PAIR-SPLITTING, PAIR-REAPING AND CARDINAL INVARIANTS OF $F_\sigma$-IDEALS

MICHAEL HRUŠÁK, DAVID MEZA-ALCÁNTARA, AND HIROAKI MINAMI

Abstract. We investigate the pair-splitting number $s_{\text{pair}}$ which is a variation of splitting number, pair-reaping number $r_{\text{pair}}$ which is a variation of reaping number and cardinal invariants of ideals on $\omega$. We also study cardinal invariants of $F_\sigma$ ideals and their upper bounds and lower bounds. As an application, we answer a question of S. Solecki by showing that the ideal of finitely chromatic graphs is not locally Katětov-minimal among ideals not satisfying Fatou’s lemma.

Introduction

The splitting number $s$ and the reaping number $r$ are cardinal invariants which play important role when we study $P(\omega)/\text{fin}$.

For $X, Y \in [\omega]^{\omega}$ we say $X$ splits $Y$ if both $X \cap Y$ and $Y \setminus X$ are infinite. We call $S \subset [\omega]^{\omega}$ a splitting family if for each $Y \in [\omega]^{\omega}$, there exists $X \in S$ such that $X$ splits $Y$. The splitting number $s$ is the least size of a splitting family.

We call $R$ a reaping family if for each $X \in [\omega]^{\omega}$, there exists $Y \in R$ such that $Y$ is not split by $X$, that is, $X \cap Y$ is finite or $Y \setminus X$ is finite.

The reaping number $r$ is the least size of a reaping family.

We shall study variations of splitting number and of reaping number and study cardinal invariants of ideals of $\omega$.

The pair-reaping number $r_{\text{pair}}$ and the pair-splitting number $s_{\text{pair}}$ are introduced in two different contexts with the same definition independently.

One is motivated by the investigation of the dual-reaping number $r_d$ and the dual-splitting number $s_d$ which are reaping number and

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splitting number for the structure of all infinite partitions of \( \omega \) ordered by “almost coarser” \( ((\omega)^\omega, \leq^*) \) respectively.

We call \( A \subset [\omega]^2 \) unbounded if for \( k \in \omega \), there exists \( a \in A \) such that \( a \cap k = \emptyset \). For \( X \in [\omega]^\omega \) and unbounded \( A \subset [\omega]^2 \), \( X \) pair-splits \( A \) if there exist infinitely many \( a \in A \) such that \( a \cap X \neq \emptyset \) and \( a \setminus X \neq \emptyset \).

We call \( S \subset [\omega]^\omega \) a pair-splitting family if for each unbounded \( A \subset [\omega]^2 \), there exists \( X \in S \) such that \( X \) pair-splits \( A \). The pair-splitting number \( s_{\text{pair}} \) is the least size of a pair-splitting family.

We call \( R \subset P([\omega]^2) \) a pair-reaping family if for each \( A \in R \), \( A \) is unbounded and for \( X \in [\omega]^\omega \), there exists \( A \in R \) such that \( X \) does not pair-split \( A \), that is, for all but finitely many \( a \in A \), \( a \cap X = \emptyset \) or \( a \subset X \). The pair-reaping number \( r_{\text{pair}} \) is the least size of a pair-reaping family.

In [13] it is proved that there is the following relationship between \( r_{\text{pair}}, s_{\text{pair}} \) and other cardinal invariants.

**Proposition 0.1.**

1. \( s_{\text{pair}} \leq \text{non}(\mathcal{M}), \text{non}(\mathcal{N}) \).
2. \( r_{\text{pair}} \geq \text{cov}(\mathcal{M}), \text{cov}(\mathcal{N}) \).
3. \( s_{\text{pair}} \geq s \).
4. \( r_{\text{pair}} \leq r, s_d \).

It is not known whether \( r_d \leq s_{\text{pair}} \) or not.

For \( G \subset \omega^\omega \), we call \( G \) a dominating family if for each \( f \in \omega^\omega \), there exists \( g \in G \) such that for all but finitely many \( n \in \omega \), \( f(n) \leq g(n) \), denoted by \( f \leq^* g \). The dominating number \( d \) is the least size of a dominating family.

For \( G \subset \omega^\omega \), we call \( G \) an unbounded family if for each \( f \in \omega^\omega \), there exists \( g \in G \) such that \( g \not\leq^* f \), that is, there exist infinitely many \( n \in \omega \) such that \( g(n) > f(n) \). The unbounded number \( b \) is the least size of an unbounded family.

\( s \leq d \) and \( r \geq b \) hold (see in [3]). Kamo proved the following statement in [13]:

**Theorem 0.2.** \( r_d \leq d \) and \( s_d \geq b \).

So we have the following diagram:
An arrow $\kappa \rightarrow \lambda$ denotes the inequality $\kappa \geq \lambda$.

In [13] by using finite support iteration of Hechler forcing, the following consistency results are proved.

**Theorem 0.3.** It is consistent that $s_{pair} < \text{add}(\mathcal{M})$. Dually it is consistent that $r_{pair} > \text{cof}(\mathcal{M})$.

$r_{pair}$ is a lower bound of $r$ and $s_d$, and $s_{pair}$ is an upper bound of $s$ (and maybe of $r_d$). So it is natural to ask the question whether $s_{pair} \leq \delta$ or not and whether $r_{pair} \geq \delta$ or not. In [14] the consistency of $s_{pair} > \delta$ and of $r_{pair} < \delta$ are shown and an upper bound of $s_{pair}$ and a lower bound of $r_{pair}$ are given.

The other source of motivation stems from the study of Borel ideals on $\omega$.

For a set $X$, we call $\mathcal{I} \subset \mathcal{P}(X)$ an **ideal on $X$** if $\mathcal{I}$ satisfies the following:

1. for $A, B \in \mathcal{I}$, $A \cup B \in \mathcal{I}$,
2. for $A, B \subset X$, $A \subset B$ and $B \in \mathcal{I}$ implies $A \in \mathcal{I}$ and
3. $X \not\in \mathcal{I}$.

In this paper we assume that all ideals on $X$ contain all finite subsets of $X$. We say an ideal $\mathcal{I}$ on $\omega$ is **tall** if for each $X \in [\omega]^\omega$ there exists $I \in \mathcal{I}$ such that $I \cap X$ is infinite.
If $\mathcal{I}$ is an ideal on $\omega$ and $Y \in [\omega]^\omega$, we denote by $\mathcal{I} \upharpoonright Y$ the ideal $\{I \cap Y : I \in \mathcal{I}\}$ on $Y$.

The topology of $\mathcal{P}(\omega)$ is induced by identifying each subset of $\omega$ with its characteristic function, where $2^\omega$ is equipped with the product topology of the discrete topology of $2 = \{0,1\}$. We call $\mathcal{I}$ a Borel ideal on $\omega$ if $\mathcal{I}$ is an ideal on $\omega$ and $\mathcal{I}$ is Borel in this topology.

Let $\mathcal{I}$ be a tall ideal on $\omega$. Then the uniformity number of $\mathcal{I}$, denoted by $\non^*(\mathcal{I})$ and the covering number of $\mathcal{I}$, denoted by $\cov^*(\mathcal{I})$ are given by

$$\non^*(\mathcal{I}) = \min\{|A| : A \subseteq [\omega]^\omega \land (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(|A \cap I| < \aleph_0)\},$$

$$\cov^*(\mathcal{I}) = \min\{|A| : A \subseteq \mathcal{I} \land (\forall X \in [\omega]^\omega)(\exists A \in \mathcal{A})(|X \cap A| = \aleph_0)\}.$$ 

The (pre)orderings on the family of ideals are crucial in describing some properties of ideals on $\omega$. For example, Cohen-destructibility of an ideal $\mathcal{I}$ on $\omega$ is equivalent to the statement $\mathcal{I}$ is smaller than the nowhere dense ideal in the Katětov order ([8, 6]).

Suppose $\mathcal{I}$ and $\mathcal{J}$ are ideals on $\omega$. Then $\mathcal{I} \leq_K \mathcal{J}$ if there exists a function $f : \omega \to \omega$ such that for each $I \in \mathcal{I}$, $f^{-1}[I] \in \mathcal{J}$. We call this ordering Katětov order.

When we investigate the Katětov order, the uniformity number of ideals and the covering number of ideals are significant.

**Proposition 0.4.** If $\mathcal{I} \leq_K \mathcal{J}$, then $\non^*(\mathcal{I}) \leq \non^*(\mathcal{J})$ and $\cov^*(\mathcal{I}) \geq \cov^*(\mathcal{J})$.

In the study, the Katětov order between the finite chromatic ideal on $[\omega]^2$, denoted by $G_{FC}$, which is an $F_\sigma$-ideal, and other Borel ideals is investigated. The pair-reaping number and the pair-splitting number are introduced as other descriptions of the uniformity number of $G_{FC}$ and the covering number of $G_{FC}$.

The encounter of these two different studies produces more general results.

In the present paper we shall investigate the relationship between $r_{\text{pair}}$, $s_{\text{pair}}$, cardinal invariants of ideals on $\omega$ and other classical cardinal invariants.

In Section 1 we shall show $r_{\text{pair}} = r_n$ for $n \geq 3$ and $s_{\text{pair}} = s_n$ for $n \geq 3$. In Section 2 we shall investigate the relation between $s_{\text{pair}}$, $r_{\text{pair}}$ and cardinal invariants of the ideal of finitely chromatic graphs. In Section 3 we shall show the consistency of $\non^*(\mathcal{I}) < b$ for $F_\sigma$-ideals on $\omega$. In Section 4 we shall answer a question by Solecki from [15].

1. $n$-SPLITTING NUMBER AND $n$-REAPING NUMBER

In this section we shall show $s_{\text{pair}} = s_n$ and $r_{\text{pair}} = r_n$ for $n \geq 2$. 
We call $A \subset [\omega]^n$ unbounded if for $k \in \omega$ there exists $a \in A$ such that $a \cap k = \emptyset$.

For $X \in [\omega]^{<\omega}$ and unbounded $A \subset [\omega]^n$, $X$ $n$-splits $A$ if there exist infinitely many $a \in A$ such that $a \cap X \neq \emptyset$ and $a \setminus X \neq \emptyset$. We call $\mathcal{S} \subset [\omega]^{<\omega}$ an $n$-splitting family if for each unbounded $A \subset [\omega]^n$ there exists $X \in \mathcal{S}$ such that $X$ $n$-splits $A$. The $n$-splitting number $s_n$ is the least size of an $n$-splitting family.

We call $\mathcal{R} \subset \mathcal{P}([\omega]^n)$ an $n$-reaping family if for each $A \in \mathcal{R}$, $A$ is unbounded and for $X \in [\omega]^{<\omega}$, there exists $A \in \mathcal{R}$ such that $X$ does not $n$-split $A$, that is, for all but finitely many $a \in A$, $a \cap X = \emptyset$ or $a \subset X$. The $n$-reaping number $r_n$ is the least size of an $n$-reaping family. So $s_{pair} = s_2$ and $r_{pair} = r_2$.

The following relations hold between $s_n$ for $n \geq 2$ and between $r_n$ for $n \geq 2$.

**Proposition 1.1.**

1. $s_{pair} = s_2 \geq s_3 \geq \ldots \geq s_n \geq \ldots$ and $s_n \geq s$ for $n \geq 2$.

2. $r_{pair} = r_2 \leq r_3 \leq \ldots \leq r_n \leq \ldots$ and $r \geq r_n$ for $n \geq 2$.

**Proof.** Fix $n \geq 2$. Let $\mathcal{S}$ be an $n$-splitting family of cardinality $s_n$. For an unbounded $A \subset [\omega]^{n+1}$, let $A^* \subset [\omega]^n$ be the collection of the initial $n$-many elements of an element of $A$. Then there exists $X \in \mathcal{S}$ which $n$-splits $A^*$. So there exist infinitely many $a \in A^*$ such that $a \cap X \neq \emptyset$ and $a \setminus X \neq \emptyset$. Since for each $a \in A^*$, there exists $a^* \in A$ such that $a \subset a^*$, there exist infinitely many $a^* \in A$ such that $a^* \cap X \neq \emptyset$ and $a^* \setminus X \neq \emptyset$. So $\mathcal{S}$ is an $(n+1)$-splitting family. Hence $s_{n+1} \leq s_n$.

We shall show $s_n \geq s$. Let $\mathcal{S}$ be an $n$-splitting family of cardinality $s_n$. For $Y \in [\omega]^{<\omega}$, fix an infinite subset $A_Y$ of $[Y]^n$ whose elements are pairwise disjoint. Then $A_Y$ is unbounded. Pick $X \in \mathcal{S}$ which $n$-splits $A_Y$. So there exist infinitely many $a \in A_Y$ such that $a \cap X \neq \emptyset$ and $a \setminus X \neq \emptyset$. Hence $|X \cap Y| = |Y \setminus X| = \omega$. So $X$ splits $Y$. Therefore $\mathcal{S}$ is a splitting family. So $s_n \geq s$.

We shall show $r_n \leq r_{n+1}$. Let $\mathcal{R}$ be an $(n+1)$-reaping family of cardinality $r_{n+1}$. Put $\mathcal{R}^*$ the set of the initial $n$-many elements of an element of $\mathcal{R}$. Given $X \in [\omega]^{<\omega}$, pick $A \in \mathcal{R}$ such that for all but finitely many $a \in A$, $a \cap X = \emptyset$ or $a \subset X$. Put $A^*$ the set of initial segments of size $n$ of elements of $A$. Then for all but finite many $a^* \in A^*$, $a^* \cap X = \emptyset$ or $a^* \subset X$. So $\mathcal{R}^*$ is an $n$-reaping family of cardinality $r_{n+1}$. Hence $r_n \leq r_{n+1}$.

We shall prove $r \geq r_n$. Let $\mathcal{R}$ be a reaping family of cardinality $r$. For each $Y \in \mathcal{R}$, fix an infinite subset $A_Y$ of $[Y]^n$ whose elements are pairwise disjoint. $\mathcal{R}^*$ is the collection of $A_Y$ with $Y \in \mathcal{R}$.
For $X \in [\omega]^\omega$, pick $Y \in \mathcal{R}$ such that $Y \setminus X$ is finite or $X \cap Y$ is finite. Then for all but finitely many $a \in A_Y$, $a \subset X$ or for all but finitely many $a \in A_Y$, $a$ does not meet $X$. So $\mathcal{R}^*$ is an $n$-reaping family of cardinality $r$. Therefore $r_n \leq r$. □

Proposition 1.1 was proved as early as $r_n$ and $s_n$ were defined. However, it was not known whether $s_{\text{pair}} = s_n$ for $n \geq 3$ or not.

Between $r_{\text{pair}}$ and $r_n$, we can prove the following statement.

**Proposition 1.2.** $r_{\text{pair}} = r_n$ for $n \geq 3$.

**Proof.** We shall prove that $r_{\text{pair}} \geq r_4$. Let $\mathcal{R}$ be a pair-reaping family of cardinality $r_{\text{pair}}$. Without loss of generality we can assume $\mathcal{R}$ is closed under finite changes, i.e., if $C \in \mathcal{R}$ and $|D \triangle C| < \aleph_0$ then $D \in \mathcal{R}$; and $A$ is pairwise disjoint for each $A \in \mathcal{R}$. Let $e_A$ be a bijection from $A$ to $\omega$. Put

$$\mathcal{R}^* = \{ C : \exists A, B \in \mathcal{R} \left( C = \{ \bigcup e_A^{-1}[b] : b \in B \} \right) \}.$$ 

We shall prove such $\mathcal{R}^*$ is a 4-reaping family. Let $X \in [\omega]^\omega$. Then we can find $A \in \mathcal{R}$ such that for all $a \in A$, $a \cap X = \emptyset$ or $a \subset X$. Then define $Y_{A,X} \subset \omega$ so that

$$n \in Y_{A,X} \text{ if } \left\{ \begin{array}{ll} e_A^{-1}(n) & \subset X \quad \text{if } \exists \infty m \in \omega \left( e_A^{-1}(m) \subset X \right), \\ e_A^{-1}(n) \cap X & = \emptyset \quad \text{otherwise} \end{array} \right.$$ 

Pick $B \in \mathcal{R}$ such that for all $b \in B$, $b \cap Y_{A,X} = \emptyset$ or $b \subset Y_{A,X}$. Let $C_{A,B} = \{ \bigcup e_A^{-1}[b] : b \in B \} \in \mathcal{R}^*$. Let $b \in B$. Since for $a \in A$, $a \cap X = \emptyset$ or $a \subset X$, $e_A^{-1}(i) \cap X = \emptyset$ or $e_A^{-1}(i) \subset X$ for $i \in b$. Since $b \cap Y_{A,X} = \emptyset$ or $b \subset Y_{A,X}$ for $b \in B$, $\bigcup e_A^{-1}[b] \cap X = \emptyset$ or $\bigcup e_A^{-1}[b] \subset X$. So $X$ does not 4-split $C_{A,B}$. Since $|\mathcal{R}^*| = r_{\text{pair}}$, $r_4 \leq r_{\text{pair}}$. By Proposition 1.2, $r_{\text{pair}} = r_4 = r_2$.

Similarly we can prove $r_{\text{pair}} = r_2n$ for $n \geq 2$. □

David Asperó conjectured that $s_{\text{pair}} = s_3$. Shizuo Kamo gave the proof. The proofs for the splitting numbers are not dual to the proofs for the reaping numbers. It might simplify in terms of Galois-Tukey connections as in [16]. However it might be difficult. In [11] and [12], Mildenberger introduced another variation of reaping numbers $r_n$ and $r_n = r_m(= r)$ holds for $n, m \in \omega$ but it is proved that there are no nice Galois-Tukey connections between Mildenberger’s reaping numbers.

**Theorem 1.3.** (Kamo) $s_{\text{pair}} = s_n$ for $n \geq 3$.

**Proof.** We shall prove $s_{\text{pair}} = s_4$. Let $\text{ZFC}^-$ be a large enough fragment of ZFC. Suppose $s_4, s_3 < s_{\text{pair}}$ holds. Let $M_0$ be a model of $\text{ZFC}^-$ such
that the cardinality is \( s_3 \) and \( M_0 \cap [\omega]^\omega \) is a 3-splitting family and 4-splitting family.

Pick an infinite subset \( A \) of \([\omega]^2\) which is not 2-split by \( M_0 \cap [\omega]^\omega \).

Without loss of generality we can assume this \( A \) is pairwise disjoint.

Let \( M_1 \) be a model of ZFC\(^-\) of cardinality \( s_3 \) which contains \( A \) and all elements of \( M_0 \). Pick \( B \) in \( M_1 \) such that \( B \) is an infinite subset of \([A]^2\) and \( B \) is not 2-split by any elements in \( M_1 \cap [A]^\omega \). We can also assume this \( B \) is pairwise disjoint.

Let \( C = \{ a \cup b : \{a, b\} \in B \} \). Since \( M_0 \cap [\omega]^\omega \) is a 4-splitting family, there exists \( X \in M_0 \cap [\omega]^\omega \) such that \( X \) 4-splits \( C \). Since \( A \) is not 2-split by \( X \), there exist infinitely many \( a \in A \) such that \( a \subset X \) or \( X \cap a = \emptyset \). So there exist infinitely many \( \{a, b\} \in B \) such that \( a \subset X \) and \( b \) does not meet \( X \). Put \( Y = \{a \in A : a \subset X \} \). Then \( Y \in M_1 \) and \( Y \) 2-splits \( B \). However, this is a contradiction to the fact \( B \) is not split by any infinite subset of \( A \) in \( M_1 \).

Similarly we can prove that \( s_{\text{pair}} = s_{2n} \) for \( n \geq 2 \). Therefore \( s_{\text{pair}} = s_n \) for \( n \in \omega \). \( \square \)

2. The Ideal of Finitely Chromatic Graphs

In this section we shall investigate the relation between the finite chromatic ideal, pair-splitting number and pair-reaping number.

The finite chromatic ideal on \([\omega]^2\) is defined by

\[
G_{FC} = \{ A \subset [\omega]^2 : \chi(\omega, A) < \aleph_0 \}
\]

where \( \chi(\omega, A) = \min\{k \in \omega : (\exists f \in k^\omega)(\forall a \in A)((|f[a]| = 2) \} \).

**Theorem 2.1.** The following conditions hold.

1. \( s_{\text{pair}} = \text{cov}^*(G_{FC}) \),
2. \( \text{non}^*(G_{FC}) \) is the minimal cardinality of a family \( A \subseteq [\omega]^\omega \) such that for any finite partition \( P \) of \( \omega \) there is an element \( A \) of \( A \) such that for every \( r \in A \) there is \( P \in \mathcal{P} \) such that \( r \subseteq P \) and
3. \( r_{\text{pair}} \leq \text{non}^*(G_{FC}) \).

**Proof.** First we shall prove \( s_{\text{pair}} \leq \text{cov}^*(G_{FC}) \). Let \( A \) be a subset of \( G_{FC} \) such that \( |A| = \text{cov}^*(G_{FC}) \) and

\[
(1) \quad (\forall X \subset [\omega]^2)(\exists A \in A) (|X| = \aleph_0 \rightarrow |A \cap X| = \aleph_0).
\]

**Claim 2.2.** If \( A \in G_{FC} \), then there exist \( n \in \omega \) and \( A_i \subset A \) for \( i < n \) such that \( A = \bigcup_{i<n} A_i \) and \( \chi(A_i) = 2 \) for \( i < n \).
Proof of Claim. Suppose $A \in \mathcal{G}_{FC}$, $k \in \omega$ and $f : \omega \to k$ such that for all $a \in A$ $|f[a]| = 2$. For $i, j < k$ with $i < j$, put $A_{i,j} = \{a \in A : f[a] = \{i, j\}\}$. Then $\chi(\omega, A_{i,j}) = 2$ and $A = \bigcup_{i,j<k,i<j} A_{i,j}$. \hfill \Box

By this claim, we can assume $\chi(\omega, A) = 2$ for $A \in \mathcal{A}$. For each $A \in \mathcal{A}$, fix $f : \omega \to 2$ so that $f$ witnesses $\chi(\omega, A) = 2$. Put $A_0 = f^{-1}(0) \cap \bigcup A$ and $A_0 = \{A_0 : A \in \mathcal{A}\}$.

Then $A_0$ is a pair-splitting family. Let $B \subset [\omega]^2$ be infinite. Since $\mathcal{A}$ satisfies (1), there is $A \in \mathcal{A}$ such that $|A \cap B| = \aleph_0$. So there exist infinitely many $b \in B$ such that $b \in A$. So there exist infinitely many $b \in B$ such that $b \cap A_0 \neq \emptyset$ and $b \setminus A_0 \neq \emptyset$. Therefore $s_{\text{pair}} \leq \text{cov}^*(\mathcal{G}_{FC})$.

We shall prove $s_{\text{pair}} \geq \text{cov}^*(\mathcal{G}_{FC})$. Let $\mathcal{S} \subset [\omega]^2$ be a pair-splitting family. For each $S \in \mathcal{S}$, put $A_S = \{a \in [\omega]^2 : a \cap S \neq \emptyset \land a \cap \omega \setminus S \neq \emptyset\}$ and $\mathcal{A}(S) = \{A_S : S \in \mathcal{S}\}$.

Then $\mathcal{A}(S)$ satisfies that for each infinite $X \subset [\omega]^2$, there exists an $A_S \in \mathcal{A}(S)$ such that $|X \cap A_S| = \aleph_0$. Let $X \subset [\omega]^2$ be infinite. Since $\mathcal{A}$ is a pair-splitting family, there exists an $S \in \mathcal{S}$ such that $S$ pair-splits $X$. So there exist infinitely many $a \in X$ such that $a \cap S \neq \emptyset$ and $a \setminus S \neq \emptyset$. Hence $|X \cap A_S| = \aleph_0$. Therefore $\text{cov}^*(\mathcal{G}_{FC}) \leq s_{\text{pair}}$.

In order to prove (2), note that if $P$ is a finite partition of $\omega$ then $G_P = \{(n, m) : (\exists a \neq b \in P)(n \in a \land m \in b)\} \in \mathcal{G}_{FC}$, and moreover, $\{G_P : P$ is a finite partition of $\omega\}$ is a base of $\mathcal{G}_{FC}$. Then, if $\mathcal{A}$ is a family as in (2) then $\mathcal{A}$ itself witnesses $\text{non}^*(\mathcal{G}_{FC})$; and if $\mathcal{B}$ is a witness of $\text{non}^*(\mathcal{G}_{FC})$ then defining $\mathcal{A}$ as the family of finite changes of elements of $\mathcal{B}$ we are done. (3) follows directly from (2). \hfill \Box

It can be easily seen that $\mathcal{G}_{FC}$ is an $F_\sigma$-ideal. In particular, $s_{\text{pair}}$ is equal to the covering number of an $F_\sigma$-ideal and $\tau_{\text{pair}}$ is bounded by the uniformity number of an $F_\sigma$-ideal.

Concerning to the covering number of $F_\sigma$-ideals and $\mathfrak{d}$, we can construct a proper forcing notion which destroys tallness of an $F_\sigma$-ideal and preserves the unbounded number.

**Theorem 2.3.** [9] For each $F_\sigma$-ideal $\mathcal{I}$, there exists a proper forcing notion $\mathbb{P}_I$ which is $\omega^\omega$-bounding and adds a new element $X$ in the extension such that $|X \cap I| < \aleph_0$ for $I \in \mathcal{I} \cap V$.

By using $\omega_2$-stage countable support iteration of $\mathbb{P}_I$, we can show the following statement.

**Corollary 2.4.** Suppose $\mathcal{I}$ is an $F_\sigma$-ideal on $\omega$. Then it is consistent that $\text{cov}^*(\mathcal{I}) > \mathfrak{d}$.

**Corollary 2.5.** It is consistent that $s_{\text{pair}} = \text{cov}^*(\mathcal{G}_{FC}) > \mathfrak{d}$.
3. The uniformities of $F_0$-ideals

The eventually different ideal is defined by
\[ \mathcal{ED} = \{ A \subset \omega \times \omega : (\exists m, n \in \omega)(\forall k > n) (|\{l : (k, l) \in A\}| \leq m)\}. \]
Define $\mathcal{ED}_{fin} = \mathcal{ED} \upharpoonright \triangle$, where $\triangle = \{ (m, n) : n \leq m \}$. On the $\text{cov}^*(\mathcal{ED})$ we have the following result.

**Lemma 3.1.** $\text{cov}^*(\mathcal{ED}) = \text{non}(\mathcal{M})$.

*Proof.* We will use the following lemma, due to Bartoszyński and Miller.

**Lemma 3.2** ([1], Lemma 2.4.8). For any cardinal $\kappa$ the following are equivalent:
\begin{align*}
(a) & \quad \kappa < \text{non}(\mathcal{M}), \\
(b) & \quad (\forall F \in [\omega^\omega]^\kappa)(\exists g \in \omega^\omega)(\exists X \in [\omega]^{\omega})(\forall f \in F)(\forall n \in X)(f(n) \neq g(n)) \\
(c) & \quad (\forall F \in [\mathcal{C}]^\kappa)(\exists g \in \omega^\omega)(\exists S \in F)(\forall n \in S)(g(n) \notin S(n))
\end{align*}

Let $\mathcal{F}$ be a subset of $\omega^\omega$ of minimal cardinality such that
\[(\forall g \in \omega^\omega)(\forall X \in [\omega]^{\omega})(\exists f \in \mathcal{F})(\exists n \in X)(f(n) = g(n))\]
(We are identifying every function $f \in \omega^\omega$ with its graph $\{(n, f(n)) : n < \omega\}$.) Define $\mathcal{A} = \mathcal{F} \cup \{\{n\} \times \omega : n < \omega\}$. Obviously $\mathcal{A} \subseteq \mathcal{ED}$. We claim that $\mathcal{A}$ is a covering family. Let $X$ be an infinite subset of $\omega \times \omega$. If there exists $n < \omega$ such that $X_n = X \cap (\{n\} \times \omega)$ is infinite, then $X_n$ is an infinite subset of an element of $\mathcal{A}$. If the set $A = \{ n < \omega : X_n \neq \emptyset \}$ is infinite then there exists $f \in \mathcal{F}$ such that $f(n) = \min(X_n)$ for infinitely many $n \in A$. Hence, $f \cap X$ is infinite.

On the other hand, let $\mathcal{A}$ be a subset of $\mathcal{ED}$ with $|\mathcal{A}| < \text{non}(\mathcal{M})$. For every $A \in \mathcal{A}$, let $n_A < \omega$ such that $|A_k| \leq n_A$ for all $k \geq n_A$, and define a slalom $S_A$ by
\[ S_A(n) = \begin{cases} 
\emptyset & \text{if } n < n_A \\
A_n & \text{if } n \geq n_A
\end{cases} \]
Note that $|\{S_A : A \in \mathcal{A}\}| \leq |\mathcal{A}|$, and by the lemma above, there exists $g \in \omega^\omega$ such that for every $A \in \mathcal{A}$, $g(n) \notin S_A(n)$, for almost all $n < \omega$. Hence, $g \cap A$ is finite for all $A \in \mathcal{A}$, and so, $\mathcal{A}$ is not a covering family. \[\square\]

**Theorem 3.3.** If $\mathcal{I}$ is a Borel ideal on $\omega$, then $\text{non}^*(\mathcal{I}) = \omega$ or $\mathcal{ED}_{fin} \leq_{K} \mathcal{I}$. So $\text{non}^*(\mathcal{I}) = \omega$ or $\text{non}^*(\mathcal{ED}_{fin}) \leq \text{non}^*(\mathcal{I})$.

*Proof.* For a Borel ideal $\mathcal{I}$, let us consider the following two-player game: In stage $k$, Player I chooses a finite subset $F_k$ of $\omega$ and then, Player II chooses a natural number $n_k \notin F_k$. 


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<table>
<thead>
<tr>
<th>I</th>
<th>$F_0 \in [\omega]^{&lt;\omega}$</th>
<th>$F_1 \in [\omega]^{&lt;\omega}$</th>
<th>$\ldots$</th>
</tr>
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<tbody>
<tr>
<td>II</td>
<td>$n_0 \notin F_0$</td>
<td>$n_1 \notin F_1$</td>
<td>$\ldots$</td>
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Player I wins if $\{n_i : i \in \omega\} \in \mathcal{I}$ and Player II wins $\{n_i : i \in \omega\} \in \mathcal{I}^+$.

**Claim 3.4.** If Player I has a winning strategy then $\mathcal{E}D_{\text{fin}} \leq_K \mathcal{I}$.

**Proof of Claim.** If Player I has a winning strategy then there is a cofinite-branching tree $T \subseteq \omega^{<\omega}$ such that every $t \in T$ is an increasing sequence and $\text{rng}(f) \in \mathcal{I}$ for all $f \in [T]$. Choose $g : \omega \to \omega$ a strictly increasing function such that if $n \in \omega$ and $t \in T$ with $\text{rng}(t) \subset g(n)$ then $[g(n+1), \infty) \subseteq \text{Succ}(t)$. Then every selector of $\{[g(n), g(n+1)) : n \in \omega\}$ is in $\mathcal{I}$, hence there exists a branch of $T$. Therefore every selector of $\{[g(n), g(n+1)) : n \in \omega\}$ is in $\mathcal{I}$.

Choose $f : \omega \to \Delta$ an injection so that for each $n \in \omega$, there exists $k \in \omega$ such that $f([g(n), g(n+1)]) \subseteq \{(k, l) : l \leq k\}$. We shall show this $f$ witnesses $\mathcal{E}D_{\text{fin}} \leq_K \mathcal{I}$. Let $I \in \mathcal{E}D_{\text{fin}}$ and $m \in \omega$ be such that for all but finitely many $k$, $\{|(k, l) : l \leq k \wedge (k, l) \in I\| \leq m$. So $f^{-1}[I]$ is a union of $m$-many selectors of $\{[g(n), g(n+1)) : n \in \omega\}$. Since every selector of $\{[g(n), g(n+1)) : n \in \omega\}$ is in $\mathcal{I}$, $f^{-1}[I] \in \mathcal{I}$ i.e., $\mathcal{E}D_{\text{fin}} \leq_K \mathcal{I}$.

**Claim 3.5.** If Player II has a winning strategy, then $\text{non}^*(\mathcal{I}) = \omega$.

**Proof of Claim.** Player II has a winning strategy if and only if there exists an infinitely-branching tree $T \subseteq \omega^{<\omega}$ such that $\text{rng}(f) \in \mathcal{I}^+$ for all $f \in [T]$.

We shall show $\{\text{succ}(t) : t \in T\}$ is a witness of $\text{non}^*(\mathcal{I})$. Assume to the contrary that there exists $I \in \mathcal{I}$ such that $|I \setminus \text{succ}(t)| = \omega$ for all $t \in T$. Then there exists $b \in [T]$ such that $\text{rng}(b) \subseteq b \in \mathcal{I}$. This is a contradiction. Therefore $\text{non}^*(\mathcal{I}) = \omega$.

By Borel determinacy this game is determined i.e., either Player I or Player II has a winning strategy. So $\mathcal{E}D_{\text{fin}} \leq_K \mathcal{I}$ or $\text{non}^*(\mathcal{I}) = \omega$.

Concerning to the cardinal invariants of $\mathcal{E}D_{\text{fin}}$, we have proved the following.

**Proposition 3.6.** The following relations hold:

1. $\text{non}^*(\mathcal{E}D_{\text{fin}}) \leq \tau$,
2. $\text{cov}(\mathcal{M}) = \min\{d, \text{non}^*(\mathcal{E}D_{\text{fin}})\}$ and
3. $\text{non}(\mathcal{M}) = \max\{b, \text{cov}^*(\mathcal{E}D_{\text{fin}})\}$.

**Proof.** For any $A \subseteq \Delta$ we will denote by $A_n = \{m \leq n : (n, m) \in A\}$. Let us prove (1). We will say that a family $\mathcal{R}$ of infinite subsets of $\omega$ is *hereditarily reaping* if for every $X \in \mathcal{R}$ and every infinite subset $Y$ of $X$ there is $R$ in $\mathcal{R}$ such that $R \subseteq Y$ or $R \subseteq X \setminus Y$. 
Lemma 3.7. \( \tau = \min\{ |R| : R \text{ is hereditarily reaping} \} \)

Proof. It will be enough to prove that there is a hereditarily reaping family with cardinality \( \tau \). Let \( Q \) be a reaping family with cardinality \( \tau \). Define \( Q_n \) by recursion on \( n < \omega \). Let \( Q_0 = Q \). Given \( Q_n \) and \( A \in Q_n \), let \( Q_{n+1} \upharpoonright A \) be a reaping family on \( A \) with cardinality \( \tau \). Put \( Q_{n+1} = \bigcup_{A \in Q_n} Q_{n+1} \upharpoonright A \). So, \( R = \bigcup_{n<\omega} Q_n \) is a hereditarily reaping family. \( \square \)

Let \( R \) be a hereditarily reaping family, and for every \( R \in R \) and \( n < \omega \) define \( X_{R,n} = \{ (m,n) : m \geq n \land m \in R \} \). We will see that \( A = \{ X_{R,n} : R \in R \land n < \omega \} \) witnesses \( \text{non}^*(ED_{fin}) \). Let \( I \) be in \( ED_{fin} \), and choose \( \{ f_i : i \leq n \} \) functions such that \( I \subseteq \bigcup_{i \leq n} f_i \). Define \( A_j = \{ k : (\exists i \leq n) (f_i(k) = j) \} \), for \( j \leq n \). If \( A_j \) is finite for some \( j \leq n \), then \( I \cap X_{R,j} \) is finite for every \( R \in R \). So we can assume \( A_j \) is infinite for \( j \leq n \). Let \( R_0 \) be in \( R \) such that \( R_0 \cap A_0 = \emptyset \) or \( R_0 \subseteq A_0 \). In general, for \( 1 \leq j \leq n \) we can find \( R_j \in R \) such that \( R_j \cap (R_{j-1} \cap A_j) = \emptyset \) or \( R_j \subseteq R_{j-1} \cap A_j \). If the first case is true for a \( j \leq n \) we are done, because for \( j \) minimal, we have that \( X_{R_j,j} \cap I = \emptyset \). Suppose that \( R_j \subseteq R_{j-1} \cap A_j \) for all \( j \leq n \). Then, for any \( k \in R_{n} \), \( I \cap (\{ k \} \times \omega) = n + 1 \), and so, \( X_{R_{n,n+1}} \cap I = \emptyset \).

In order to prove (2) we will need the following lemma, due to Bartoszyński and Miller.

Lemma 3.8 ([1], Lemma 2.4.2). For any cardinal \( \kappa \) the following conditions are equivalent:

(i) \( \kappa < \text{cov}(\mathcal{M}) \)

(ii) \( (\forall F \in [\omega^\omega]^\kappa)(\forall G \in [[\omega^\omega]^\kappa])(\exists f \in F)(\forall X \in G)(\exists \in n \in X)(f(n) = g(n)) \).

Let \( \kappa \) be a subset of \([\Delta]^\kappa\) with \( |\kappa| < \text{cov}(\mathcal{M}) \). For every \( X \in \kappa \) define \( G_X = \{ n < \omega : X \cap (\{ n \} \times \omega) \neq \emptyset \} \) and let \( f_X \in \omega^\omega \) be a function such that \( f_X(n) \in X \cap (\{ n \} \times \omega) \). By the previous lemma, there is a function \( g \in \omega^\omega \) such that \( f_X(n) = g(n) \) for infinitely many elements \( n \) of \( G_X \), for all \( X \in \kappa \). Then, \( \Delta \cap g \) is an element of \( ED_{fin} \) having an infinite intersection with every element of \( \kappa \), proving \( |\kappa| < \text{non}^*(ED_{fin}) \). So \( \text{cov}(\mathcal{M}) \leq \text{non}^*(ED_{fin}) \). In addition, it is a well known fact that \( \text{cov}(\mathcal{M}) \leq d \). Therefore \( \text{cov}(\mathcal{M}) \leq \min\{d, \text{non}^*(ED_{fin})\} \).

We shall show \( \min\{d, \text{non}^*(ED_{fin})\} \leq \text{cov}(\mathcal{M}) \). Let \( \kappa \) be a cardinal lower than \( d \) and \( \text{non}^*(ED_{fin}) \). We will prove and use the following lemma.

Lemma 3.9. Let \( \kappa \) be an infinite cardinal. The following conditions are equivalent.
Proof. Suppose that $\kappa$ satisfies (b) and let $B$ be a family of $\kappa$ infinite subsets of $\Delta$. For every $B \in B$, let $X_B = \{ n : B_n \neq \emptyset \}$ and $f_B : \omega \to \omega$ such that $(n, f_B(n)) \in B$ if $n \in X_B$, and $f_B(n) = 0$ if not. The families $\mathcal{F} = \{ f_B : B \in B \}$ and $\mathcal{A} = \{ X_B : B \in B \}$ have cardinality $\kappa$, and so, there exists a function $g \in \omega^\omega$ such that for all $B \in B$ there are infinitely many $n \in X_B$ such that $g(n) = f_B(n)$, showing that $g$ has an infinite intersection with $B$.

On the other hand assume that $\kappa < \text{non}^*(\mathcal{E} \mathcal{D}_{fin})$, $\mathcal{F} \subseteq \omega^\omega$ and $\mathcal{A} \subseteq [\omega]^\omega$ have cardinality $\kappa$, and $\mathcal{F}$ is bounded by an increasing function $h \in \omega^\omega$. We will identify every $f \in \mathcal{F}$ with a subset of an $\mathcal{E} \mathcal{D}_{fin}$-positive subset $\Delta'$ of $\Delta$, as follows: Define $X = h[\omega]$, $\Delta' = \prod_{n \in X} n$, $A' = h[A]$ if $A \in \mathcal{A}$, and for $f \in \mathcal{F}$, define $f' : X \to \omega$ by $f'(n) = f(h^{-1}(n))$. So, $\mathcal{F}' = \{ f' : f \in \mathcal{F} \}$ is a family of infinite subsets of $\Delta'$. Let $B = \{ f' : f \in \mathcal{F} \cap A \in \mathcal{A} \}$. Since $|B| = \kappa$, there exists $I \in \mathcal{E} \mathcal{D}_{fin}$ such that $I \cap B$ is infinite for all $B \in B$. Let $\{ g_i : i \leq N \}$ be a set of functions in $\omega^\omega$ such that $I \subseteq \bigcup_{i \leq N} g_i$. Define $B_{f,A} = \{ n \in A' : f'(n) = g_i(n) \}$, for some $i \leq N$ such that $|\{ f' \upharpoonright A' \cap g_i \}| = \aleph_0$, and define $C = \{ B_{f,A} : f \in \mathcal{F} \cap A \in \mathcal{A} \}$. By Proposition 3.6 (1) $|C| \leq \kappa < \tau$, and so, there exists $Y \in [\omega]^\omega$ such that $|Y \cap B_{f,A}| = \omega = |B_{f,A} \setminus Y|$. We can find a partition $\{ Y_0, Y_1 \}$ of $Y$ such that $|Y_0 \cap B_{f,A}| = \aleph_0 = |Y_1 \cap B_{f,A}|$, for all $f \in \mathcal{F}$ and for all $A \in \mathcal{A}$, and inductively, we can find a partition $\{ Y_0, Y_1, \ldots, Y_n \}$ of $Y$ such that for every $i \leq n$, $|B_{f,A} \cap Y_i| = \aleph_0$. Now, we define $g(n) = g_i(n)$ if $n \in Y_i$ and $g(n) = 0$ if $n \notin Y$. Given $f$ and $A$, if $i \leq n$ is such that $B_{f,A} = \{ n \in A' : f'(n) = g_i(n) \}$ then $f'(n) = g(n)$ for infinitely many $n \in Y_i \cap A'$, and so, $f(n) = g(h(n))$ for infinitely many $n \in h^{-1}[Y_i] \cap A$. 

Let us prove that $\kappa < \text{cov}(\mathcal{M})$ when $\kappa < \text{min}\{ \text{non}^*(\mathcal{E} \mathcal{D}_{fin}) \}$, by using Lemma 3.8. Let $F$ and $G$ be families such that $F \in [\omega^\omega]^\kappa$ and $G \in [[\omega^\omega]]^\kappa$.

Claim 3.10. There exists $h \in \omega^\omega$ such that for all $X \in G$ and for all $f \in F$, $f(n) < h(n)$ for infinitely many $n \in X$.

Proof of the Claim. For all $f \in F, X \in G$, let $e_X$ be the enumeration of $X$ and let $h_{f,X} \in \omega^\omega$ be such that $h_{f,X}(n) \geq f(e_X(i))$ for all $i \leq n$. Since $\kappa < \text{d}$, there is a function $h$ which is not dominated by $\{ h_{f,X} : X \in G \land f \in F \}$. This $h$ does the work. 

}\end{proof}
Now, for every \( f \in F \) define \( f' \in \omega^\omega \) such that \( f'(n) = f(n) \) if \( f(n) < h(n) \) and \( f'(n) = 0 \) otherwise; for every \( f \in F \) and for every \( X \in G \) define \( C_{f,X} = \{ n \in X : f(n) < h(n) \} \), \( A = \{ C_{f,X} : f \in F \land X \in G \} \) and \( F = \{ f' : f \in F \} \). \( F \) is bounded and so, by Lemma 3.9, there is \( g \in \omega^\omega \) such that for all \( f \in F \) and for all \( A \in A \), \( g(n) = f'(n) \) for infinitely many \( n \in A \) and in consequence, \( g(n) = f(n) \) for infinitely many \( n \in C_{f,X} \subset X \) for every \( X \in G \). Therefore \( \kappa < \text{cov}(M) \) by Lemma 3.9.

We shall prove (3). It is well known that \( b \leq \text{non}(M) \) and note that \( \text{ED} \leq \text{K} \), \( \text{ED}_{\text{fin}} \) and so, \( \text{cov}^*(\text{ED}_{\text{fin}}) \leq \text{cov}^*(\text{ED}) = \text{non}(M) \). So \( \max\{b, \text{cov}^*(\text{ED}_{\text{fin}})\} \leq \text{non}(M) \).

To show \( \max\{b, \text{cov}^*(\text{ED}_{\text{fin}})\} \geq \text{non}(M) \), we are going to use the following lemma.

**Lemma 3.11** ([1], Theorem 2.4.7). \( \text{non}(M) \) is the size of the smallest family \( F \subseteq \omega^\omega \) such that for every \( g \in \omega^\omega \) there is an element \( f \) of \( F \) such that \( f(n) = g(n) \) for infinitely many \( n \in \omega \).

Let \( \kappa \) be a cardinal greater than \( \text{cov}^*(\text{ED}_{\text{fin}}) \) and greater than \( b \). Let \( G = \{ f_\alpha : \alpha < \kappa \} \) be an unbounded family of functions in \( \omega^\omega \), and let \( G_\alpha \) a witness of \( \text{cov}^*(\text{ED}_{\text{fin}}) \) in \( \Delta_\alpha = \{ (n, m) : m \leq f_\alpha(n) \} \), for all \( \alpha < \kappa \). Without loss of generality we can assume that every element of \( I \) of \( G_\alpha \) is the graph of a function in \( \omega^\omega \). We will prove that \( F = \bigcup_{\alpha < \kappa} G_\alpha \) is such that for every \( g \in \omega^\omega \) there is \( f \in F \) such that \( f(n) = g(n) \) for infinitely many \( n \in \omega \). Given \( g \in \omega^\omega \), let \( \alpha < \kappa \) be such that \( f_\alpha \not\leq^* g \). Then, \( g \cap \Delta_\alpha \) is infinite and so, there is \( I \in G_\alpha \) such that \( I \cap (g \cap \Delta_\alpha) \) is infinite. Since \( I \) is the graph of a function in \( F \), we are done.

By Proposition 3.6, it is consistent that \( \text{non}^*(\text{ED}_{\text{fin}}) < b \). For example if the ground model satisfies Martin axiom, then the random forcing corresponding to the product space \( 2^{\omega_1} \) forces \( \text{non}^*(\text{ED}_{\text{fin}}) = \text{cov}(M) = \omega_1 < b = c \). However, we cannot use this argument to show the consistency of \( \text{non}^*(I) < b \) for every \( F_\sigma \)-ideal \( I \) because \( \text{cov}(N) \leq \text{non}^*(\mathcal{G}_{\text{FC}}) \) and the random forcing corresponding to the product space \( 2^{\omega_1} \) forces \( \text{cov}(N) = c \) whenever the ground model satisfies Martin axiom.

However, \( F_\sigma \)-ideals on \( \omega \) have the following good property.

**Theorem 3.12.** [10] \( I \) is an \( F_\sigma \)-ideal on \( \omega \) if and only if \( I = \text{Fin}(\varphi) \) for some lower semi-continuous submeasure \( \varphi \), where \( \text{Fin}(\varphi) = \{ A \subset \omega : \varphi(A) < \infty \} \). Here \( \varphi : \mathcal{P}(\omega) \to [0, \infty] \) is a lower semi-continuous submeasure if

1. \( \varphi(\emptyset) = 0 \),
2. whenever \( X, Y \subset \omega \) and \( X \subset Y \), \( \varphi(X \cup Y) \leq \varphi(X) + \varphi(Y) \),
(3) \( \varphi(\{n\}) < \infty \) for \( n \in \omega \) and
(4) \( \varphi(A) = \lim_{n \to \infty} \varphi(A \cap n) \) for every \( A \subset \omega \).

To show the consistency of \( \text{non}^*(I) < b \), we shall use the Laver forcing \( \mathbb{L} \). \( \mathbb{L} \) is defined by \( T \in \mathbb{L} \) if \( T \subset \omega^\omega \) is a tree and for \( s \in T \) with \( \text{stem}(T) \subset s \), \( |\text{succ}_T(s)| = \aleph_0 \). \( \mathbb{L} \) is ordered by inclusion. Then \( \mathbb{L} \) adds an unbounded real.

**Proposition 3.13.** Let \( G \) be a \( \mathbb{L} \)-generic over \( V \) and \( f_G = \bigcup \{ \text{stem}(T) : T \in G \} \). Then \( f_G \in \omega^\omega \) and \( f_G \) dominates for all \( g \in \omega^\omega \cap V \).

Therefore, if \( \mathbb{L}_{\omega_2} \) is an \( \omega_2 \)-stage countable support iteration of Laver forcing, then \( V^{\mathbb{L}_{\omega_2}} \models b = c \).

By Proposition 3.13 it is enough to show that \( \mathbb{L}_{\omega_2} \) preserves \( \text{non}^*(I) \) for each \( F_\sigma \)-ideal \( I \) on \( \omega \). We shall use the Laver property.

**Definition 4.** [4] A forcing notion \( \mathbb{P} \) have the Laver property if for every \( H : \omega \to \omega \in V \)

\[
\Vdash \left( \forall f \in \prod_{n \in \omega} H(n) \cap V[\dot{G}] \right) \left( \exists A : \omega \to \omega^\omega \in V \right)

(\forall n \in \omega) (f(n) \in A(n) \land |A(n)| \leq 2^n).
\]

The Laver property has the following good property.

**Theorem 4.1.** [4] The Laver property is preserved under countable support iteration of proper forcing notions.

**Theorem 4.2.** [1, p353] The Laver forcing \( \mathbb{L} \) has the Laver property.

So \( \mathbb{L}_{\omega_2} \) has the Laver property.

**Theorem 4.3.** If \( I \) is an \( F_\sigma \)-ideal on \( \omega \), then it is consistent that \( \text{non}^*(I) < b \).

**Proof.** Let \( I \) be an \( F_\sigma \)-ideal and let \( \varphi \) be a lower semi-continuous submeasure such that \( I = \text{Fin}(\varphi) \).

If a forcing notion \( \mathbb{P} \) has the Laver property, then \( \mathbb{P} \) has the following good property:

**Lemma 4.4.** If \( \mathbb{P} \) has the Laver property, then

\[
\Vdash_\mathbb{P} "(\forall X \in I \cap V[\dot{G}]) (\exists A \in [\omega]^\omega \cap V) (|X \cap A| < \aleph_0) ".
\]

**Proof of Lemma.** Let \( p \in \mathbb{P} \) and let \( \dot{X} \) be a \( \mathbb{P} \)-name such that \( \Vdash_\mathbb{P} "\dot{X} \in I" \). Without loss of generality we can assume that there exists \( n \in \omega \) such that \( p \Vdash_\mathbb{P} "\varphi(\dot{X}) < n" \).
Claim 4.5. Let $\varphi : \mathcal{P}(\omega) \rightarrow [0, \infty]$ be a lower semi-continuous submeasure such that $\text{Fin}(\varphi) = \mathcal{I}$ for some $F_\sigma$-ideal on $\omega$. For each $k \in \omega$ and $l \in \omega$, there exists $m \in \omega$ such that $\varphi([l, m]) > k$.

Proof of Claim. Since $[l, \infty) \not\subseteq \mathcal{I}$, $\varphi([l, \infty)) = \infty$. Because $\varphi$ has the lower semi-continuous, there exists $m > l$ such that $\varphi([l, m]) > k$. □

Let $\Pi = \{I_j : j \in \omega\}$ be an interval partition of $\omega$ such that $\varphi(I_j) > 2^j \cdot n$. By the Laver property, there exist $q \leq p$ and $A : \omega \rightarrow \bigcup_{j \in \omega} \mathcal{P}(2^j) \subseteq V$ such that for $j \in \omega$, $A(j) \subset 2^j$ and $|A(j)| \leq 2^j$ and $q \models \forall j \in \omega \left( \exists Y \mid I_j \in A(j) \right)$. Without loss of generality we can assume $\varphi(J) \leq n$ for $J \in A(j)$ and for $j \in \omega$. By the finite subadditivity of $\varphi$, $\varphi(\bigcup_{j \in \omega} A(j)) \leq \sum_{j \in \omega} \varphi(J(j)) \leq 2^j \cdot n$. So $I_j \setminus A_j \neq \emptyset$ for $j \in \omega$. Choose $y_j \in I_j \setminus \bigcup_{j \in \omega} A(j)$ for $j \in \omega$. Put $Y = \{y_j : j \in \omega\}$. Then $q \models \forall j \in \omega \left( \exists Y \subseteq \omega \cap V \mid \forall y \in Y3Y \in [\omega]^\omega \cap V \left( (X \cap Y) < \aleph_0 \right) \right)$.

So if the ground model satisfies CH, then $V^{1\rightarrow 2} \models [\omega]^\omega \cap V$ witnesses $\text{non}^*(\mathcal{I})$. Therefore it is consistent $\text{non}^*(\mathcal{I}) < b$. □

In [7] Masaru Kada introduced a cardinal invariant associated with the Laver property.

We call a function $f : [\omega]^{<\omega} \rightarrow \omega$ a slalom. Let $\mathcal{S}$ be the collection of slaloms such that $\forall \phi \in \mathcal{S} \forall n \in \omega(|\phi(n)| \leq 2^n)$. $I$ is the smallest cardinal $\kappa$ such that for every $n \in \omega$ there is a set $\Phi \in \mathcal{S}$ with cardinality $\kappa$ such that, for every $f \in \omega$ with $f(n) < h(n)$ for all $n < \omega$, there is $f \in \Phi$ such that for all but finitely many $n \in \omega$, we have $f(n) \in \phi(n)$.

Pawlikowski showed that the dual notion to the definition of $I$ characterizes $\text{trans-add}(\mathcal{N})$, transitive additivity of the null ideal (see [1, p.91]). That is, $\text{trans-add}(\mathcal{N})$ is the smallest size of $\leq^*_{\text{bounded family}} F \subset \omega^\omega$ such that for every $\phi \in \mathcal{S}$ there is $f \in F$ such that for infinitely many $n \in \omega$, $f(n) \notin \phi(n)$.

As the proof of Theorem 4.3 we can prove the following statement.

Corollary 4.6. If $\mathcal{I}$ is an $F_\sigma$-ideal, then

1. $\text{non}^*(\mathcal{I}) \leq I$ and
2. $\text{cov}^*(\mathcal{I}) \geq \text{trans-add}(\mathcal{N})$.

Proof of Corollary. 1. Let $\mathcal{I}$ be an $F_\sigma$-ideal on $\omega$ and let $\varphi$ be a lower semi-continuous submeasure such that $\text{Fin}(\varphi) = \mathcal{I}$. Choose $\Pi = \{I_j : j \in \omega\}$ an interval partition of $\omega$ such that $\varphi(I_j) > 2^j \cdot j$. Choose $\Phi$ a family of functions from $\omega$ to $\bigcup_{j \in \omega} \mathcal{P}(2^j)$ such that

i. $|\Phi| \leq I,$

ii. for each $j \in \omega$ and $\phi \in \Phi$, $\phi(j) \in 2^j$ and $|\phi(j)| \leq 2^j$ and
iii. for each $X \in [\omega]^\omega$, there exists $\phi \in \Phi$ such that for all but finitely many $j \in \omega$, $X \cap I_j \in \phi(j)$.

Without loss of generality we can assume that for each $\phi \in \Phi$ and each $j \in \omega$, $J \in \phi(j)$ implies $\varphi(J) \leq j$. For each $j \in \omega$ and $\phi \in \Phi$, $\varphi(\bigcup \phi(j)) \leq \sum_{j \in \phi(j)} \varphi(J) \leq 2^j \cdot j$. So for each $j \in J$, $I_j \setminus \bigcup \phi(j) \neq \emptyset$.

For each $\phi \in \Phi$, choose $X_\phi \in [\omega]^{\omega}$ such that $X_\phi \cap I_j \setminus \bigcup \phi(j) \neq \emptyset$. Put $A = \{X_\phi : \phi \in \Phi\}$. We shall show for each $I \in \mathcal{I}$, there exists $X \in A$ such that $|A \cap I| < \aleph_0$.

Let $I \in \mathcal{I}$ and let $n \in \omega$ such that $\varphi(I) < n$. Choose $m \in \omega$ and $\phi \in \Phi$ so that for $j \geq m$, $I \cap I_j \in \phi(j)$. Then for $j \geq \max n, m$, $X_\phi \cap I_j \cap I = \emptyset$. So $|X_\phi \cap I| < \aleph_0$. Hence non$^*(\mathcal{I}) \leq 1$.

2. Let $\mathcal{I}$ be an $F_\sigma$-ideal. Let $A \subset \mathcal{I}$ such that $|A| < \text{trans-add}(\mathcal{N})$. Let $\Pi = \langle I_j : j \in \omega \rangle$ be an interval partition of $\omega$ such that $\varphi(I_j) > 2^j \cdot j$.

Since $|A| < \text{trans-add}(\mathcal{N})$, there exists $\phi : \omega \to \bigcup_{j \in \omega} \mathcal{P}(2^I)$ such that

i. for each $j \in \omega$, $\phi(j) \subset \mathcal{P}(2^I)$,
ii. for each $j \in \omega$, $|\phi(j)| \leq 2^j$ and
iii. for each $I \in A$ for all but finitely many $j \in \omega$, $I \cap I_j \in \phi(j)$.

Without loss of generality we can assume that for each $j \in \omega$ and $J \in \phi(j)$, $\varphi(J) < j$. By the finite subadditivity of $\varphi$, $\varphi(\bigcup \phi(j)) \leq \sum_{j \in \phi(j)} \varphi(J) \leq 2^j \cdot j$ for each $j \in \omega$. So $I_j \setminus \bigcup \phi(j) \neq \emptyset$ for $j \in \omega$.

Choose $X_\phi \in [\omega]^{\omega}$ such that $X_\phi \cap I_j \setminus \bigcup \phi(j)$ for $j \in \omega$. For each $I \in A$, there exists $m \in \omega$ such that $j \geq m$ implies $I \cap I_j \in \phi(j)$. Then $j \geq m$ implies $I \cap I_j \cap X_\phi = \emptyset$. So $|I \cap X_\phi| < \aleph_0$. Therefore trans$^\ast(\mathcal{N}) \leq \text{cov}^\ast(\mathcal{I})$. \hfill \Box
Corollary 4.7.  
(1) It is consistent \( \mathfrak{t}_{\text{pair}} < \mathfrak{b} \).
(2) \( \mathfrak{t}_{\text{pair}} \leq I \) and \( \mathfrak{s}_{\text{pair}} \geq \text{trans-add}(\mathcal{N}) \).

Question 4.8.  
(1) \( \mathfrak{t}_{d} \leq \mathfrak{s}_{\text{pair}} \) ?
(2) If \( \mathcal{I} \) is a Borel ideal, then \( \text{non}^*(\mathcal{I}) \leq \text{cof}(\mathcal{N}) \) ?

5. Fatou’s Lemma and a Question of Solecki

In this section we answer a question of S. Solecki related to the Katetov order by using cardinal invariants of Borel ideals.

For a sequence of \( (a_n)_{n \in \omega} \) of real numbers and an ideal \( \mathcal{I} \) on \( \omega \),
\[
\lim_{\mathcal{I}} \inf a_n = \sup \{ r \in \mathbb{R} : \{ n \in \omega : a_n < r \} \in \mathcal{I} \}.
\]

Let \( (X, \mathcal{B}, \mu) \) be a \( \sigma \)-finite measure space with \( \mu \) defined on \( \sigma \)-algebra \( \mathcal{B} \). Let \( f_n : X \to [0, \infty] \) be a sequence of \( \mu \)-measurable functions and let \( \mathcal{I} \) be an ideal on \( \omega \). We say that Fatou’s lemma holds on \( \langle f_n : n \in \omega \rangle \) with respect to \( \mathcal{I} \) if
\[
\int_{\mathcal{I}} \liminf f_n \, d\mu \leq \liminf_{\mathcal{I}} \int f_n \, d\mu
\]
where \( \int \) is the lower integral, i.e., if \( g \geq 0 \), then
\[
\int g \, d\mu = \sup \left\{ \int f \, d\mu : f \leq g \text{ and } f \text{ is } \mu\text{-measurable} \right\}.
\]
Let $\mathcal{I}$ be an ideal on $\omega$. We say that Fatou’s lemma holds for $\mathcal{I}$ if Fatou’s lemma holds with respect to $\mathcal{I}$ for any sequence $\langle f_n : n \in \omega \rangle$ of measurable functions from $X$ to $[0, \infty]$ on any $\sigma$-finite measure space.

The ideal $\mathcal{S}$ is a critical (locally minimal in the Katětov order) among the ideals which satisfy Fatou’s lemma. Let $\Omega = \{ U \in \text{Clopen}(2^\omega) : \mu(U) = \frac{1}{2} \}$. $\mathcal{S}$ is an ideal on $\Omega$ generated by the set $\{ I_x : x \in 2^\omega \}$ where $I_x = \{ U \in \Omega : x \in U \}$.

**Theorem 5.1.** [15] Let $\mathcal{I}$ be a Borel ideal on $\omega$.

$\mathcal{I}$ does not satisfy Fatou’s lemma if and only if there exists $X \in \mathcal{I}^+$ such that $\mathcal{S} \leq_K \mathcal{I} \upharpoonright X$.

Concerning this theorem, Solecki asked the following question.

**Question 5.2.** [15] Can $\mathcal{S}$ be replaced by $\mathcal{G}_{FC}$?

When we think about question related to the Katětov order, cardinal invariants of ideals are significant.

**Theorem 5.3.** $\text{cov}^*(\mathcal{S}) = \text{non}(\mathcal{N})$.

To prove this theorem, we will use the following lemmas.

**Lemma 5.4.** [5] For any $\{ U_n : n \in \omega \} \subset \Omega$,

$$\mu(\{ x \in 2^\omega : \exists n (x \in U_n) \}) \geq \frac{1}{2}.$$  

*Proof of Lemma.* Assume to the contrary that there exists $\{ U_n : n \in \omega \} \in [\Omega]^{\omega}$ with $\mu(\{ x \in 2^\omega : \exists n \in \omega (x \in U_n) \}) < \frac{1}{2}$. Then there exists a compact set $K \subset 2^\omega$ such that $\mu(K) > \frac{1}{2}$ and $K$ is disjoint with $\{ x \in 2^\omega : \exists n \in \omega (x \in U_n) \}$. Let $\delta = \mu(K) - \frac{1}{2} > 0$. Then $\mu(K \cap U_n) \geq \frac{1}{2}$ for each $n \in \omega$.

For each $k \in \omega$, define $A_k \subset K$ by

$$A_k = \{ x \in K : \{ n \in \omega : x \in U_n \} = k \}.$$

Then $\mu(K) = \sum_{k \in \omega} \mu(A_k)$. So there exists $m \in \omega$ such that $\sum_{k \geq m} \mu(A_k) < \frac{\delta}{2}$. For each $n < m$, choose a compact subset $C_n$ of $A_n$ so that $\mu(A_n \setminus C_n) \leq \frac{\delta}{2m}$.

Put $C = \bigcup_{n < m} C_n$. Then $\mu(\bigcup_{n < m} A_n \setminus C) \leq \frac{\delta}{2}$. Since

$$\mu(K \setminus C) = \sum_{n \geq m} \mu(A_n) + \mu(\bigcup_{n < m} A_n \setminus C) < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

$\mu(C \cap U_n) \geq \mu(C) + \frac{1}{2} - 1 > 0$ for $n \in \omega$. However, $\sum_{n \in \omega} \mu(C \cap U_n) \leq m \cdot \mu(C) < \infty$ by $C_n \subset A_n$ for $n < m$. This is a contradiction. Therefore $\mu(\{ x \in 2^\omega : \exists n (x \in U_n) \}) \geq \frac{1}{2}$.  

□
Lemma 5.5. Given \(X \subset 2^\omega\).

(1) If \(\mu^*(X) < \frac{1}{2}\), then \(\{I_x : x \in X\}\) does not witness to \(\text{cov}^*(\mathcal{S})\).

(2) If \(\{I_x : x \in X\}\) does not witness to \(\text{cov}^*(\mathcal{S})\), then \(\mu^*(X) \leq \frac{1}{2}\).

Proof of Lemma. (1). Assume \(\mu^*(X) < \frac{1}{2}\). By the definition of the outer measure, there exists a compact subset \(K\) of \(2^\omega\) such that \(\mu(K) = \frac{1}{2}\) and \(K \cap X = \emptyset\).

Let \(\{U_n : n \in \omega\}\) be a strictly decreasing sequence of open sets such that \(K = \bigcap_{n \in \omega} U_n\). Choose \(V_n \in \Omega\) such that \(V_n \notin \{V_i : i < n\}\) and \(V_n \subset U_n\). Let \(Y = \{V_n : n \in \omega\}\).

Since \(K \cap X = \emptyset\), for each \(x \in X\), there exists \(n \in \omega\) such that \(x \notin U_n\). So \(|Y \cap I_x| < \omega\) for every \(x \in X\).

(2). Suppose \(\{I_x : x \in X\}\) does not witness to \(\text{cov}^*(\mathcal{S})\). Choose \(Y = \{U_n : n \in \omega\} \in [\Omega]^\omega\) such that \(|I_x \cap Y| < \omega\). By Lemma 5.4, \(\mu((x \in 2^\omega : |I_x \cap Y| = \omega)) = \frac{1}{2}\). So

\[
\mu^*(X) \leq \mu((x \in 2^\omega : |I_x \cap Y| < \omega)) \leq \frac{1}{2}.
\]

\(\square\)

Proof of Theorem 5.3. Firstly we shall show \(\text{cov}^*(\mathcal{S}) \leq \text{non}(\mathcal{N})\).

Let \(X\) be a non-null set with \(\mu^*(X) > 0\).

Claim 5.6. There exists \(Y \subset 2^\omega\) such that \(|Y| = |X|\) and \(\mu^*(Y) = 1\).

Then \(\{I_x : x \in Y\}\) is a witness to \(\text{cov}^*(\mathcal{S})\) by Lemma 5.5.

Next we shall show \(\text{cov}^*(\mathcal{S}) \geq \text{non}(\mathcal{N})\). Let \(\kappa < \text{non}(\mathcal{N})\) and let \(X \subset 2^\omega\) with \(|X| = \kappa\). Then \(\mu^*(X) = 0\). By Lemma 5.5, \(\{I_x : x \in X\}\) does not witness to \(\text{cov}^*(\mathcal{S})\). So \(\kappa < \text{cov}^*(\mathcal{S})\). Therefore \(\text{non}(\mathcal{N}) \leq \text{cov}^*(\mathcal{S})\). \(\square\)

Corollary 5.7. \(\mathcal{G}_{FC} \geq_K \mathcal{S}\) but \(\mathcal{G}_{FC} \not\leq_K \mathcal{S}\).

Proof. \(\mathcal{G}_{FC} \geq_K \mathcal{S}\) is proved in [15]. We shall only show \(\mathcal{G}_{FC} \not\leq_K \mathcal{S}\).

In the Cohen model, \(\text{cov}^*(\mathcal{G}_{FC}) = s_{pair} < \text{cov}^*(\mathcal{S}) = \text{non}(\mathcal{N})\) since \(s_{pair} \leq \text{non}(\mathcal{M})\) [13]. By Proposition 0.4, \(\mathcal{G}_{FC} \not\leq_K \mathcal{S}\) in the Cohen model. By absoluteness of the Katětov order on Borel ideals, \(ZFC \vdash \mathcal{G}_{FC} \not\leq_K \mathcal{S}\). \(\square\)

We need to find a Borel ideal \(\mathcal{I}\) such that \(\mathcal{I} \geq_K \mathcal{S}\) but for every \(X \in \mathcal{I}^+, \mathcal{I} \upharpoonright X \not\approx_K \mathcal{G}_{FC}\).

\(\text{nwd}\) denotes the ideal of nowhere dense subsets of \(\mathbb{Q}\).

By the Sierpiński’s characterization of \(\mathbb{Q}\) we have the following.

Theorem 5.8. \([2]\) \(\text{nwd} \simeq_K \text{nwd} \upharpoonright X\) for every \(X \in \text{nwd}^+\).
Given a forcing notion $\mathbb{P}$, we say an ideal $\mathcal{I}$ on $\omega$ is $\mathbb{P}$-indestructible if $\mathbb{P}$ does not add an infinite subset of $\omega$ which is almost disjoint from every element of $\mathcal{I}$. We say an ideal $\mathcal{I}$ is $\mathbb{P}$-destructible if $\mathcal{I}$ is not $\mathbb{P}$-indestructible. The ideal $nwd$ is important when we think which ideals on $\omega$ are Cohen-destructible.

**Theorem 5.9.** [8, 6] $\mathcal{I}$ is Cohen-destructible if and only if $\mathcal{I} \leq_{K} nwd$.

Using this theorem, we can decide the Katětov order between $\mathcal{G}_{FC}$ and $nwd$ and between $\mathcal{S}$ and $nwd$.

**Theorem 5.10.**

1. $\mathcal{S} \leq_{K} nwd$.
2. $\mathcal{G}_{FC} \not\leq_{K} nwd$.

**Proof.** Since adding $c^{+}$-many Cohen reals enlarges $\text{cov}^{*}(\mathcal{S}) = \text{non}(\mathcal{N}) \geq \text{cov}(\mathcal{M})$, Cohen forcing destroys $\mathcal{S}$. By Theorem 5.9, $\mathcal{S} \leq_{K} nwd$.

However, adding $\omega_{2}$-many Cohen reals implies that $\text{cov}^{*}(\mathcal{G}_{FC}) = s_{\text{pair}} \leq \text{non}(\mathcal{M}) = \omega_{1}$, while $\text{cov}^{*}(nwd) \geq \omega_{2}$. Hence $\mathcal{G}_{FC}$ is Cohen-indestructible. So $\mathcal{G}_{FC} \not\leq_{K} nwd$. □

By Theorem 5.8 and 5.10, $\mathcal{S}$ can not be replaced by $\mathcal{G}_{FC}$ in Theorem 5.1. So the answer of Question 5.2 is in the negative.

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**References**


Instituto de Matemáticas, UNAM, Apartado Postal 61-3, Xangari, 58089, Morelia, Michoacán, México.

E-mail address: michael@matmor.unam.mx

E-mail address: dmeza@matmor.unam.mx

Kurt Gödel Research Center for Mathematical Logic, Währinger Strasse 25, A-1090 Wien, Austria

E-mail address: minami@kurt.scitec.kobe-u.ac.jp