Winter 2015, Introduction to mathematical logic

Series 1

Exercise 1 Let $n, m \in \mathbb{N}$. Let $\alpha_{i,j}, \beta_i, \gamma$ be sentential formulas where $i \leq n$ and $j \leq m$. Prove

$$\neg \bigvee_{i \leq n} \beta_i \iff \bigwedge_{i \leq n} \neg \beta_i$$

$$(\bigvee_{i \leq n} \beta_i) \land \gamma \iff \bigvee_{i \leq n} (\beta_i \land \gamma)$$

$$\bigwedge \bigvee_{i \leq n, j \leq m} \alpha_{i,j} \iff \bigvee f_{i \leq n} \bigwedge_{i \leq n} \alpha_{i,f(i)}$$

where $F$ is the set of functions from $\{0, \ldots, n\}$ to $\{0, \ldots m\}$.

Exercise 2 What is the meaning of the functions $f$ and $g$ that satisfy the following for all $\alpha, \beta \in L$, all $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow, |\}$ and all $n \in \mathbb{N}, n \geq 1$?

(a)

$$f((\alpha \ast \beta)) = f(\alpha) + f(\beta) + 3$$

$$f(\neg \alpha) = f(\alpha) + 3$$

$$f(A_n) = 1$$

(b)

$$g((\alpha \ast \beta)) = g(\alpha) + f(\beta) + 1$$

$$g(\neg \alpha) = g(\alpha) + 1$$

$$g(A_n) = 0$$

Exercise 3 For $n \geq 1$ let $L_n := \{\alpha \in L \mid BKS(\alpha) \subseteq \{A_1, \ldots, A_n\}\}$. Compute the largest possible cardinality of a set $X \subseteq L_n$ such that the elements of $X$ are pairwise non-equivalent.
Exercise 4  Let $X$ be a set and $\mathcal{P}(X)$ its powerset. Let $B \subseteq \mathcal{P}(X)$ be finite. Let $K$ contain the functions

\[
\begin{align*}
(x, y) & \mapsto x \cap y \\
(x, y) & \mapsto x \cup y \\
x & \mapsto X \setminus x
\end{align*}
\]

Prove $|C(B, K)| \leq 2^{2^{|B|}}$.

Hint: Let $B = \{b_1, \ldots, b_n\}$. Show that each $x \in C(B, K)$ is a union of atoms, i.e., sets of the form $b_{i_1}^{\epsilon_1} \cap \cdots \cap b_{i_n}^{\epsilon_n}$ for $\epsilon_i \in \{0, 1\}$; here we write $y^1 := y$ and $y^0 := X \setminus y$.

Exercise 5 (Compactness theorem for sentential logic)  A set of sentential formulas $\Gamma$ is satisfiable if there exists a complete truth assignment $S$ such that $\bar{S}(\alpha) = T$ for all $\alpha \in \Gamma$. Prove that a set of sentential formulas $\Gamma$ is satisfiable if and only if every finite subset of $\Gamma$ is satisfiable.

Hint: enumerate $\Gamma$ as $\alpha_0, \alpha_1, \alpha_2, \ldots$. Consider a tree made of partial truth assignments that satisfy the first $n$ formulas (for some $n$). Apply König’s Lemma.