**Winter 2015, Introduction to mathematical logic**

**Series 13**

**Exercise 1** For a model $\mathcal{M}$ of ZFC let

$$\omega^{\mathcal{M}} := \{a \in M \mid \mathcal{M} \models x \in \omega[a]\},$$

where "$x \in \omega$" abbreviates a suitable $\{\in\}$-formula.

Show that every model $\mathcal{M}$ of ZFC has an elementary extension $\mathcal{N}$ such that $\omega^{\mathcal{N}} \setminus \omega^{\mathcal{M}}$ is nonempty.

Show that there exists a nonempty $X \subseteq \omega^{\mathcal{N}}$ without $\in^{\mathcal{N}}$-minimal element (i.e. for all $a \in X$ there is $b \in X$ such that $b \in^{\mathcal{N}} a$.)

Work in ZF.

**Exercise 2 (Tarski)** Show that (AC) is equivalent to the statement that for every infinite $x$ there is a bijection from $x \times x$ onto $x$.

*Hint:* suppose $f$ is a bijection from $(x \cup \mathcal{H}(x))^2$ onto $x \cup \mathcal{H}(x)$. Show that for every $a \in x$ there is $\alpha \in \mathcal{H}(x)$ such that $f(\langle a, \alpha \rangle) \in \mathcal{H}(x)$. Let $\alpha_a$ be the minimal such $\alpha$. Then show $a \mapsto \langle \alpha_a, f(\langle a, \alpha_a \rangle)\rangle$ is an injection of $x$ into $\mathcal{H}(x) \times \mathcal{H}(x)$. Conclude that $x$ is well-orderable.

**Exercise 3** Let $\lambda$ be a limit ordinal. Show that

$$\mathrm{cf}(\lambda) = \min\{\alpha \in ON \mid \exists f \in {}^{\alpha}\lambda : \bigcup \mathrm{ran}(f) = \lambda\}.$$

**Exercise 4** Recall Exercise 2 from Series 12.

(a) If $\kappa$ is weakly inaccessible, then $\kappa$ is the $\kappa$-th fixed point of $\aleph$.

(b) If $\kappa$ is strongly inaccessible, then $\kappa$ is the $\kappa$-th fixed point of $\beth$.