Winter 2014, Introduction to mathematical logic

Series 2

Exercise 1 Let L be a language. Let T be a set of L-formulas which is *deductively closed* in the sense that $\varphi \in T$ whenever $T \vdash \varphi$. Let Term_L be the set of all L-terms.

Show that

$$t \sim t' \quad \Longleftrightarrow \quad t \doteq t' \in T$$

defines an equivalence relation on Term_L.

For $t \in \mathit{Term}_L$ let $[t] := \{t' \in \mathit{Term}_L \mid t \sim t'\}$ denote the equivalence class of t. Define an L-structure $\mathfrak{T}(T)$ with universe $\{[t] \mid t \in \mathit{Term}_L\}$ as follows:

$$\begin{array}{rcl} ([t_1], \ldots, [t_r]) \in R^{\mathfrak{T}(T)} & \Longleftrightarrow & Rt_1 \cdots t_r \in T, \\ f^{\mathfrak{T}(T)}([t_1], \ldots, [t_r]) & := & [ft_1 \cdots t_r], \\ c^{\mathfrak{T}(T)} & := & [c], \end{array}$$

where $r \geq 1$ is a natural, $R \in L$ is an r-ary relation symbol, $f \in L$ is an r-ary function symbol and $c \in L$ is a constant $c \in L$.

Show that $\mathfrak{T}(T)$ is well-defined.

Let β be an assignment in $\mathfrak{T}(T)$ such that $\beta(x) = [x]$ for every variable x. Show that

1. $t^{\mathfrak{T}(T)}[\beta] = [t]$ for every $t \in \mathit{Term}_L$;
2. $\mathfrak{T}(T) \models \varphi[\beta] \Longleftrightarrow \varphi \in T$ for every atomic L-formula φ;

Exercise 2 This continues the above exercise. Prove:

1. If γ is an assignment in A sich that $\mathfrak{A} \models T[\gamma]$, then

   $$h([t]) := t^{\mathfrak{A}}[\gamma]$$

   defines an homomorphism from $\mathfrak{T}(T)$ into $\mathfrak{A}$.
2. $\mathfrak{T}(T) \models \varphi$ if $\varphi \in T$ is an L-sentence of the form $\forall \bar{x}((\varphi_0 \wedge \ldots \wedge \varphi_\ell) \rightarrow \varphi_{\ell+1})$ where each φ_i is an atomic L-formul;

3. In particular, if $L = \{^{-1}, e, \circ\}$ is the language of groups and T is the set of L-formulas logically implied by the group axioms, then $\mathfrak{T}(T)$ is a group such that for every finite or countable group $\mathfrak{G}$ there is a *surjective* homomorphism from $\mathfrak{T}(T)$ onto $\mathfrak{G}$.

Exercise 3 Let L be a language and T an L-theory. Call an L-sentence *T-provable* if it is an element of T or a tautology or an equality axiom or an $\exists$-axiom or can be obtained from T-provable formulas via Modus Ponens or $\exists$-introduction.

Show that an L-sentence φ T-provable if and only if $T \vdash \varphi$.