Winter 2015, Introduction to mathematical logic

Series 3

Exercise 1 Let Γ be a theory. Let T be the set of terms in the language $\mathcal{L}(\Gamma)$ of Γ. Then

$$t \sim t' \iff \Gamma \vdash t = t'$$

defines an equivalence relation on T. For $t \in T$ let

$$[t] := \{t' \in T \mid t \sim t'\}$$

denote the equivalence class of t. Define a structure $\mathcal{T}_\Gamma$ for $\mathcal{L}(\Gamma)$ with universe

$$\{[t] \mid t \in T\}$$

as follows:

$$\begin{aligned} ([t_1], \dots, [t_r]) \in R^{\mathcal{T}_\Gamma} &\iff \Gamma \vdash Rt_1 \cdots t_r, \\ f^{\mathcal{T}_\Gamma}([t_1], \dots, [t_r]) &:= [ft_1 \cdots t_r], \\ c^{\mathcal{T}_\Gamma} &:= [c], \end{aligned}$$

where $r \in \mathbb{N}$ is a natural, R is an r-ary relation symbol of $\mathcal{L}(\Gamma)$, f is an r-ary function symbol of $\mathcal{L}(\Gamma)$ and c is a constant of $\mathcal{L}(\Gamma)$.

Prove:

(a) $\mathcal{T}_\Gamma$ is well-defined;

(b) if $t \in T$ has variables $\{x_1, \dots, x_k\}$ and $t_1, \dots, t_k \in T$, then

$$t(x_1/[t_1], \dots, x_k/[t_k])^{\mathcal{T}_\Gamma} = [t(x_1/t_1, \dots, x_k/t_k)];$$

(c) if φ is an atomic $\mathcal{L}(\Gamma)$-formula with free variables $\{x_1, \dots, x_k\}$ and $t_1, \dots, t_k \in T$, then

$$\mathcal{T}_\Gamma \models \varphi(x_1/[t_1], \dots, x_k/[t_k]) \iff \Gamma \vdash \varphi(x_1/t_1, \dots, x_k/t_k).$$

Exercise 2 This continues the above exercise. Prove:

(a) Let $\mathcal{A} \models \Gamma$. Let γ be a map from the set of variables into A. Set

$$h([t]) := t(x_1/\gamma(x_1), \ldots, x_k/\gamma(x_k))^{\mathcal{A}}$$

for each $t \in T$ with free variables $\{x_1, \ldots, x_k\}$. Show that h is a homomorphism from $\mathcal{T}_\Gamma$ into $\mathcal{A}$.

(b) Let φ be an $\mathcal{L}(\Gamma)$-sentence of the form $\forall \bar{x}((\varphi_0 \wedge \ldots \wedge \varphi_\ell) \to \varphi_{\ell+1})$ where each φ_i is atomic. Show:

$$\text{if } \Gamma \vdash \varphi, \text{ then } \mathcal{T}_\Gamma \models \varphi.$$

(c) Let $\mathcal{L}$ be the first-order language containing function symbols $+$ (binary), $-$ (unary) and a constant 0. Give a theory Γ in this language whose models are precisely the groups (with $+$ denoting the group operation, $-$ the inverse and 0 the neutral element).

Show that $\mathcal{T}_\Gamma$ is a group such that for every finite or countable group $\mathcal{G}$ there is a surjective homomorphism from $\mathcal{T}_\Gamma$ onto $\mathcal{G}$.

Exercise 3 Let $\mathcal{L}$ be as in the previous exercise (c). Show:

(a) There exists a theory Γ' whose models are precisely the infinite groups.

(b) If Γ'' is a theory whose models contain arbitrarily large finite groups, then there is an infinite group that is a model of Γ''.

(c) An $\mathcal{L}$-sentence that holds in all infinite groups, also holds in some finite group. Conclude that a theory Γ' as in (a) cannot be finite.

Exercise 4 Let $\mathcal{L}$ be an arbitrary first-order language. Let Γ, Γ' be theories in this language. Assume $\Gamma \cup \Gamma'$ does not have a model. Show that there exists an $\mathcal{L}$-sentence φ such that $\Gamma \vdash \varphi$ and $\Gamma' \vdash \neg\varphi$.