Winter 2015, Introduction to mathematical logic

Series 11

Work in ZF, in particular, use the axiom of foundation.

Exercise 1 If x, y are transitive sets such that $(x, \in) \cong (y, \in)$, then $x = y$.

Exercise 2 A set is an ordinal if and only if it is transitive and all its members are transitive.

Exercise 3 Show that for all sets x, y:

1. $x \subseteq \mathrm{trcl}(x)$.
2. if $x \subseteq y$ and y is transitive, then $\mathrm{trcl}(x) \subseteq y$.
3. $\mathrm{trcl}(x) = \mathrm{trcl}(\mathrm{trcl}(x))$.
4. if $y \in x$, then $\mathrm{trcl}(y) \subseteq \mathrm{trcl}(x)$.
5. $y \in \mathrm{trcl}(x)$ if and only if there exists an *$\in$-path from y to x*, that is, a sequence $(z_0, \ldots, z_n) \in \mathrm{trcl}(\{x\})^{<\omega}$ where $n \in \omega$, such that $z_0 = y$ and $z_n = x$ and $z_i \in z_{i+1}$ for all $i < n$.

Exercise 4 (Class form of foundation (Gödel)) Let $\varphi(x)$ be a formula of set theory. Prove

$$\exists x \varphi(x) \to \exists y(\varphi(y) \land \forall z(z \in y \to \neg\varphi(z))).$$

Hint: choose x such that $\varphi(x)$ and consider $\mathrm{trcl}(x)$.