Winter 2014, Introduction to mathematical logic

Series 1

Exercise 1 Prove or refute the following claim.

> For every language L, every L-structure $\mathfrak{A}$, every assignment β in A and all L-formulas φ, ψ: if variable x does not appear free in ψ and if $\mathfrak{A} \models (\varphi \to \psi)[\beta]$, then $\mathfrak{A} \models (\exists x\varphi \to \psi)[\beta]$.

Exercise 2 Let L be a language and $\mathfrak{A}, \mathfrak{B}$ be L-structures. We say $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$, and write $\mathfrak{A} \subseteq \mathfrak{B}$, if $A \subseteq B$ and for all $r \geq 1$:

(i) $R^{\mathfrak{B}} \cap A^r = R^{\mathfrak{A}}$ for all r-ary relation symbols $R \in L$;

(ii) $f^{\mathfrak{B}} \upharpoonright_{A^r} = f^{\mathfrak{A}}$ for all r-ary function symbols $f \in L$;

(iii) $c^{\mathfrak{B}} = c^{\mathfrak{A}}$ for all constants $c \in L$.

An L-formula is *quantifier-free* if the symbols $\exists, \forall$ do not occur in it. An L-formula φ is *universal* if $\varphi = \forall x_1 \cdots \forall x_k \psi$ for variables $x_1, \ldots, x_k$ and ψ a quantifier-free L-formula.

Let $\mathfrak{A} \subseteq \mathfrak{B}$, β an assignment in A and let φ be a universal L-formula. Show that $\mathfrak{B} \models \varphi[\beta]$ implies $\mathfrak{A} \models \varphi[\beta]$.

Exercise 3 Let L be a language and $\mathfrak{A}, \mathfrak{B}$ be L-structures. An *homomorphism form $\mathfrak{A}$ to $\mathfrak{B}$* is a function $h : A \to B$ such that for every $r \geq 1$, every $(a_1, \ldots, a_r) \in A^r$ and every r-ary relation symbol $R \in L$ and every r-ary function symbol $f \in L$ and every constant $c \in L$:

(i) if $(a_1, \ldots, a_r) \in R^{\mathfrak{A}}$, then $(h(a_1), \ldots, h(a_r)) \in R^{\mathfrak{B}}$;

(ii) $h(f^{\mathfrak{A}}(a_1, \ldots, a_r)) = f^{\mathfrak{B}}(h(a_1), \ldots, h(a_r))$;

(iii) $h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$.

An L-formula φ is *positive existential* if it can be built from atomic L-formulas via $\wedge, \vee$ and existential quantification $\exists x$.

Let h is an homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, β an assignment in A and let φ be a positive existential L-formula. Show that $\mathfrak{A} \models \varphi[\beta]$ implies $\mathfrak{B} \models \varphi[h \circ \beta]$.

Hint: induction on φ.

Exercise 4 let L be a language. An L-formula is in *prenex normal form* if it has the form $Q_1x_1 \cdots Q_kx_k\ \psi$ where $Q_1, \dots, Q_k \in \{\exists, \forall\}$ are quantifiers, $x_1, \dots, x_k$ are variables and ψ is quantifier free L-formula.

Two L-formulas φ and ψ are *logically equivalent* if $(\varphi \leftrightarrow \psi)$ is valid.

Prove that every L-formula is logically equivalent to an L-formula in prenex normal form.

Hint: induction on φ. Use that the formulas $(Qx\varphi \wedge \psi)$ and $Qx(\varphi \wedge \psi\frac{y}{x})$ are logically equivalent where Q is a quantifier and y is a "new" variable.