Winter 2015, Introduction to mathematical logic

Series 12

Exercise 1 Work in ZF. Show $\mathcal{H}(\alpha)$ is the smallest cardinal strictly bigger than α. Show that the range of $\aleph$ consists exactly in the infinite cardinals.

Exercise 2 Work in ZF. A function $F : ON \to ON$ is *normal* if it is

1. *increasing*, i.e. $F(\alpha) < F(\beta)$ for all $\alpha < \beta \in ON$, and
2. *continuous*, i.e. $F(\lambda) = \bigcup_{\beta<\lambda} F(\beta)$ for all limit ordinals λ.

Show that for every α there is $\beta > \alpha$ such that $F(\beta) = \beta$, i.e. β is a *fixed point of* F. Show that if x is a set of fixed points of F, then $\bigcup x$ is a fixed point of F.

Exercise 3 Work in ZFC. Assume X, Y are topological spaces that are compact and Hausdorff. A continuous function $f : X \to Y$ is *irreducible* if it is surjective but $f \restriction Z$ is not surjective for any closed $Z \subsetneq X$. Show that for every continuous $f : X \to Y$ there exists a closed $X' \subseteq X$ such that $f \restriction X'$ is irreducible.

Hint: use Zorn's Lemma.

Exercise 4 Work in ZFC. Show that the following are equivalent for every set x.

1. x is infinite.
2. $\omega \preccurlyeq x$.
3. x is *Dedekind-infinite*, i.e. there exists an injection f of x into x which is not surjective.