Winter 2015, Introduction to mathematical logic

Series 1

Exercise 1 Let $n, m \in \mathbb{N}$. Let $\alpha_{i,j}, \beta_i, \gamma$ be sentential formulas where $i \leq n$ and $j \leq m$. Prove

$$\begin{aligned} \neg \bigvee_{i \leq n} \beta_i &\Leftrightarrow \bigwedge_{i \leq n} \neg \beta_i \\ (\bigvee_{i \leq n} \beta_i) \wedge \gamma &\Leftrightarrow \bigvee_{i \leq n} (\beta_i \wedge \gamma) \\ \bigwedge_{i \leq n} \bigvee_{j \leq m} \alpha_{i,j} &\Leftrightarrow \bigvee_{f \in F} \bigwedge_{i \leq n} \alpha_{i,f(i)} \end{aligned}$$

where F is the set of functions from $\{0, \ldots, n\}$ to $\{0, \ldots m\}$.

Exercise 2 What is the meaning of the functions f and g that satisfy the following for all $\alpha, \beta \in \mathcal{L}$, all $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow, |\}$ and all $n \in \mathbb{N}, n \geq 1$?

(a)

$$\begin{aligned} f((\alpha * \beta)) &= f(\alpha) + f(\beta) + 3 \\ f((\neg \alpha)) &= f(\alpha) + 3 \\ f(\mathtt{A}_n) &= 1 \end{aligned}$$

(b)

$$\begin{aligned} g((\alpha * \beta)) &= g(\alpha) + f(\beta) + 1 \\ g((\neg \alpha)) &= g(\alpha) + 1 \\ g(\mathtt{A}_n) &= 0 \end{aligned}$$

Exercise 3 For $n \geq 1$ let $\mathcal{L}_n := \{\alpha \in \mathcal{L} \mid BKS(\alpha) \subseteq \{\mathtt{A}_1, \ldots, \mathtt{A}_n\}\}$. Compute the largest possible cardinality of a set $X \subseteq \mathcal{L}_n$ such that the elements of X are pairwise non-equivalent.

Exercise 4 Let X be a set and $\mathcal{P}(X)$ its powerset. Let $B \subseteq \mathcal{P}(X)$ be finite. Let K contain the functions

$$\begin{aligned} (x, y) &\mapsto x \cap y \\ (x, y) &\mapsto x \cup y \\ x &\mapsto X \setminus x \end{aligned}$$

Prove $|C(B, K)| \leq 2^{2^{|B|}}$.

Hint: Let $B = \{b_1, \ldots, b_n\}$. Show that each $x \in C(B, K)$ is a union of *atoms*, i.e., sets of the form $b_1^{\epsilon_1} \cap \cdots \cap b_n^{\epsilon_n}$ for $\epsilon_i \in \{0, 1\}$; here we write $y^1 := y$ and $y^0 := X \setminus y$.

Exercise 5 (Compactness theorem for sentential logic) A set of sentential formulas Γ is *satisfiable* if there exists a complete truth assignment S such that $\bar{S}(\alpha) = T$ for all $\alpha \in \Gamma$. Prove that a set of sentential formulas Γ is satisfiable if and only if every finite subset of Γ is satisfiable.

Hint: enumerate Γ as $\alpha_0, \alpha_1, \alpha_2, \ldots$. Consider a tree made of partial truth assignments that satisfy the first n formulas (for some n). Apply König's Lemma.