SUSLIN FORCING AND PARAMETRIZED $\diamondsuit$ PRINCIPLES

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Abstract. By using finite support iteration of Suslin c.c.c forcing notions we construct several models which satisfy some $\diamondsuit$-like principles while other cardinal invariants are larger than ω_1.

§1. Introduction. This work is about parametrized diamond principles, a broad framework of $\diamondsuit$-like principles introduced by Moore, Hrušák and Džamonja in [7] to analyze systematically $\diamondsuit$ and its consequences.

For our purpose call a triple (A, B, E) a Borel invariant if

1. $|A|, |B| \leq \mathfrak{c}$,
2. $E \subset A \times B$,
3. for each $a \in A$ there exists $b \in B$ such that $(a, b) \in E$,
4. for each $b \in B$ there exists $a \in A$ such that $(a, b) \notin E$ and,
5. A, B and E are Borel sets in some Polish space.

If a triple (A, B, E) is a Borel invariant, then its evaluation $\langle A, B, E\rangle$ is given by

$$\langle A, B, E\rangle = \min\{|X| : X \subset B \text{ and } \forall a \in A \exists b \in X\ (aEb)\}.$$

We call $F : 2^{<\omega_1} \to A$ a Borel function if $F \restriction 2^\alpha$ is a Borel function for $\alpha < \omega_1$. Then $\diamondsuit(A, B, E)$ is the following statement:

$\diamondsuit(A, B, E)$ For all Borel $F : 2^{<\omega_1} \to A$ there exists $g : \omega_1 \to B$ such that for every $f : \omega_1 \to 2$ the set $\{\alpha \in \omega_1 : F(f \restriction \alpha)Eg(\alpha)\}$ is stationary.

The witness g for given F in this statement is called $\diamondsuit(A, B, E)$-sequence for F.

Note. When we deal with a Borel invariant whose evaluation is a well-known cardinal invariant, we will use the cardinal invariant to denote the Borel invariant (e.g., we will use $\diamondsuit(\text{add}(\mathcal{N}))$ to denote $\diamondsuit(\mathcal{N}, \mathcal{N}, \not\supset)$).

In [7] Moore, Hrušák and Džamonja introduced several methods for constructing parametrized diamond principles.

THEOREM 1.1. *[7] Let $\mathbb{C}(\omega_1)$ and $\mathbb{B}(\omega_1)$ be the Cohen and random forcing corresponding to the product space 2^{ω_1}. Then $V^{\mathbb{C}(\omega_1)} \models$ "$\diamondsuit(non(\mathcal{M}))$" and $V^{\mathbb{B}(\omega_1)} \models$ "$\diamondsuit(non(\mathcal{N}))$".*

In [6] by using ω_1-stage finite support iteration several models which satisfy CH and some $\diamondsuit(A, B, E)$ while others fail are constructed. For countable support iteration, there is a general theorem to construct $\diamondsuit(A, B, E)$.

THEOREM 1.2. *[7] Suppose that $\langle \mathcal{Q}_\alpha : \alpha < \omega_2\rangle$ is a sequence of Borel partial orders such that for each $\alpha < \omega_2$ $\mathcal{Q}_\alpha$ is equivalent to $\wp(2)^+ \times \mathcal{Q}_\alpha$ as a forcing* *notion and let $\mathcal{P}_{\omega_2}$ be the countable support iteration of this sequence. If $\mathcal{P}_{\omega_2}$ is proper and (A, B, E) is a Borel invariant then $\mathcal{P}_{\omega_2}$ forces $\langle A, B, E\rangle \leq \omega_1$ iff $\mathcal{P}_{\omega_2}$ forces $\diamondsuit(A, B, E)$.*

This result is best possible because the following proposition holds.

PROPOSITION 1.3. *Let (A, B, E) be a Borel invariant. If $\diamondsuit(A, B, E)$ holds, then $\langle A, B, E\rangle \leq \omega_1$.*

In this paper we shall prove the consistency of $\diamondsuit(\mathfrak{x}) + \mathfrak{y} = \omega_2$ for several pairs $(\mathfrak{x}, \mathfrak{y})$ of cardinal invariants of the continuum. As mentioned above (Theorem1.2) this has been achieved before by Moore, Hrušák and Džamonja in [7]. They used countable support iteration to show $\diamondsuit(\mathfrak{x}) + \mathfrak{y} = \omega_2$.

Our approach is completely different from theirs. We shall use finite support iteration of Suslin c.c.c forcing notions to prove the consistency of $\diamondsuit(\mathfrak{x})+\mathfrak{y} = \omega_2$. In addition, our results are more general. We can obtain the consistency of $\diamondsuit(\mathfrak{x}) + \mathfrak{y} = \kappa$, not just of $\diamondsuit(\mathfrak{x}) + \mathfrak{y} = \omega_2$.

Along the way, new preservation results for finite support iteration are established. These are interesting in their own right.

The present paper is organized as follows. In section 2, we will show some properties of Suslin c.c.c forcing. Section 3 is devoted to prove preservation results for finite support iteration of some Suslin c.c.c forcing notions.

In section 4, we shall present several models satisfying parametrized diamond principles by using ω_2-stage finite support iteration of Suslin c.c.c forcing notions.

§2. Suslin c.c.c forcing and complete embedding. In this section we will study some properties of a family of c.c.c forcing notions which have a nice definition.

DEFINITION 2.1. *[1, p.168] A forcing notion $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}}\rangle$ has a Suslin definition if $\mathbb{P} \subset \omega^\omega$, $\leq_{\mathbb{P}} \subset \omega^\omega \times \omega^\omega$ and $\perp_{\mathbb{P}} \subset \omega^\omega \times \omega^\omega$ are Σ^1_1.*
$\mathbb{P}$ is Suslin c.c.c if $\mathbb{P}$ is c.c.c and has a Suslin definition.

DEFINITION 2.2. *[1, p.168] Let $M \models ZFC^*$. A Suslin c.c.c forcing $\mathbb{P}$ is in M if all the parameters used in the definitions of $\mathbb{P}$, $\leq_{\mathbb{P}}$ and $\perp_{\mathbb{P}}$ are in M.*

We will interpret a Suslin c.c.c forcing notion in forcing extensions by using its code rather than by taking the ground model forcing notion.

DEFINITION 2.3. *Let $\mathbb{A}$ and $\mathbb{B}$ be forcing notions. Then $i : \mathbb{A} \to \mathbb{B}$ is a complete embedding if*

1. *whenever $a, a' \in \mathbb{A}$ and $a \leq a'$, then $i(a) \leq i(a')$,*
2. *for all a_1, $a_2 \in \mathbb{A}$, $a_1 \perp a_2$ if and only if $i(a_1) \perp i(a_2)$ and*
3. *whenever $\mathcal{A}$ is a maximal antichain in $\mathbb{A}$, then $i[\mathcal{A}]$ is a maximal antichain in $\mathbb{B}$.*

If there is a complete embedding from $\mathbb{A}$ to $\mathbb{B}$, then we write $\mathbb{A} \lessdot \mathbb{B}$.

LEMMA 2.4. *Assume $\mathbb{A} \lessdot \mathbb{B}$ and $\mathcal{P}$ is a Suslin c.c.c forcing notion. Then $\mathbb{A} * \dot{\mathcal{P}} \lessdot \mathbb{B} * \dot{\mathcal{P}}$ where $\dot{\mathcal{P}}$ are names for interpretations of the code for the Suslin c.c.c forcing notion in each model.*

Proof. Let $i : \mathbb{A} \to \mathbb{B}$ be a complete embedding. Define $\hat{i} : \mathbb{A} * \dot{\mathcal{P}} \to \mathbb{B} * \dot{\mathcal{P}}$ by $\hat{i}(\langle a, \dot{x}\rangle) = \langle i(a), i_*(\dot{x})\rangle$ where i_* is the class map from $\mathbb{A}$-names to $\mathbb{B}$-names induced by i (see [4, p.222]). We will show if $\mathcal{A} \subset \mathbb{A} * \dot{\mathcal{P}}$ is a maximal antichain, then $\hat{i}[\mathcal{A}]$ is also a maximal antichain.

Let $\mathcal{A} = \{\langle a_\alpha, \dot{p}_\alpha\rangle : \alpha < \kappa\}$ be a maximal antichain of $\mathbb{A} * \dot{\mathcal{P}}$. It is clear $\hat{i}[\mathcal{A}]$ is an antichain. Assume there exists $\langle b, \dot{p}\rangle \in \mathbb{B} * \dot{\mathcal{P}}$ such that $\langle b, \dot{p}\rangle$ is incompatible with $\hat{i}(\langle a_\alpha, \dot{p}_\alpha\rangle)$ for all $\alpha < \kappa$. Let G be a $(\mathbb{B}, V)$-generic such that $b \in G$ and let $H = i^{-1}[G]$. Let $\mathcal{A}' = \{\dot{p}_\alpha[H] : i(a_\alpha) \in G\} \in V[H]$.

Claim 2.4.1. *$V[H] \models$ "$\mathcal{A}'$ is a maximal antichain".*

Proof of claim. Firstly we shall show $\mathcal{A}'$ is an antichain. Suppose $\dot{p}_\alpha[H]$, $\dot{p}_\beta[H] \in \mathcal{A}'$. Since a_α, $a_\beta \in H$, a_α is compatible with a_β . Since $\langle a_\alpha, \dot{p}_\alpha\rangle$ is incompatible with $\langle a_\beta, \dot{p}_\beta\rangle$, for all $r \le a_\alpha, a_\beta$ there exists $s \le r$ such that $s \Vdash$ "$\dot{p}_\alpha$ is incompatible with $\dot{p}_\beta$". So $\dot{p}_\alpha[H]$ is incompatible with $\dot{p}_\beta[H]$. Hence $\mathcal{A}'$ is an antichain.

Next we shall show maximality of $\mathcal{A}'$. Assume to the contrary that there exists $q \in \mathcal{P}$ such that q is incompatible with $\dot{p}_\alpha[H]$ for any $\dot{p}_\alpha[H] \in \mathcal{A}'$. So there exist $a \in H$ and an $\mathbb{A}$-name $\dot{q}$ such that $\dot{q}[H] = q$ and $a \Vdash$ "$\forall\alpha < \kappa(a_\alpha \in \dot{H} \to \dot{q}$ is incompatible with $\dot{p}_\alpha)$". Hence $\langle a, \dot{q}\rangle$ is incompatible with $\langle a_\alpha, \dot{p}_\alpha\rangle$ for $\alpha < \kappa$. However, this contradicts the maximality of $\mathcal{A}$.

⊣

Since $V[H] \models$ "$\mathcal{A}'$ is a maximal antichain in $\mathcal{P}$" and the statement "$\mathcal{A}'$ is a maximal antichain in $\mathcal{P}$" is a Π^1_1 statement with parameter $\mathcal{A}'$, $\mathcal{P}$, $\le_{\mathcal{P}}$ and $\perp_{\mathcal{P}}$, $V[G] \models$ "$\mathcal{A}'$ is a maximal antichain in $\mathcal{P}$" by Π^1_1-absoluteness. However, this is a contradiction to the fact $V[G] \models$ "$\dot{p}[G]$ is incompatible with $i_*(\dot{p}_\alpha)[G]$" for $i(a_\alpha) \in G$.

⊣

Theorem 2.5. *Let $\langle Q_\alpha : \alpha < \kappa\rangle$ be a sequence of Suslin c.c.c forcing notions. Let $\mathbb{P}_\kappa$ be the limit of the finite support iteration of $\langle \mathbb{P}_\alpha, \dot{Q}_\alpha : \alpha < \kappa\rangle$. Then $\mathbb{A} \lessdot \mathbb{B}$ implies $\mathbb{A} * \dot{\mathbb{P}}_\kappa \lessdot \mathbb{B} * \dot{\mathbb{P}}_\kappa$*

Proof. By induction on κ. The limit stage is clear. The successor stage follows from the above lemma.

⊣

Corollary 2.6. *Let $\langle Q_\alpha : \alpha < \kappa\rangle$ be a sequence of Suslin c.c.c forcing notions. Let $I \subset \kappa$. Then $\mathbb{P}_I \lessdot \mathbb{P}_\kappa$ where $\mathbb{P}_I$ is the limit of the iteration of $\langle \mathbb{P}^\alpha_I, \dot{R}_\alpha : \alpha < \kappa\rangle$ where $\Vdash_{\mathbb{P}^\alpha_I} \dot{R}_\alpha = \begin{cases} \dot{Q}_\alpha & \alpha \in I \\ \{1\} & \text{otherwise.}\end{cases}$*

§3. Preservation results. In this section we shall show some preservation results for finite support iteration of Suslin c.c.c forcing notions. We deal with well-known Suslin forcing notions.

Definition 3.1. (1) *The Hechler forcing notion is defined as follows:*

$$\langle s, f\rangle \in \mathbb{D} \text{ if } s \in \omega^{<\omega}, f \in \omega^\omega \text{ and } s \subset f.$$

It is ordered by

$$\langle s, f\rangle \leq \langle t, g\rangle \text{ if } s \supset t \text{ and } g \leq f.$$

(2) *The eventually different forcing notion is defined as follows:*

$$\langle s, H\rangle \in \mathbb{E} \text{ if } s \in \omega^{<\omega} \text{ and } H \in [\omega^\omega]^{<\omega}.$$

It is ordered by $\langle s, H\rangle \leq \langle t, G\rangle$ *if* $s \supset t$, $H \supset G$ *and*

$$\text{for all } g \in G \text{ for all } j \in [|t|, |s|)\, s(j) \neq g(j).$$

(3) *Let* $\boldsymbol{Borel}(2^\omega)$ *be the smallest* σ*-algebra containing all open subsets of* 2^ω. *Let* μ *be the standard product measure on* 2^ω *and let* $\mathcal{N} = \{A \in \boldsymbol{Borel}(2^\omega) : \mu(A) = 0\}$. *For* $A, B \in \boldsymbol{Borel}(2^\omega)$ *let* $A \cong_{\mathcal{N}} B$ *if* $A \triangle B \in \mathcal{N}$. *Let* $[A]_{\mathcal{N}}$ *be the equivalence class of the set* A *with respect to the equivalence relation* $\cong_{\mathcal{N}}$.

Define the random forcing notion by

$$\mathbb{B} = \{[A]_{\mathcal{N}} : A \in \boldsymbol{Borel}(2^\omega)\}.$$

It is ordered by $[A]_{\mathcal{N}} \leq [B]_{\mathcal{N}}$ *if* $A \setminus B \in \mathcal{N}$.

Notice that $\mathbb{D}$, $\mathbb{E}$ and $\mathbb{B}$ are Suslin c.c.c.

The proof of the following result is similar to the argument showing that finite support iteration of Hechler forcings preserves $\mathrm{cov}(\mathcal{N})$.

Theorem 3.2. *Let* $\Pi = \langle I_n : n \in \omega\rangle$ *be a partition of* ω *into finite intervals* I_n *with* $|I_n| = n + 1$ *for* $n \in \omega$. *Suppose* γ *is an ordinal and* $\mathbb{P}$ *is a forcing notion which has a* $\mathbb{P}$*-name* $\dot{c}$ *such that for all* $x \in 2^\omega \cap V$, $\Vdash_{\mathbb{P}}$ "$\exists^\infty n\, (x \restriction I_n = \dot{c} \restriction I_n)$". *Let* $\dot{x}$ *be a* $\mathbb{D}_\gamma$*-name such that* $\Vdash_{\mathbb{D}_\gamma}$ "$\dot{x} \in 2^\omega$". *Then* $\Vdash_{\mathbb{P}*\dot{\mathbb{D}}_\gamma}$ "$\exists^\infty n\, (\dot{c} \restriction I_n = \dot{x} \restriction I_n)$".

More precisely we should write $\Vdash_{\mathbb{P}*\dot{\mathbb{D}}_\gamma}$ "$\exists^\infty n\, (\dot{c} \restriction I_n = i_*(\dot{x}) \restriction I_n)$" *where* i_* *is the canonical map from* $\mathbb{D}_\gamma$*-names to* $\mathbb{P} * \dot{\mathbb{D}}_\gamma$*-names induced by the complete embedding* $i : \mathbb{D}_\gamma \to \mathbb{P} * \dot{\mathbb{D}}_\gamma$.

Proof. We proceed by induction on γ.

First step

Let $\dot{x}$ be a $\mathbb{D}$-name such that $\Vdash_{\mathbb{D}}$ "$\dot{x} \in 2^\omega$". Let $\dot{c}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\exists^\infty n \in \omega\, (\dot{c} \restriction I_n = x \restriction I_n)$" for all $x \in V \cap 2^\omega$. Let $(p_0, \dot{q}_0) \in \mathbb{P} * \dot{\mathbb{D}}$ and $m \in \omega$.

It suffices to show there exist $(p_1, \dot{q}_1) \leq_{\mathbb{P}*\dot{\mathbb{D}}} (p_0, \dot{q}_0)$ and $n \geq m$ such that $(p_1, \dot{q}_1) \Vdash_{\mathbb{P}*\dot{\mathbb{D}}}$ "$\dot{x} \restriction I_n = \dot{c} \restriction I_n$".

Without loss of generality we can assume $p_0 \Vdash_{\mathbb{P}}$ "$\dot{q}_0 = \langle \check{s}, \dot{g}\rangle$" for some $s \in \omega^{<\omega}$.

Claim 3.2.1. *Let* $\dot{x}$ *be a* $\mathbb{D}$*-name such that* $\Vdash_{\mathbb{D}}$ "$\dot{x} \in 2^\omega$". *Then for each* $s \in \omega^{<\omega}$, *there exists* $x_s \in \omega^\omega \cap V$ *such that*

$$\forall j \in \omega \forall f \in \omega^\omega\, (f \supset s \to \neg\langle s, f\rangle \Vdash_{\mathbb{D}} \text{"} \dot{x} \restriction I_j \neq x_s \restriction I_j) .$$

Proof of Claim. It suffices to show that for each $s \in \omega^{<\omega}$ and $j \in \omega$, there exists $\sigma \in 2^{I_j}$ such that for each $f \in \omega^\omega$ with $s \subset f$, $\neg\langle s, f\rangle \Vdash_{\mathbb{D}}$ "$\dot{x} \restriction I_j \neq \sigma$".

Assume to the contrary that there exist $s \in \omega^{<\omega}$ and $j \in \omega$ such that for all $\sigma \in 2^{I_j}$, there exists $f_\sigma \in \omega^\omega$ with $s \subset f_\sigma$ such that $\langle s, f_\sigma\rangle \Vdash_{\mathbb{D}}$ "$\dot{x} \restriction I_j \neq \sigma$". Let

$f \in \omega^\omega$ such that $s \subset f$ and $f_\sigma \leq f$. Then $\langle s, f\rangle \leq \langle s, f_\sigma\rangle$ for $\sigma \in 2^{I_j}$. Therefore $\langle s, f\rangle \Vdash_{\mathbb{D}}$ " $\dot{x} \restriction I_j \notin 2^{I_j}$". This is a contradiction.

$\dashv$

Let $x_s \in V \cap 2^\omega$ such that

$$\forall j \in \omega \forall g' \in \omega^\omega \left(g' \supset s \to \neg\langle s, g'\rangle \Vdash_{\mathbb{D}} \text{“ } \dot{x} \restriction I_j \neq x_s \restriction I_j\text{”}\right).$$

Let $r \leq p_0$ such that $r \Vdash_{\mathbb{P}}$ " $x_s \restriction I_n = \dot{c} \restriction I_n$" for some $n \geq m$. Then fix $\langle r_k : k \in \omega\rangle$ a decreasing sequence in $\mathbb{P}$ and $g^* \in 2^\omega \cap V$ such that $r_0 \leq_{\mathbb{P}} r$ and $r_k \Vdash_{\mathbb{P}}$ " $\dot{g} \restriction (|s| + k) = g^* \restriction (|s| + k)$".

By definition of x_s there is $\langle t, h\rangle \leq_{\mathbb{D}} \langle s, g^*\rangle$ such that $\langle t, h\rangle \Vdash_{\mathbb{D}}$ " $x_s \restriction I_n = \dot{x} \restriction I_n$". Since $\langle t, h\rangle \leq_{\mathbb{D}} \langle s, g^*\rangle$, $g^*(l) \leq t(l)$ for $l \in [|s|, |t|)$. Since $r_{|t|} \Vdash_{\mathbb{P}}$ " $\forall i \in |t| \, (\dot{g}(i) = g^*(i) \leq t(i))$", $r_{|t|} \Vdash_{\mathbb{P}}$ " $\langle t, h\rangle$ is compatible with $\langle s, \dot{g}\rangle$". Put $p_1 = r_{|t|}$ and choose a $\mathbb{P}$-name $\dot{q}_1$ so that $p_1 \Vdash_{\mathbb{P}}$ " $\dot{q}_1 \in \mathbb{D}$ and $\dot{q}_1 \leq_{\mathbb{D}} \langle s, \dot{g}\rangle, \langle t, h\rangle$". Then $(p_1, \dot{q}_1) \leq_{\mathbb{P}*\dot{\mathbb{D}}} (p_0, \dot{q}_0)$ and $p_1 \Vdash_{\mathbb{P}}$ " $x_s \restriction I_n = \dot{c} \restriction I_n$" by $p_1 \leq_{\mathbb{P}} r$ and $p_1 \Vdash_{\mathbb{P}}$ " $\dot{q}_1 \Vdash_{\mathbb{D}} x_s \restriction I_n = \dot{x} \restriction I_n$" by $p_1 \Vdash_{\mathbb{P}}$ " $\dot{q}_1 \leq_{\mathbb{D}} \langle t, h\rangle$". Therefore $(p_1, \dot{q}_1) \Vdash_{\mathbb{P}*\dot{\mathbb{D}}}$ " $\dot{x} \restriction I_n = x_s \restriction I_n = \dot{c} \restriction I_n$".

Successor step:

Suppose the lemma holds for γ. Let $\dot{x}$ be a $\mathbb{D}_{\gamma+1}$-name such that $\Vdash_{\mathbb{D}_{\gamma+1}}$ " $\dot{x} \in 2^\omega$". Let $(p_0, \dot{q}_0) \in \mathbb{P} * \dot{\mathbb{D}}_{\gamma+1}$ and $m \in \omega$. Without loss of generality we can assume $(p_0, \dot{q}_0 \restriction \gamma) \Vdash_{\mathbb{P}*\dot{\mathbb{D}}_\gamma}$ " $\dot{q}_0(\gamma) = \langle \check{s}, \dot{g}\rangle$" for some $s \in \omega^{<\omega}$.

Let $\dot{x}_s$ be a $\mathbb{D}_\gamma$-name such that

$$\Vdash_{\mathbb{D}_\gamma} \text{“ } \forall j \in \omega \forall \dot{g}' \in \dot{\omega^\omega} \left(\dot{g}' \supset \check{s} \to \neg\langle \check{s}, \dot{g}'\rangle \Vdash_{\mathbb{D}} \text{“ } \dot{x}_s \restriction I_j \neq \dot{x} \restriction I_j\text{”}\right)\text{”}.$$

By induction hypothesis there are $(p', \dot{q}') \in \mathbb{P} * \dot{\mathbb{D}}_\gamma$ and $n \geq m$ such that $(p', \dot{q}') \leq_{\mathbb{P}*\dot{\mathbb{D}}_\gamma} (p_0, \dot{q}_0 \restriction \gamma)$ and $(p', \dot{q}') \Vdash_{\mathbb{P}*\dot{\mathbb{D}}_\gamma}$ " $\dot{x}_s \restriction I_n = \dot{c} \restriction I_n$".

Since $\mathbb{D}_\gamma \lessdot \mathbb{P} * \dot{\mathbb{D}}_\gamma$, there is a $\mathbb{D}_\gamma$-name $\dot{Q}$ for a partial order such that $\mathbb{P} * \dot{\mathbb{D}}_\gamma \cong \mathbb{D}_\gamma * \dot{Q}$. Let q^* be the projection of $(p', \dot{q}')$ to $\mathbb{D}_\gamma$.

Define $\mathbb{D}_\gamma$-names $\dot{g}^*$ and $\langle \dot{r}_k : k \in \omega\rangle$ such that

(i) $\Vdash_{\mathbb{D}_\gamma}$ " $\dot{g}^* \in \omega^\omega$ and $\dot{r}_k \in \dot{Q}$ " for $k \in \omega$,
(ii) $(q^*, \dot{r}_0) \leq (p', \dot{q}')$,
(iii) $\Vdash_{\mathbb{D}_\gamma}$ " $\dot{r}_{k+1} \leq_{\dot{Q}} \dot{r}_k$" for $k \in \omega$ and
(vi) $\Vdash_{\mathbb{D}_\gamma}$ "$\dot{r}_k \Vdash_{\dot{Q}}$ " $\dot{g}(k) = \dot{g}^*(k)$" ".

Let $q_1^* \leq_{\mathbb{D}_\gamma} q^*$, $t \in \omega^{<\omega}$ and let $\dot{h}$ be a $\mathbb{D}_\gamma$-name such that $\Vdash_{\mathbb{D}_\gamma}$ " $\dot{h} \in \omega^\omega$" and $q_1^* \Vdash_{\mathbb{D}_\gamma}$ "$\langle \check{t}, \dot{h}\rangle \leq_{\mathbb{D}} \langle s, \dot{g}^*\rangle$ and $\langle \check{t}, \dot{h}\rangle \Vdash_{\mathbb{D}}$ " $\dot{x} \restriction I_n = \dot{x}_s \restriction I_n$"". Since $(q_1^*, \dot{r}_{|t|}) \Vdash_{\mathbb{D}_\gamma * \dot{Q}}$ " $\forall i \in |t| \left(\dot{g}(i) = \dot{g}^*(i) \leq \dot{h}(i)\right)$", $(q_1^*, \dot{r}_{|t|}) \Vdash_{\mathbb{D}_\gamma * \dot{Q}}$ " $\langle t, \dot{h}\rangle$ is compatible with $\langle s, \dot{g}\rangle$".

Choose $(p_1, \dot{q}_1) \in \mathbb{P} * \dot{\mathbb{D}}_{\gamma+1}$ so that $(p_1, \dot{q}_1 \restriction \gamma) = (q_1^*, \dot{r}_{|t|})$ and $(p_1, \dot{q}_1 \restriction \gamma) = (q^*, \dot{r}_{|t|}) \Vdash_{\mathbb{D}_\gamma * \dot{Q}}$ " $\dot{q}_1(\gamma) \in \mathbb{D}$ and $\dot{q}_1(\gamma) \leq_{\mathbb{D}} \langle t, \dot{h}\rangle, \langle s, \dot{g}\rangle$". Then $(p_1, \dot{q}_1 \restriction \gamma) \Vdash_{\mathbb{P}*\dot{\mathbb{D}}_\gamma}$ " $\dot{c} \restriction I_n = \dot{x}_s \restriction I_n$" and $\dot{q}_1(\gamma) \Vdash_{\dot{\mathbb{D}}}$ " $\dot{x}_s \restriction I_n = \dot{x} \restriction I_n$". Therefore $(p_1, \dot{q}_1) \Vdash_{\mathbb{P}*\dot{\mathbb{D}}_{\gamma+1}}$ " $\dot{c} \restriction I_n = \dot{x} \restriction I_n$".

Limit step:

Suppose γ is a limit ordinal and for $\beta < \gamma$ the lemma holds. Without loss of generality we can assume the cofinality of γ is ω. Let $\langle \gamma_i : i \in \omega\rangle$ be a strictly increasing sequence converging to γ. Let $(p_0, \dot{q}_0) \in \mathbb{P} * \dot{\mathbb{D}}_\gamma$, $m \in \omega$ and $\dot{x}$ be a $\mathbb{D}_\gamma$-name such that $\Vdash_{\mathbb{D}_\gamma}$ " $\dot{x} \in 2^\omega$ ". Suppose $(p_0, \dot{q}_0) \in \mathbb{P} * \dot{\mathbb{D}}_{\gamma_j}$.

In $V^{\mathbb{D}_{\gamma_j}}$ let $\langle r_k : k \in \omega \rangle$ be a decreasing sequence in $\mathbb{D}_{[\gamma_j, \gamma)}$ such that $r_k \Vdash_{\mathbb{D}_{[\gamma_j,\gamma)}}$ "$\dot{x} \restriction I_k = x_j \restriction I_k$" where $x_j \in 2^\omega \cap V^{\mathbb{D}_{\gamma_j}}$.

Back in V let $\dot{r}_k$ and $\dot{x}_j$ be $\mathbb{D}_{\gamma_j}$-names such that $\Vdash_{\mathbb{D}_{\gamma_j}}$ "$\langle \dot{r}_k : k \in \omega \rangle$ and $\dot{x}_j$ satisfies the above".

By induction hypothesis there exist $\langle p', \dot{q}' \rangle \leq_{\mathbb{P} * \dot{\mathbb{D}}_{\gamma_j}} \langle p_0, \dot{q}_0 \rangle$ and $n \geq m$ such that $(p', \dot{q}') \Vdash_{\mathbb{P} * \dot{\mathbb{D}}_{\gamma_j}}$ " $\dot{c} \restriction I_n = \dot{x}_j \restriction I_n$". Put $p_1 = p'$ and $\Vdash_{\mathbb{P}}$ " $\dot{q}_1 = \dot{q}'^\frown \dot{r}_n$"

Then $\langle p_1, \dot{q}_1 \rangle \Vdash_{\mathbb{P} * \dot{\mathbb{D}}_\gamma}$ " $\dot{c} \restriction I_n = \dot{x}_j \restriction I_n = \dot{x} \restriction I_n$".

⊣

The proof of the following result is similar to the argument showing that finite support iteration of eventually different forcings preserves unbounded families.

THEOREM 3.3. *Suppose γ is an ordinal and $\mathbb{P}$ is a forcing notion which has a $\mathbb{P}$-name $\dot{c}$ such that $\Vdash_{\mathbb{P}}$ "$\exists^\infty n\, (x(n) < \dot{c}(n))$" for $x \in \omega^\omega \cap V$. Let $\dot{x}$ be a $\mathbb{E}_\gamma$-name such that $\Vdash_{\mathbb{E}_\gamma}$ " $\dot{x} \in \omega^\omega$". Then $\Vdash_{\mathbb{P} * \dot{\mathbb{E}}_\gamma}$ " $\exists^\infty n\, (\dot{x}(n) < \dot{c}(n))$".*

PROOF. We proceed by induction on γ. We shall only prove the successor step. The rest of the proof is similar to the proof of Theorem 3.2.

Successor step:
Suppose the lemma holds for γ. Let $\dot{x}$ be a $\mathbb{E}_{\gamma+1}$-name such that $\Vdash_{\mathbb{E}_{\gamma+1}}$ " $\dot{x} \in \omega^\omega$". Let $(p_0, \dot{q}_0) \in \mathbb{P} * \dot{\mathbb{E}}_{\gamma+1}$ and $m \in \omega$. Without loss of generality we can assume $(p_0, \dot{q}_0 \restriction \gamma) \Vdash_{\mathbb{P} * \dot{\mathbb{E}}_\gamma}$ "$\dot{q}_0(\gamma) = \langle s, \dot{F} \rangle$ and $\dot{F} = \{\dot{f}_j : j < l\}$" for some $l \in \omega$ and $s \in \omega^{<\omega}$.

CLAIM 3.3.1. *[1, p367] Let $\dot{x}$ be a $\mathbb{E}$-name such that $\Vdash_{\mathbb{E}}$ " $\dot{x} \in \omega^\omega$". For each $s \in \omega^\omega$, $l \in \omega$ and $i \in \omega$, put*

$$x_{s,l}(i) = \min\{j \in \omega : \forall H \subset \omega^\omega \text{ with } |H| = l\, (\neg \langle s, H \rangle \Vdash_{\mathbb{E}} \text{ “} \dot{x}(i) > j \text{”})\}.$$

Then $x_{s,l} \in \omega^\omega$.

PROOF OF CLAIM. Fix $s \in \omega^{<\omega}$, $l \in \omega$ and $i \in \omega$. For $t \in \omega$ with $s \subset t$ put

$$A_t = \{H \in (\omega^\omega)^l : \forall f \in H \forall k \in [|s|, |t|)\, (f(k) \neq t(k))\}.$$

Then $(\omega^\omega)^l = \bigcup\{A_t : t \in \omega^{<\omega} \wedge s \subset t \wedge \exists G \in [\omega^\omega]^{<\omega}\, (\langle t, G \rangle \text{ decides } \dot{x}(i))\}$.

We assume ω is equipped with the cofinite topology and $(\omega^\omega)^l$ is equipped with the product topology. Since ω is compact in the topology, $(\omega^\omega)^l$ is also compact by Tychonoff's theorem.

Since $\{A_t : t \in \omega^{<\omega} \wedge s \subset t \wedge \exists G \in [\omega^\omega]^{<\omega}\, (\langle t, G \rangle \text{ decides } \dot{x}(i))\}$ is an open covering of $(\omega^\omega)^l$, there exist finitely many $t_0,\, t_1,\, \ldots\, t_{n-1}$ such that $(\omega^\omega)^k = A_{t_0} \cup A_{t_1} \cup \ldots \cup A_{t_{n-1}}$.

Pick $G_0,\, G_1,\, \ldots,\, G_n \in (\omega^\omega)^k$ and $j_0, j_1, \ldots, j_n \in \omega$ so that $\langle t_m, G_m \rangle \Vdash_{\mathbb{E}}$ " $\dot{x}(i) = j_m$" for $m < n$. Put $x_{s,l}(i) = \max\{j_m : m < n\}$. Then $x_{s,l}(i)$ is as desired:

For each $\langle s, H \rangle$ with $H \in (\omega^\omega)^l$, there is t_m with $m < n$ such that $H \in A_{t_m}$. Since $H \in A_{t_m}$, $\langle t_m, G_m \cup H \rangle \leq \langle t_m, G_m \rangle, \langle s, H \rangle$ and $\langle t_m, G_m \cup H \rangle \Vdash_{\mathbb{E}}$ " $\dot{x}(i) = j_m \leq x_{s,l}(i)$". Therefore $\neg \langle s, H \rangle \Vdash_{\mathbb{E}}$ " $\dot{x}(i) > x_{s,l}(i)$".

⊣

Apply this claim in $V^{\mathbb{E}_\gamma}$ for $\dot{x}$ and put $\dot{x}_{s,l}$ a $\mathbb{E}_\gamma$-name such that

$$\Vdash_{\mathbb{E}_\gamma} \text{“ } \dot{x}_{s,l}(i) = \min\{j : \forall \dot{H} \subset \omega^\omega \text{ with } |\dot{H}| = l \left(\neg\langle s, \dot{H}\rangle \Vdash \dot{x}(i) > j\right)\}\text{”}.$$

By induction hypothesis there are $(p', \dot{q}') \in \mathbb{P} * \dot{\mathbb{E}}_\gamma$ and $n \geq m$ such that $(p', \dot{q}') \leq_{\mathbb{P}*\dot{\mathbb{E}}_\gamma} (p_0, \dot{q}_0 \restriction \gamma)$ and $(p', \dot{q}') \Vdash_{\mathbb{P}*\dot{\mathbb{E}}_\gamma}$ “ $\dot{c}(n) > \dot{x}_{s,l}(n)$”. Since $\mathbb{E}_\gamma \lessdot \mathbb{P} * \dot{\mathbb{E}}_\gamma$, there is a $\mathbb{E}_\gamma$-name $\dot{Q}$ for a partial order such that $\mathbb{P} * \dot{\mathbb{E}}_\gamma \cong \mathbb{E}_\gamma * \dot{Q}$. Let q^* be a projection of $(p', \dot{q}')$ to $\mathbb{E}_\gamma$. Find $\mathbb{E}_\gamma$-names $\langle \dot{r}_k : k \in \omega\rangle$ and $\dot{F}^*$ such that

(i) $\Vdash_{\mathbb{E}_\gamma}$ “ $\dot{F}^* = \{\dot{f}^*_j : j < l\} \subset \omega^\omega$ and $\dot{r}_k \in \dot{Q}$” for $k \in \omega$,
(ii) $(q^*, \dot{r}_0) \leq (p', \dot{q}')$,
(iii) $\Vdash_{\mathbb{E}_\gamma}$ “ $\dot{r}_{k+1} \leq_{\dot{Q}} \dot{r}_k$” for $k \in \omega$ and,
(iv) $(q^*, \dot{r}_k) \Vdash_{\mathbb{E}_\gamma * \dot{Q}}$ “ $\forall j < l \left(\dot{f}^*_j(k) = \dot{f}_j(k)\right)$” for $k \in \omega$.

Then there are $q_1^* \leq_{\mathbb{E}_\gamma} q^*$, $t \in \omega^{<\omega}$ and a $\mathbb{E}_\gamma$-name $\dot{G}$ such that $q_1^* \Vdash_{\mathbb{E}_\gamma}$ “$\langle t, \dot{G}\rangle \leq_{\mathbb{E}} \langle s, \dot{F}^*\rangle$ and $\langle t, \dot{G}\rangle \Vdash_{\dot{\mathbb{E}}}$ “ $\dot{x}(n) \leq \dot{x}_{s,l}(n)$””.

Since $(q^*, \dot{r}_{|t|}) \Vdash_{\mathbb{E}_\gamma * \dot{Q}}$ “$\forall j < l \; \forall k < |t| \left(\dot{f}_j(k) = \dot{f}^*_j(k)\right)$” and $q_1^* \Vdash_{\mathbb{E}_\gamma}$ “ $\forall j < n \forall k \in [|s|, |t|) \left(\dot{f}^*_j(k) \neq t(k)\right)$”, $(q_1^*, \dot{r}_{|t|}) \Vdash_{\mathbb{E}_\gamma * \dot{Q}}$ “$\langle t, \dot{G}\rangle$ is compatible with $\langle s, \dot{F}\rangle$”.

Choose $(p_1, \dot{q}_1) \in \mathbb{P} * \dot{\mathbb{E}}_{\gamma+1}$ so that $(p_1, \dot{q}_1 \restriction \gamma) = (q_1^*, \dot{r}_{|t|})$ and $(p_1, \dot{q}_1 \restriction \gamma) \Vdash_{\mathbb{P}*\dot{\mathbb{E}}_\gamma}$ “ $\dot{q}_1(\gamma) \leq_{\mathbb{E}} \langle s, \dot{F}\rangle, \langle t, \dot{G}\rangle$”. Then $(p_1, \dot{q}_1 \restriction \gamma) \Vdash_{\mathbb{P}*\dot{\mathbb{E}}_\gamma}$ “ $\dot{x}_{s,l}(n) < \dot{c}(n)$ and $\dot{q}_1(\gamma) \Vdash_{\dot{\mathbb{E}}}$ “ $\dot{x}(n) \leq \dot{x}_{s,l}(n)$””. Therefore $(p_1, \dot{q}_1) \Vdash_{\mathbb{P}*\dot{\mathbb{E}}_{\gamma+1}}$ “ $\dot{x}(n) < \dot{c}(n)$”.

⊣

The proof of the following result is similar to the argument showing that finite support iteration of random forcings preserves unbounded families.

THEOREM 3.4. *Suppose γ is an ordinal and $\mathbb{P}$ is a forcing notion which has a $\mathbb{P}$-name $\dot{c}$ such that $\Vdash_{\mathbb{P}}$ “ $\exists^\infty n\,(x(n) < \dot{c}(n))$” for $x \in \omega^\omega \cap V$. Let $\dot{x}$ be a $\mathbb{B}_\gamma$-name such that $\Vdash$ “ $\dot{x} \in \omega^\omega$”. Then $\Vdash_{\mathbb{P}*\dot{\mathbb{B}}_\gamma}$ “ $\exists^\infty n\,(\dot{x}(n) < \dot{c}(n))$”.*

PROOF. We proceed by induction on γ. We shall prove only the successor step.

Successor step:

Suppose the lemma holds for γ. Let μ be a measure on $\mathbb{B}$. Let $\dot{x}$ be a $\mathbb{B}_{\gamma+1}$-name such that $\Vdash_{\mathbb{B}_{\gamma+1}}$ “ $\dot{x} \in \omega^\omega$”.

CLAIM 3.4.1. *Let $\dot{m}$ be a $\mathbb{B}$-name such that $\Vdash_{\mathbb{B}}$ “ $\dot{m} \in \omega$”. Then for each n, there exists $l \in \omega$ such that $\mu([\![\dot{m} \leq \check{l}]\!]) \geq 1 - \frac{1}{n}$.*

Apply this claim in $V^{\mathbb{B}_\gamma}$ for $\dot{x}(n)$ for $n < \omega$ and choose $\dot{x}^*$ a $\mathbb{B}_\gamma$-name such that

$$\Vdash_{\mathbb{B}_\gamma} \mu([\![\dot{x}(k) \leq \dot{x}^*(k)]\!]_{\dot{\mathbb{B}}}) \geq 1 - \frac{1}{2^k}.$$

Let $(p_0, \dot{q}_0) \in \mathbb{P} * \dot{\mathbb{B}}_{\gamma+1}$ and $m \in \omega$. Without loss of generality we can assume $(p_0, \dot{q}_0 \restriction \gamma) \Vdash_{\mathbb{P}*\dot{\mathbb{B}}_\gamma}$ “ $\mu(\dot{q}_0(\gamma)) \geq \dfrac{1}{2^l}$” for some $l \in \omega$. By induction hypothesis there are $(p', \dot{q}') \in \mathbb{P} * \dot{\mathbb{B}}_\gamma$ and $n \geq m, l$ such that $(p', \dot{q}') \leq_{\mathbb{P}*\dot{\mathbb{B}}_\gamma} (p, \dot{q} \restriction \gamma)$ and $(p', \dot{q}') \Vdash_{\mathbb{P}*\dot{\mathbb{B}}_\gamma}$ “ $\dot{x}^*(n) < \dot{c}(n)$”. Put $(p_1, \dot{q}_1) \in \mathbb{P} * \dot{\mathbb{B}}_{\gamma+1}$ so that $(p_1, \dot{q}_1 \restriction \gamma) =$

$(p', \dot{q}')$ and $(p_1, \dot{q}_1 \restriction \gamma) \Vdash_{\mathbb{P}*\dot{\mathbb{B}}_\gamma}$ " $\dot{q}_1(\gamma) \leq_{\mathbb{B}} \dot{q}_0(\gamma)$ and $\dot{q}_1(\gamma) \leq [\![\dot{x}(n) \leq \dot{x}^*(n)]\!]_{\mathbb{B}}$". Then $(p_1, \dot{q}_1 \restriction \gamma) \Vdash_{\mathbb{P}*\dot{\mathbb{B}}_\gamma}$ "$\dot{x}^*(n) < \dot{c}(n)$ and $\dot{q}_1(\gamma) \Vdash_{\mathbb{B}} \dot{x}(n) \leq \dot{x}^*(n)$"". Therefore $(p_1, \dot{q}_1) \Vdash_{\mathbb{P}*\dot{\mathbb{B}}_{\gamma+1}}$ " $\dot{x}(n) \leq \dot{x}^*(n) < \dot{c}(n)$".

$\dashv$

We shall show a preservation theorem for finite support iteration of complete Boolean algebras with a strictly positive finitely additive measure, where we say that a Boolean algebra $\mathcal{B}$ has a strictly positive finitely additive measure μ if μ is a function from $\mathcal{B}$ to $[0,1]$ such that

1. $\mu(\mathbf{0}_{\mathcal{B}}) = 0$,
2. $\mu(\mathbf{1}_{\mathcal{B}}) = 1$,
3. for every finite pairwise disjoint subset $\{a_i : i \in I\}$ of $\mathcal{B}$,
$$\mu(\bigvee_{i \in I} a_i) = \sum_{i \in I} \mu(a_i) \text{ and}$$
4. $a \neq \mathbf{0}_{\mathcal{B}}$ implies $\mu(a) > 0$.

Note that if a Boolean algebra has a strictly positive finitely additive measure, then the Boolean algebra is c.c.c.

Let $\mathbb{LOC} = \{\phi : \phi : \omega \to \omega^{<\omega}$ and $\exists k \in \omega \forall n \in \omega \left(|\phi(n)| \leq n^k\right)\}$. Define $\phi \not\sqsupset x$ if $\exists^\infty n\, (\phi(n) \not\ni x(n))$ for $\phi \in \mathbb{LOC}$ and $x \in \omega^\omega$.

THEOREM 3.5. *Suppose γ is an ordinal and $\mathbb{P}$ is a forcing notion which has a $\mathbb{P}$-name $\dot{c}$ such that $\Vdash_{\mathbb{P}}$ " $\exists^\infty n\, (\phi(n) \not\ni \dot{c}(n))$" for $\phi \in \mathbb{LOC} \cap V$. Let $\mathcal{B}_\gamma$ be a γ-stage finite support iteration of complete Boolean algebras with strictly positive finitely additive measure μ and which is Suslin c.c.c for each γ. Let $\dot{\phi}$ be a $\mathcal{B}_\gamma$-name such that $\Vdash_{\mathcal{B}_\gamma}$ " $\dot{\phi} \in \mathbb{LOC}$". Then $\Vdash_{\mathbb{P}*\dot{\mathcal{B}}_\gamma}$ " $\dot{\phi} \not\sqsupset \dot{c}$".*

PROOF. We proceed by induction on γ. We shall prove only the successor step.

Successor step:
Suppose for γ the lemma holds. Let $\dot{\phi}$ be a $\mathcal{B}_{\gamma+1}$-name such that $\Vdash_{\mathcal{B}_{\gamma+1}}$ " $\dot{\phi} \in \mathbb{LOC}$". Let $\dot{\psi}_i$ $(i < \omega)$, $\dot{p}_i$ $(i < \omega)$ and $\dot{k}_i$ $(i < \omega)$ be $\mathcal{B}_\gamma$-names such that

- $\Vdash_{\mathcal{B}_\gamma}$ " $\dot{\psi}_i \in \mathbb{LOC}$, $\dot{p}_i \in \dot{\mathcal{B}}$ and $\dot{k}_i \in \omega$" for $i < \omega$,
- $\Vdash_{\mathcal{B}_\gamma}$ "$\dot{p}_i \Vdash_{\dot{\mathcal{B}}} \forall n \in \omega \left(\dot{\phi}(n) \leq n^{\dot{k}_i}\right)$" and
- $\Vdash_{\mathcal{B}_\gamma}$ " $\dot{\psi}_i(n) = \{j : \mu\left([\![j \in \dot{\phi}(n)]\!]_{\dot{\mathcal{B}}} \wedge \dot{p}_i\right) \geq \frac{1}{n} \cdot \mu(\dot{p}_i)\}$".

CLAIM 3.5.1. $\Vdash_{\mathcal{B}_\gamma} \left|\dot{\psi}_i(n)\right| \leq n^{\dot{k}_i+1}$.

PROOF OF CLAIM. Since $\Vdash_{\mathcal{B}_\gamma}$ " $\sum_{j \in \omega} \mu([\![j \in \phi(n)]\!] \wedge \dot{p}_i) \leq n^{\dot{k}_i} \cdot \mu(\dot{p}_i)$", $\Vdash_{\mathcal{B}_\gamma}$ " $\left|\dot{\psi}_i(n)\right| \leq \frac{n^{\dot{k}_i} \cdot \mu(\dot{p}_i)}{\frac{1}{n}\mu(\dot{p}_i)} = n^{\dot{k}_i+1}$".

$\dashv$

Let $m \in \omega$ and $(p_0, \dot{q}_0) \in \mathbb{P} * \dot{\mathcal{B}}_{\gamma+1}$. Without loss of generality we can find $i \in \omega$ and $n_i \in \omega$ such that $(p, \dot{q} \restriction \gamma) \Vdash_{\mathbb{P}*\dot{\mathcal{B}}_\gamma}$ "$\mu\left(\dot{q}(\gamma) \wedge \dot{p}_i\right) \geq \frac{1}{n_i}$". By induction hypothesis there exist $(p', \dot{q}') \leq_{\mathbb{P}*\dot{\mathcal{B}}_\gamma} (p, \dot{q} \restriction \gamma)$ and $n \geq n_i, m$ such that $(p', \dot{q}') \Vdash_{\mathbb{P}*\dot{\mathcal{B}}_\gamma}$ " $\dot{c}(n) \notin \dot{\psi}_i(n)$". Without loss of generality we can assume p' decides $\dot{c}(n)$ and

$p' \Vdash_{\mathcal{B}}$ "$\dot{c}(n) = l$" for some $l \in \omega$. Since $(p', \dot{q}') \Vdash_{\mathbb{P}*\dot{\mathcal{B}}_\gamma}$ " $l \notin \dot{\psi}_i(n)$", $(p', \dot{q}') \Vdash_{\mathbb{P}*\dot{\mathcal{B}}_\gamma}$ " $\mu\left(\llbracket l \in \dot{\phi}(n)\rrbracket_{\dot{\mathcal{B}}} \wedge \dot{p}_i\right) < \frac{1}{n}$". So $(p', \dot{q}') \Vdash_{\mathbb{P}*\dot{\mathcal{B}}_\gamma}$ " $\mu\left(\llbracket l \notin \dot{\phi}(n)\rrbracket_{\dot{\mathcal{B}}} \wedge \dot{p}_i \wedge \dot{q}(\gamma)\right) > 0$". Put $(p_1, \dot{q}_1) \in \mathbb{P} * \dot{\mathcal{B}}_{\gamma+1}$ so that $(p_1, \dot{q}_1 \restriction \gamma) = (p', \dot{q}')$ and $(p_1, \dot{q}_1 \restriction \gamma) \Vdash_{\mathbb{P}*\dot{\mathcal{B}}_\gamma}$ " $\dot{q}_1(\gamma) = \llbracket l \notin \dot{\phi}(n)\rrbracket_{\dot{\mathcal{B}}} \wedge \dot{p}_i \wedge \dot{q}(\gamma)$". Then $(p_1, \dot{q}_1) \Vdash_{\mathbb{P}*\dot{\mathcal{B}}_{\gamma+1}}$ " $\dot{c}(n) = l \notin \dot{\phi}(n)$". $\dashv$

§4. Construction of Parametrized $\diamondsuit$ principles. We shall construct several models by finite support iteration of Suslin c.c.c forcing notions.

If two Borel invariants (A_1, B_1, E_1),(A_2, B_2, E_2) are comparable in the Borel Tukey order, then $\diamondsuit(A_1, B_1, E_1)$ and $\diamondsuit(A_2, B_2, E_2)$ satisfy some relation:

Definition 4.1. (Borel Tukey ordering [2]) Given a pair of Borel invariants (A_1, B_1, E_1) and (A_2, B_2, E_2), we say that $(A_1, B_1, E_1) \leq^B_T (A_2, B_2, E_2)$ if there exist Borel maps $\phi : A_1 \to A_2$ and $\psi : B_2 \to B_1$ such that $(\phi(a), b) \in E_2$ implies $(a, \psi(b)) \in E_1$.

Proposition 4.2. [7] Let (A_1, B_1, E_1) and (A_2, B_2, E_2) be Borel invariants. Suppose $(A_1, B_1, E_1) \leq^B_T (A_2, B_2, E_2)$ and $\diamondsuit(A_2, B_2, E_2)$ holds. Then $\diamondsuit(A_1, B_1, E_1)$ holds.

Concerning $\leq^B_T$, we know the following holds.

(Cichoń's diagram)

$$\begin{array}{ccccccc}
(\mathbb{R}, \mathcal{N}, \in) & \longleftarrow & (\mathcal{M}, \mathbb{R}, \not\ni) & \longleftarrow & (\mathcal{M}, \subset) & \longleftarrow & (\mathcal{N}, \subset) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & (\omega^\omega, \not\geq^*) & \longleftarrow & (\omega^\omega, \leq^*) & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(\mathcal{N}, \not\supset) & \longleftarrow & (\mathcal{M}, \not\supset) & \longleftarrow & (\mathbb{R}, \mathcal{M}, \in) & \longleftarrow & (\mathcal{N}, \mathbb{R}, \not\ni)
\end{array}$$

(The direction of the arrow is from larger to smaller in the Borel Tukey order).

Hence the following holds:

$$\begin{array}{ccccccc}
\diamondsuit(\mathrm{cov}(\mathcal{N})) & \longleftarrow & \diamondsuit(\mathrm{non}(\mathcal{M})) & \longleftarrow & \diamondsuit(\mathrm{cof}(\mathcal{M})) & \longleftarrow & \diamondsuit(\mathrm{cof}(\mathcal{N})) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & \diamondsuit(\mathfrak{b}) & \longleftarrow & \diamondsuit(\mathfrak{d}) & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\diamondsuit(\mathrm{add}(\mathcal{N})) & \longleftarrow & \diamondsuit(\mathrm{add}(\mathcal{M})) & \longleftarrow & \diamondsuit(\mathrm{cov}(\mathcal{M})) & \longleftarrow & \diamondsuit(\mathrm{non}(\mathcal{N}))
\end{array}.$$

(The direction of the arrow is the direction of the implication).

We call this diagram "Cichoń's diagram for parametrized diamonds". We will deal with Borel invariants in Cichoń's diagram.

Theorem 4.3. *Let κ be an ordinal with $cf(\kappa) > \omega_1$. Let $\mathbb{D}_\kappa$ be the κ-stage finite support iteration of $\mathbb{D}$. Then $V^{\mathbb{D}_\kappa} \models \diamondsuit(cov(\mathcal{N}))$.*

Proof. Let $\Pi = \langle I_n : n \in \omega \rangle$ be a partition of ω into finite intervals I_n with $|I_n| = n+1$ for $n \in \omega$. Define a relation $=^\infty_\Pi$ so that $x =^\infty_\Pi y$ if there exist infinitely many $n \in \omega$ such that $x \restriction I_n = y \restriction I_n$. We will show $V^{\mathbb{D}_\kappa} \models \diamondsuit(2^\omega, =^\infty_\Pi)$.

Let $\dot{F}$ be a $\mathbb{D}_\kappa$-name such that $\Vdash_{\mathbb{D}_\kappa}$ " $\dot{F} : 2^{<\omega_1} \to 2^\omega$". Since $\mathbb{D}_\kappa$ has the c.c.c, a real $\dot{r}_\alpha$ coding the Borel function $\dot{F} \restriction 2^\alpha$ appears at an intermediate stage. By $cf(\kappa) > \omega_1$ we can assume $\dot{F}$ is a $\mathbb{D}_\beta$-name for some $\beta < \kappa$. Since the cofinality of the oreder type of $[\beta, \kappa)$ is $cf(\kappa) > \omega_1$ for $\beta < \kappa$ and $\mathbb{D}_\kappa = \mathbb{D}_\beta * \dot{\mathbb{D}}_{[\beta,\kappa)}$, we can assume $\dot{F}$ is a Borel function in the ground model. Let F be a Borel function in the ground model. Let $\dot{c}_\alpha$ be a $\mathbb{D}_{\omega_1}$-name such that $\Vdash_{\mathbb{D}_{\omega_1}}$ " $\exists^\infty n\, (\dot{c}_\alpha \restriction I_n = \dot{x} \restriction I_n)$" for $\dot{x} \in 2^\omega \cap V^{\mathbb{D}_\alpha}$. We can obtain such $\dot{c}_\alpha$. For example let $\dot{c}_\alpha$ be a $\mathbb{D}_{\omega_1}$-name for a Cohen real over $V^{\mathbb{D}_\alpha}$.

We shall show $\Vdash_{\mathbb{D}_\kappa}$ " $\langle \dot{c}_\alpha : \alpha < \omega_1 \rangle$ is a $\diamondsuit(2^\omega, =^\infty_\Pi)$-sequence for F". Let $\dot{f}$ be a $\mathbb{D}_\kappa$-name such that $\Vdash_{\mathbb{D}_\kappa}$ " $\dot{f} : \omega_1 \to 2$". Then the following claim holds:

Claim 4.3.1. *Define $C_{\dot{f}} \subset \omega_1$ by*

$$C_{\dot{f}} = \{\alpha < \omega_1 : \dot{f} \restriction \alpha \text{ is a } \mathbb{D}_{\alpha \cup [\omega_1, \kappa)}\text{-name}\}.$$

Then $C_{\dot{f}}$ contains a club.

Remark 4.3.2. *More precisely we should write*

$$C_{\dot{f}} = \{\alpha < \omega_1 : \text{there exists a } \mathbb{D}_{\alpha \cup [\omega_1,\kappa)}\text{-name } \dot{x}_\alpha \text{ such that } \Vdash_{\mathbb{D}_\kappa} \text{“} \dot{f} \restriction \alpha = i_*(\dot{x}_\alpha) \text{”}\}$$

where i_* is the class function from $\mathbb{D}_{\alpha \cup [\omega_1,\kappa)}$-names to $\mathbb{D}_\kappa$-names induced by the complete embedding $i : \mathbb{D}_{\alpha \cup [\omega_1,\kappa)} \lessdot \mathbb{D}_\kappa$. For convenience we will think of a $\mathbb{D}_\kappa$-name $\dot{x}$ as $\mathbb{D}_I$-name if there exists a $\mathbb{D}_I$-name $\dot{y}$ such that $\Vdash_{\mathbb{D}_\kappa}$ " $\dot{x} = i_{I*}(\dot{y})$" where i_I is the complete embedding from $\mathbb{D}_I$ to $\mathbb{D}_\kappa$ defined by Corollary 2.6.

For $\alpha \in C_{\dot{f}}$, $F(\dot{f} \restriction \alpha)$ is a $\mathbb{D}_{\alpha \cup [\omega_1,\kappa)}$-name because $\dot{f} \restriction \alpha$ is $\mathbb{D}_{\alpha \cup [\omega_1,\kappa)}$-name and $F \in V$.

In $V^{\mathbb{D}_\alpha}$, $F(\dot{f} \restriction \alpha)$ is $\mathbb{D}_{[\omega_1,\kappa)}$-name such that $\Vdash_{\mathbb{D}_{[\omega_1,\kappa)}}$ " $F(\dot{f} \restriction \alpha) \in 2^\omega$" and $\dot{c}_\alpha$ is a $\mathbb{D}_{[\alpha,\omega_1)}$-name such that $\Vdash_{\mathbb{D}_{[\alpha,\omega_1)}}$ " $\exists^\infty n \in \omega (x \restriction I_n = \dot{c}_\alpha \restriction \alpha)$" for $x \in 2^\omega \cap V^{\mathbb{D}_\alpha}$. By Theorem 3.2, $\Vdash_{\mathbb{D}_{[\alpha,\kappa)}}$ " $\exists^\infty n \in \omega (F(\dot{f} \restriction \alpha) \restriction I_n = \dot{c}_\alpha \restriction I_n)$".

Back in V, $\Vdash_{\mathbb{D}_\kappa}$ " $\exists^\infty n \in \omega (F(\dot{f} \restriction \alpha) \restriction I_n = \dot{c}_\alpha \restriction I_n)$" for $\alpha \in C_{\dot{f}}$. Since $C_{\dot{f}}$ contains a club subset of ω_1, $\Vdash_{\mathbb{D}_\kappa}$ " $\langle \dot{c}_\alpha : \alpha \in \omega_1 \rangle$ is a $\diamondsuit(2^\omega, =^\infty_\Pi)$-sequence for F".

Let $\phi : 2^\omega \to \mathcal{N}$ be the function such that

$$\phi(x) = \{y \in 2^\omega : \exists^\infty n\, (x \restriction I_n = y \restriction I_n)\}.$$

Then $\phi : 2^\omega \to \mathcal{N}$ and the identity function $id : 2^\omega \to 2^\omega$ witness $(2^\omega, \mathcal{N}, \in) \leq^B_T (2^\omega, =^\infty_\Pi)$ (see [3, Theorem 5.11]). So $V^{\mathbb{D}_\kappa} \models \diamondsuit(2^\omega, \mathcal{N}, \in)$.

$\dashv$

Theorem 4.4. *Let κ be an ordinal with $cf(\kappa) > \omega_1$. Let $\mathbb{E}_\kappa$ be the κ-stage finite support iteration of $\mathbb{E}$. Then $V^{\mathbb{E}_\kappa} \models \diamondsuit(cov(\mathcal{N}))$ and $\diamondsuit(\mathfrak{b})$.*

Proof. $\Vdash_{\mathbb{E}_\kappa}$ " $\diamondsuit(\text{cov}(\mathcal{N}))$" is similar to the proof of Theorem 4.3. To prove $V^{\mathbb{E}_\kappa} \models \diamondsuit(\mathfrak{b})$, it suffices to show $\Vdash_{\mathbb{E}_\kappa}$ " there exists a $\diamondsuit(\omega^\omega, \omega^\omega, {}^* \not\geq)$-sequence for F" for each Borel function $F \in V$.

For each $\alpha < \omega_1$, let $\dot{c}_\alpha$ be a $\mathbb{E}_{\omega_1}$-name such that $\Vdash_{\mathbb{E}_{\omega_1}}$ " $\exists^\infty n \in \omega\, (\dot{x}(n) < \dot{c}_\alpha(n))$" for each $\mathbb{E}_\alpha$-name $\dot{x}$ such that $\Vdash_{\mathbb{E}_\alpha}$ " $\dot{x} \in \omega^\omega$". Let $F : 2^{<\omega_1} \to \omega^\omega$ be a Borel function in V. Let $\dot{f}$ be a $\mathbb{E}_\kappa$-name such that $\Vdash_{\mathbb{E}_\kappa}$ " $\dot{f} : \omega_1 \to 2$". Put $C_{\dot{f}} = \{\alpha < \omega_1 : \dot{f} \restriction \alpha$ is a $\mathbb{E}_{\alpha\cup[\omega_1,\kappa)}$-name$\}$. Then $C_{\dot{f}}$ contains a club subset of ω_1.

For $\alpha \in C_{\dot{f}}$, $F(\dot{f} \restriction \alpha)$ is a $\mathbb{E}_{\alpha\cup[\omega_1,\kappa)}$-name such that $\Vdash_{\mathbb{E}_{\alpha\cup[\omega_1,\kappa)}}$ " $F(\dot{f} \restriction \alpha) \in \omega^\omega$". By Theorem 3.3, $\alpha \in C_{\dot{f}}$ implies $\Vdash_{\mathbb{E}_\kappa}$ " $\exists^\infty n \in \omega(F(\dot{f} \restriction \alpha)(n) < \dot{c}_\alpha(n))$". So $\Vdash_{\mathbb{E}_\kappa}$ " $\langle \dot{c}_\alpha : \alpha < \omega_1 \rangle$ is a $\diamondsuit(\omega^\omega, \omega^\omega, \not\geq^*)$-sequence for F".

$\dashv$

THEOREM 4.5. *Let κ be an ordinal with $cf(\kappa) > \omega_1$. Let $\mathbb{B}_\kappa$ be the κ-stage finite support iteration of $\mathbb{B}$. Then $V^{\mathbb{B}_\kappa} \models \diamondsuit(\mathfrak{b})$.*

PROOF. It suffices to show $\Vdash_{\mathbb{B}_\kappa}$ " there exists a $\diamondsuit(\omega^\omega, \omega^\omega, ^* \not\geq)$-sequence for F" for each Borel function $F \in V$.

For $\alpha < \omega_1$, let $\dot{c}_\alpha$ be a $\mathbb{B}_{\omega_1}$-name such that $\Vdash_{\mathbb{B}_{\omega_1}}$ "$\exists n \in \omega\, (\dot{x}(n) < \dot{c}_\alpha(n))$" for each $\mathbb{B}_\alpha$-name $\dot{x}$ such that $\Vdash_{\mathbb{B}_\alpha}$ " $\dot{x} \in \omega^\omega$".

Let $F : 2^{<\omega_1} \to \omega^\omega$ be a Borel function in V. Let $\dot{f}$ be a $\mathbb{B}_\kappa$-name such that $\Vdash_{\mathbb{B}_\kappa}$ " $\dot{f} : \omega_1 \to 2$". Put $C_{\dot{f}} = \{\alpha < \omega_1 : \dot{f} \restriction \alpha$ is a $\mathbb{B}_{\alpha\cup[\omega_1,\kappa)}$-name$\}$. Then $C_{\dot{f}}$ contains a club subset of ω_1.

For $\alpha \in C_{\dot{f}}$, $F(\dot{f} \restriction \alpha)$ is a $\mathbb{B}_{\alpha\cup[\omega_1,\kappa)}$-name such that $\Vdash_{\mathbb{B}_{\alpha\cup[\omega_1,\kappa)}}$ " $F(\dot{f} \restriction \alpha) \in \omega^\omega$". By Theorem 3.4, $\alpha \in C_{\dot{f}}$ implies $\Vdash_{\mathbb{B}_\kappa}$ " $\exists^\infty n \in \omega(F(\dot{f} \restriction \alpha) < \dot{c}_\alpha(n))$". So $\Vdash_{\mathbb{B}_\kappa}$ " $\langle \dot{c}_\alpha : \alpha < \omega_1 \rangle$ is a $\diamondsuit(\omega^\omega, \omega^\omega, \not\geq^*)$-sequence for F".

$\dashv$

THEOREM 4.6. *Let κ be an ordinal with $cf(\kappa) > \omega_1$. Let $(\mathbb{B} * \dot{\mathbb{D}})_\kappa$ be the κ-stage finite support iteration of $\mathbb{B} * \dot{\mathbb{D}}$. Then $V^{(\mathbb{B} * \dot{\mathbb{D}})_\kappa} \models \diamondsuit(add(\mathcal{N}))$.*

PROOF. We shall show $V^{(\mathbb{B} * \dot{\mathbb{D}})_\kappa} \models \diamondsuit(\mathbb{LOC}, \omega^\omega, \not\ni)$. Without loss of generality we can assume $\mathbb{B} * \dot{\mathbb{D}}$ is a complete Boolean algebra with strictly positive finitely additive measure μ [1, p319 Lemma 6.5.18].

For $\alpha < \omega_1$, let $\dot{c}_\alpha$ be a $(\mathbb{B} * \dot{\mathbb{D}})_{\omega_1}$-name such that $\Vdash_{(\mathbb{B} * \dot{\mathbb{D}})_{\omega_1}}$ " $\dot{\phi} \not\ni \dot{c}_\alpha$" for each $(\mathbb{B} * \dot{\mathbb{D}})_\alpha$-name $\dot{\phi}$ such that $\Vdash_{(\mathbb{B} * \dot{\mathbb{D}})_\alpha}$ " $\dot{\phi} \in \mathbb{LOC}$".

To prove $V^{(\mathbb{B} * \dot{\mathbb{D}})_\kappa} \models \diamondsuit(\mathbb{LOC}, \omega^\omega, \not\ni)$, it suffices to show that for each Borel function $F : 2^{<\omega_1} \to \mathbb{LOC} \in V$, $\Vdash_{(\mathbb{B} * \dot{\mathbb{D}})_\kappa}$ " $\langle \dot{c}_\alpha : \alpha < \omega_1 \rangle$ is a $\diamondsuit(\mathbb{LOC}, \omega^\omega, \not\ni)$-sequence for F".

Let $F : 2^{<\omega_1} \to \mathbb{LOC}$ be a Borel function in V. Let $\dot{f}$ be a $(\mathbb{B} * \dot{\mathbb{D}})_\kappa$-name such that $\Vdash_{(\mathbb{B} * \dot{\mathbb{D}})_\kappa}$ " $\dot{f} : \omega_1 \to 2$". Put $C_{\dot{f}} = \{\alpha < \omega_1 : \dot{f} \restriction \alpha$ is a $(\mathbb{B} * \dot{\mathbb{D}})_{\alpha\cup[\omega_1,\kappa)}$-name$\}$. Then $C_{\dot{f}}$ contains a club subset of ω_1.

For $\alpha \in C_{\dot{f}}$, $F(\dot{f} \restriction \alpha)$ is a $(\mathbb{B} * \dot{\mathbb{D}})_{\alpha\cup[\omega_1,\kappa)}$-name such that $\Vdash_{(\mathbb{B} * \dot{\mathbb{D}})_{\alpha\cup[\omega_1,\kappa)}}$ " $F(\dot{f} \restriction \alpha) \in \mathbb{LOC}$". By Theorem 3.5, $\alpha \in C_{\dot{f}}$ implies $\Vdash_{(\mathbb{B} * \dot{\mathbb{D}})_\kappa}$ " $F(\dot{f} \restriction \alpha) \not\ni \dot{c}_\alpha$". So $\Vdash_{(\mathbb{B} * \dot{\mathbb{D}})_\kappa}$ " $\langle \dot{c}_\alpha : \alpha < \omega_1 \rangle$ is a $\diamondsuit(\mathbb{LOC}, \omega^\omega, \not\ni)$-sequence for F". So we have $V^{(\mathbb{B} * \dot{\mathbb{D}})_\kappa} \models \diamondsuit(\mathbb{LOC}, \omega^\omega, \not\ni)$.

We shall show $V^{(\mathbb{B} * \dot{\mathbb{D}})_\kappa} \models \diamondsuit(\mathcal{N}, \mathcal{N}, \not\supset)$. Let $\{C_{i,j}\}$ be a family of independent open sets with $\mu(C_{i,j}) = \frac{1}{(i+1)^2}$ for all i, j. Let $\Phi : \omega^\omega \to \mathcal{N}$ be the function such that

$$\Phi(f) = \bigcup_n \bigcap_{i \geq n} C_{i,f(i)}.$$

For each $B \in \mathcal{N}$ fix a compact set $K_B \subset \omega^\omega \setminus B$ with $\mu(K_B \cap U) > 0$ for any open set U with $K_B \cap U \neq \emptyset$. Let $\{\sigma_n^B : n \in \omega\}$ list all $\sigma \in \omega^{<\omega}$ with $K_B \cap [\sigma] \neq \emptyset$. Put

$$g(B,n,i) = \{j : K_B \cap [\sigma_n^B] \cap C_{i,j} = \emptyset\}$$

for $i, n \in \omega$. Fix $k(B,n)$ such that

$$|g(B,n,i)| \leq \frac{(i+1)^2}{2^{n+1}}$$

for $i \geq k(B,n)$. Define $\Psi : \mathcal{N} \to \mathbb{LOC}$ by

$$\Psi(B)(i) = \bigcup_{k(B,n) \leq i} g(B,n,i).$$

Then Ψ and Φ witness $(\mathcal{N}, \mathcal{N}, \not\supset) \leq_B^T (\mathbb{LOC}, \omega^\omega, \not\ni)$ (see [1, Theorem 2.3.9]). So $V^{(\mathbb{B} * \dot{\mathbb{D}})_\kappa} \models \diamondsuit(\mathcal{N}, \mathcal{N}, \not\supset)$.

$\dashv$

Corollary 4.7. *Each of the following are relatively consistent with ZFC:*

(i) $\mathfrak{c} = add(\mathcal{M}) = \omega_2 + \diamondsuit(cov(\mathcal{N}))$ *(see Diagram 1).*
(ii) $\mathfrak{c} = non(\mathcal{M}) = cov(\mathcal{M}) = \omega_2 + \diamondsuit(\mathfrak{b}) + \diamondsuit(cov(\mathcal{N}))$ *(see Diagram 2).*
(iii) $\mathfrak{c} = cov(\mathcal{N}) = cov(\mathcal{M}) = \omega_2 + \diamondsuit(\mathfrak{b})$ *(see Diagram 3).*
(iv) $\mathfrak{c} = cov(\mathcal{N}) = add(\mathcal{M}) = \omega_2 + \diamondsuit(add(\mathcal{N}))$ *(see Diagram 4).*

Proof. (i) Suppose $V \models$ CH. By Theorem 4.3 $V^{\mathbb{D}_{\omega_2}} \models \diamondsuit(\mathrm{cov}(\mathcal{N}))$. Since $\mathbb{D}_{\omega_2}$ adds ω_2-many dominating reals and Cohen reals, $V^{\mathbb{D}_{\omega_2}} \models \mathfrak{c} = \mathfrak{b} = \mathrm{cov}(\mathcal{M}) = \omega_2$. Since $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ (see [1], [5]),

$$V^{\mathbb{D}_{\omega_2}} \models \diamondsuit(\mathrm{cov}(\mathcal{N})) + \mathfrak{c} = \mathrm{add}(\mathcal{M}) = \omega_2.$$

Cichoń's diagram for parametrized diamonds looks as follows where an ω_2 means the corresponding evaluation of the Borel invariant is ω_2 while the parametrized diamond principle for the others hold.

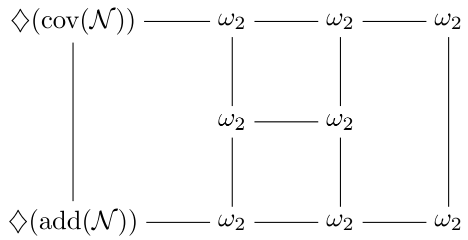

Diagram 1.

(ii) Suppose $V \models$ CH. By Theorem 4.4 $V^{\mathbb{E}_{\omega_2}} \models \diamondsuit(\mathrm{cov}(\mathcal{N})) + \diamondsuit(\mathfrak{b})$. Since $\mathbb{E}_{\omega_2}$ adds ω_2 many Cohen and eventually different reals, $\mathfrak{c} = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \omega_2$. Hence

$$V^{\mathbb{E}_{\omega_2}} \models \diamondsuit(\mathrm{cov}(\mathcal{N})) + \diamondsuit(\mathfrak{b}) + \mathfrak{c} = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}).$$

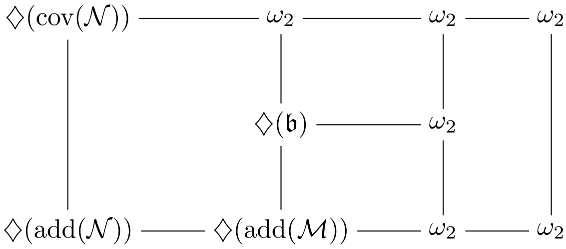

Diagram 2.

(iii) Suppose $V \models$ CH. By Theorem 4.5 $V^{\mathbb{B}_{\omega_2}} \models \diamondsuit(\mathfrak{b})$. Since $\mathbb{B}_{\omega_2}$ adds ω_2 many Cohen and random reals, $V^{\mathbb{B}_{\omega_2}} \models \mathfrak{c} = \mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \omega_2$. Hence

$$V^{\mathbb{B}_{\omega_2}} \models \diamondsuit(\mathfrak{b}) + \mathfrak{c} = \mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \omega_2.$$

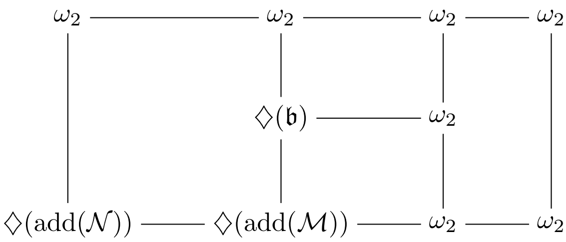

Diagram 3.

(iv) Suppose $V \models$ CH. By Theorem 4.6 $V^{(\mathbb{B}*\dot{\mathbb{D}})_{\omega_2}} \models \diamondsuit(\mathrm{add}(\mathcal{N}))$. Since $(\mathbb{B}*\dot{\mathbb{D}})_{\omega_2}$ adds ω_2 many random, Cohen and dominating reals, $\mathfrak{c} = \mathrm{cov}(\mathcal{N}) = \mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\} = \omega_2$. Hence

$$V^{(\mathbb{B}*\dot{\mathbb{D}})_{\omega_2}} \models \diamondsuit(\mathrm{add}(\mathcal{N})) + \mathfrak{c} = \mathrm{cov}(\mathcal{N}) = \mathrm{add}(\mathcal{M}) = \omega_2.$$

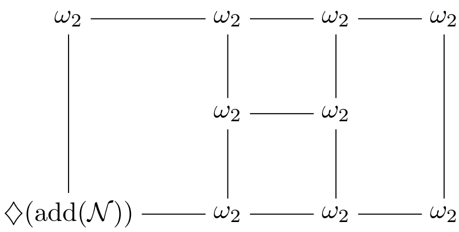

Diagram 4.

⊣

Hrušák asked the following question after a talk I gave at the 33rd Winter School on Abstract Analysis -Section of Topology held in the Czech Republic (2005 January).

Question 4.8 (Hrušák). *Let $\mathbb{A}$ be a amoeba forcing. Then $V^{\mathbb{A}_{\omega_2}} \models \diamondsuit(\mathfrak{s})$?*

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