Winter 2015, Introduction to mathematical logic

Series 10

Let $\mathcal{L}$ be the language of Peano arithmetic.

Exercise 1 Let T be an $\mathcal{L}$-theory. Let f be a unary function symbol outside $\mathcal{L}$, and $\psi(x, y)$ a Σ^0_1-formula with free variables x, y. Assume T proves both $\forall x \exists y \psi$ and $\forall x \forall y \forall y' (\psi(x, y) \wedge \psi(x, y') \rightarrow y = y')$.

Let $T(f)$ be the $\mathcal{L} \cup \{f\}$-theory $T \cup \{\psi(x, f(x))\}$. Bounded formulas are defined for the language $\mathcal{L} \cup \{f\}$ as they are for $\mathcal{L}$. The classes $\Sigma^0_t(f)$ and $\Pi^0_t(f)$ are defined as Σ^0_t and Π^0_t but for the language $\mathcal{L} \cup \{f\}$.

Sow that for all $t \geq 1$ there exists a map $\varphi \mapsto \varphi'$ from $\Sigma^0_t(f)$-formulas to Σ^0_t-formulas such that $T(f)$ proves $(\varphi \leftrightarrow \varphi')$.

Hint: first map each bounded $\mathcal{L} \cup \{f\}$-formula φ to an $\mathcal{L}$-formula φ' such that T proves $(\varphi \leftrightarrow \varphi')$ and φ' is Δ^0_1 *in* T, that is, T proves both $(\varphi' \leftrightarrow \chi_0)$ and $(\varphi' \leftrightarrow \chi_1)$ for some Σ^0_1-formula χ_0 and some Π^0_1-formula χ_1.

A closed $\mathcal{L}$-formula φ is *true* if $\mathbb{N} \models \varphi$. Recall that PA proves all true φ in Σ^0_1. Let T be an $\mathcal{L}$-theory containing PA. Assume $\mathit{Proof}_T(x, y)$ is Δ^0_1 in PA and such that for all $\mathcal{L}$-formulas ψ:

$$T \vdash \psi \iff \mathbb{N} \models \exists x \mathit{Proof}_T(x, \underline{\ulcorner \psi \urcorner})$$

Let $\mathit{Thm}_T(y)$ be the formula $\exists x \mathit{Proof}_T(x, y)$.

Note that it is *not* assumed that T is true.

Exercise 2 Show that for all all $\mathcal{L}$-formulas ψ:

$$T \vdash \psi \iff \text{there exists } n \in \mathbb{N} : T \vdash \mathit{Proof}_T(\underline{n}, \underline{\ulcorner \psi \urcorner})$$

Exercise 3 (Gödel) Let γ be a closed $\mathcal{L}$-formula such that PA proves

$$\gamma \leftrightarrow \neg \mathit{Thm}_T(\underline{\ulcorner \gamma \urcorner})$$

Show:

1. If T is consistent, then $T \not\vdash \gamma$.
2. If T is ω-consistent, then $T \not\vdash \neg\gamma$.

That T is ω-*consistent* means that there is no $\mathcal{L}$-formula $\varphi(x)$ with free variable x such that $T \vdash \exists x \varphi(x)$ and $T \vdash \neg\varphi(\underline{n})$ for all $n \in \mathbb{N}$.

Exercise 4 (Rosser) Let ρ be a closed $\mathcal{L}$-formula such that PA proves

$$\rho \leftrightarrow \forall x (\mathit{Proof}_T(x, \underline{\ulcorner \rho \urcorner}) \rightarrow \exists z < x \ \mathit{Proof}_T(z, \underline{\ulcorner \neg\rho \urcorner}))$$

Show that, if T is consistent, then $T \nvdash \rho$ and $T \nvdash \neg\rho$.