Winter 2015, Introduction to mathematical logic

Series 4

Exercise 1 (Herbrand's Theorem) Let Γ be a theory such that every sentence in Γ is *universal*, i.e. of the form $\forall x_1 \cdots \forall x_k \ \psi$ for ψ quantifier-free.

(a) Assume $\mathcal{M}$ is a model of Γ. Show that the set of $t^{\mathcal{M}}$ where t is an $\mathcal{L}(\Gamma)$-term without variables is either empty or the universe of a substructure $\mathcal{N}$ of $\mathcal{M}$. Further show that $\mathcal{N} \models \Gamma$.

(b) Let φ be quantifier-free with free variables x, y and assume $\Gamma \vdash \forall x \exists y \varphi$. Show that there exist finitely many $\mathcal{L}(\Gamma)$-terms $t_1, \ldots, t_n$ with no variable except possibly x such that

$$\Gamma \vdash \varphi(y/t_1) \vee \cdots \vee \varphi(y/t_k).$$

Hint: Add a new constant c to the language and let T be the set of terms in the new language without variables. Assuming that the conclusion above fails, show that $\Gamma \cup \{\neg\varphi(x/c, y/t) \mid t \in T\}$ is consistent (Compactness theorem).

Exercise 2 Let $\mathcal{L}$ be a first-order language with a binary relation symbol $<$ and no other non-logical symbol. A structure $\mathcal{M} = (M, <^{\mathcal{M}})$ for $\mathcal{L}$ is a *well-order* if $<^{\mathcal{M}}$ is a linear order on M such that for every $X \in \mathcal{P}(M) \setminus \{\emptyset\}$ there is $m \in X$ such that $m <^{\mathcal{M}} m'$ (more precisely, $(m, m') \in <^{\mathcal{M}}$) for all $m' \in X \setminus \{m\}$

(a) Show that a structure $\mathcal{M}$ for $\mathcal{L}$ is a well-order if and only if $\mathcal{M}$ is a linear order and there does not exist a sequence $(m_n)_{n \in \mathbb{N}}$ such that $m_{n+1} <^{\mathcal{M}} m_n$ for all $n \in \mathbb{N}$.

(b) There is no theory Γ in the language $\mathcal{L}$ such that for all structures $\mathcal{M}$ for $\mathcal{L}$:

$$\mathcal{M} \models \Gamma \iff \mathcal{M} \text{ is a well-order.}$$

Hint: Compactness theorem.

Exercise 3 Let $\mathcal{L}$ be the first-order language with binary function symbols $+, \cdot$, a unary function symbol $-$, and constants $0, 1$.

(a) For p either 0 or prime define an $\mathcal{L}$-theory Γ_{ACF_p} whose models are precisely the algebraically closed fields of characteristic p.

(b) Show that an $\mathcal{L}$-sentence holds in some algebraically closed field of characteristic 0 if it holds in all algebraically closed fields of sufficiently large characteristic.

Hint: You can use the fact that Γ_{ACF_0} is complete. Apply the Compactness theorem.

Exercise 4 Let $\mathcal{N}$ be a structure for the language $\mathcal{L}$. A substructure $\mathcal{M}$ of $\mathcal{N}$ is called *elementary* if for all $k \in \mathbb{N}$, all $\mathcal{L}$-formulas φ with free variables $x_1, \ldots, x_k$ and all $m_1, \ldots, m_k \in M$:

$$\mathcal{M} \models \varphi(x_1/m_1, \ldots, x_k/m_k) \iff \mathcal{N} \models \varphi(x_1/m_1, \ldots, x_k/m_k).$$

Assume that $\mathcal{L}$ is enumerable and show that every infinite structure $\mathcal{N}$ for $\mathcal{L}$ has an enumerable elementary substructure.

Hint: construct an enumerable set $A \subseteq N$ such that for every $\varphi(x_1, \ldots, x_k, y)$ and every $a_1, \ldots, a_k \in A$: if N contains some b such that

$$\mathcal{N} \models \varphi(x_1/a_1, \ldots, x_k/a_k, y/b),$$

then also A contains such a b.