$. Then $T \Vdash \forall n \in \omega(\dot{f}(n) \leq g(n))$. So $\mathbb{P}$ is ω^ω-bounding. Also this claim say $\mathbb{P}$ satisfies Baumgartner's Axiom A. Hence $\mathbb{P}$ is proper.

Hence the ω_2-stage countable support iteration of $\mathbb{P}$ is ω^ω-bounding by theorem 1.2. Therefore if $V \models CH$, then the ω_2-stage countable support iteration of $\mathbb{P}$ forces $\omega^\omega \cap V$ is a dominating family. So the ω_2-stage countable support iteration of $\mathbb{P}$ forces $\mathfrak{d} = \omega_1$. Hence it is consistent that $\mathfrak{s}_{pair} > \mathfrak{d}$. □

Since $\mathfrak{s} \leq \mathfrak{d}$ (see[2]), we have the following corollary.

Corollary 1.1 *It is consistent that* $\mathfrak{s} < \mathfrak{s}_{pair}$.

2 pair-reaping number and unbounded number

To show the consistency of $\mathfrak{r}_{pair} < \mathfrak{b}$, we shall use the Laver forcing $\mathbb{L}$. $\mathbb{L}$ is defined by $T \in \mathbb{L}$ if $T \subset \omega^{<\omega}$ is a tree and for $s \in T$ with $stem(T) \subset s$, $|succ_T(s)| = \aleph_0$. $\mathbb{L}$ is ordered by inclusion. Then $\mathbb{L}$ adds an unbounded real.

Proposition 2.1 *Let G be a $\mathbb{L}$-generic over V and $f_G = \bigcup\{stem(T) : T \in G\}$. Then $f_G \in \omega^\omega$ and f_G dominates for all $g \in \omega^\omega \cap V$.*

Therefore if $\mathbb{L}_{\omega_2}$ is ω_2-stage countable support iteration of Laver forcing, then $V^{\mathbb{L}_{\omega_2}} \models \mathfrak{b} = \mathfrak{c}$.

By using ω_2-stage countable support iteration of Laver forcing, we shall construct ZFC model which satisfies $\mathfrak{r}_{pair} < \mathfrak{b}$.

Theorem 2.1 *It is consistent* $\mathfrak{r}_{pair} < \mathfrak{b}$.

By proposition 2.1 it is enough $\mathbb{L}$ preserves $\mathfrak{r}_{pair}$. We shall use the Laver property.

Definition 2.1 *[4] A forcing notion $\mathbb{P}$ have the Laver property if for every $H : \omega \to \omega \in V$*

$$\Vdash \forall f \in (\Pi_{n\in\omega} H(n)) \cap V[\dot{G}] \exists A : \omega \to \omega^{<\omega} \in V \forall n \in \omega \left(f(n) \in A(n) \wedge |A(n)| \leq 2^n \right).$$

Theorem 2.2 *[4] The Laver property is preserved under countable support iteration of proper forcing notions.*

Theorem 2.3 *[1, pp353] The Laver forcing $\mathbb{L}$ has the Laver property.*

So $\mathbb{L}_{\omega_2}$ has the Laver property. If forcing notion $\mathbb{P}$ has the Laver property, then $\mathbb{P}$ has the following good property:

Lemma 2.1 *Let $\mathbb{P}$ be a forcing notion satisfying the Laver property. Then $\Vdash_{\mathbb{P}} \forall \dot{X} \in V[\dot{G}] \exists A \in V(\dot{X}$ doesn't pair-split $A)$.*

Proof Let $p \in \mathbb{P}$. Let $\Pi = \langle I_n : n \in \omega \rangle$ be an interval partition of ω such that $|I_n| = 2^{2^n} + 1$. Then $\langle \dot{X} \restriction I_n : n \in \omega \rangle \in \Pi_{n \in \omega} 2^{I_n}$. By the Laver property there exists $q \leq_{\mathbb{P}} p$ such that $\langle A_n : n \in \omega \rangle \in V$ such that $A_n \subset 2^{I_n}$, $|A_n| \leq 2^n$ and $q \Vdash \forall n \in \omega (\dot{X} \restriction I_n \in A_n)$. For each $n \in \omega$ $\{\langle \sigma(k) : \sigma \in A_n \rangle : k \in A_n\}$ is at most 2^{2^n}-many element. But $|I_n| = 2^{2^n} + 1$. So there exists k_0^n and k_1^n in I_n such that $k_0^n \neq k_1^n$ and $\langle \sigma(k_0^n) : \sigma \in A_n \rangle = \langle \sigma(k_1^n) : \sigma \in A_n \rangle$. Put $a_n = \{k_0^n, k_1^n\}$ and $A = \{a_n : n \in \omega\} \in V$. Then $q \Vdash \dot{X} \restriction I_n \cap a_n = \emptyset$ or $a_n \subset \dot{X} \restriction I_n$ for $n \in \omega$. Therefore $q \Vdash \dot{X}$ doesn't pair-split A. □

Proof of theorem 2.1 Suppose $V \models CH$. By theorem 2.2 and 2.3 $\mathbb{L}_{\omega_2}$ has the Laver property. By lemma 2.1 for each $X \in [\omega]^\omega \cap V^{\mathbb{L}_{\omega_2}}$ there exists an unbounded $A \subset [\omega]^2$ such that $V^{\mathbb{L}_{\omega_2}} \models X$ doesn't pair-split A. So $\{A \subset [\omega]^2 : A \text{ unbounded}\} \cap V$ is pair-reaping family. Since $V \models CH$, $\{A \subset [\omega]^2 : A \text{ unbounded}\} \cap V$ has the cardinality at most ω_1. Therefore $V^{\mathbb{L}_{\omega_2}} \models \mathfrak{r}_{pair} < \mathfrak{b}$.

□

Since $\mathfrak{r} \geq \mathfrak{b}$ (see[2]), we have the following corollary.

Corollary 2.1 *It is consistent that* $\mathfrak{r} > \mathfrak{r}_{pair}$.

In [5] Masaru Kada introduces a cardinal invariant associated with the Laver property.

Let $\mathcal{S}$ be the collection of functions ϕ from ω to $[\omega]^{<\omega}$ such that $|\phi(n)| \leq n+1$. $\mathfrak{l}$ is the smallest cardinal κ such that for every $h \in \omega^\omega$ there is a set $\Phi \subset \mathcal{S}$ with cardinality κ so that, for every $f \in \omega$ with $f(n) < h(n)$ for all $n < \omega$, there is $\phi \in \Phi$ such that for all but finitely many $n \in \omega$ we have $f(n) \in \phi(n)$.

As the proof of theorem 2.1 we can prove the following statement.

Corollary 2.2 $\mathfrak{r}_{pair} \leq \mathfrak{l}$.

Pawlikowski shows that the dual notion to the definition of $\mathfrak{l}$ is the characterization of trans-add($\mathcal{N}$), transitive additivity of null ideal (see [1, pp91]). That is, trans-add($\mathcal{N}$) is the smallest size of $\leq^*$-bounded family $F \subset \omega^\omega$ such that for every $\phi \in \mathcal{S}$ there is $f \in F$ such that for infinitely many $n \in \omega$ such that $f(n) \notin \phi(n)$.

Then the dual inequality to the corollary 2.2 holds.

Proposition 2.2 $\mathfrak{s}_{pair} \geq$ *trans-add*$(\mathcal{N})$.

It is known the following relation between trans-add($\mathcal{N}$) and $\mathfrak{d}$.

Theorem 2.4 *[6] It is consistent that trans-add($\mathcal{N}$) $> \mathfrak{d}$.*

By theorem 2.4 and proposition 2.2 it is consistent that $\mathfrak{s}_{pair} > \mathfrak{d}$.

3 Further results

In this section we mention the development of above results in the paper [3] written by Hrušák, Meza-Alcántara and the author.

Hrušák and Meza-Alcántara study cardinal invariants of ideals on ω and they define the pair-splitting number and the pair-reaping number independently of the author and they showed the pair-splitting number and the pair-reaping number are described as cardinal invariants of an ideal on ω.

Let $\mathcal{I}$ be an ideal on ω. Define the cardinal invariants associate with $\mathcal{I}$ by

$$\begin{aligned} \mathrm{cov}^*(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I} \wedge \forall I \in \mathcal{I} \exists A \in \mathcal{A}(|A \cap I| = \aleph_0)\} \\ \mathrm{non}^*(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subset [\omega]^\omega \wedge \forall I \in \mathcal{I} \exists A \in \mathcal{A}\,(|A \cap I| < \aleph_0)\}. \end{aligned}$$

Theorem 3.1 *[3] Let $\mathcal{G}_{FC}$ be an ideal on $[\omega]^2$ defined by*

$$\mathcal{G}_{FC} = \{A \subset [\omega]^2 : \chi(\omega, A) < \aleph_0\}$$

where $\chi(\omega, A) = \min\{k \in \omega : \exists f : \omega \to k \forall a \in A(|f[a]| = 2)\}$.

Then non$^*(\mathcal{G}_{FC}) = \mathfrak{r}_{pair}$ *and cov*$^*(\mathcal{G}_{FC}) = \mathfrak{s}_{pair}$.

From now on we assume 2^ω is equipped with product topology and the topology of $\mathcal{P}(\omega)$ is induced by identification of each subset of ω with its characteristic function.

Then $\mathcal{G}_{FC}$ is an F_σ-ideal on $[\omega]^2$. As theorem 2.4, 1.1 and theorem 2.1 we can show the following theorem.

Theorem 3.2 *Suppose $\mathcal{I}$ is an F_σ-ideal on ω.*

1. *[6] It is consistent that $\mathfrak{d} <$ cov$^*(\mathcal{I})$.*
2. *[3] It is consistent that $\mathfrak{b} >$ non$^*(\mathcal{I})$.*

Also the following statement holds as corollary 2.2 and proposition 2.2.

Corollary 3.1 *Suppose $\mathcal{I}$ is an F_σ-ideal.*

1. *If non$^*(\mathcal{I}) \neq \omega$, then non$^*(\mathcal{I}) \leq \mathfrak{l}$.*
2. *If non$^*(\mathcal{I}) \neq \omega$, then cov$^*(\mathcal{I}) \geq$ trans-add$(\mathcal{I})$.*

So many results in section 1 and 2 follows from theorem 3.2 and corollary 3.1.

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