Winter 2015, Introduction to mathematical logic

Series 5

Exercise 1  Let $\mathcal{I} = (I, <^\mathcal{I})$ be a partial order such that any two elements have a common upper bound, that is,

$$(I, <^\mathcal{I}) = \forall x \forall y \exists z((x < z \lor x = z) \land (y < z \lor y = z)).$$

Let $\mathcal{L}$ be a language and $(\mathcal{A}_i)_{i \in I}$ a family of structures for $\mathcal{L}$ such that $\mathcal{A}_i$ is a substructure of $\mathcal{A}_j$ whenever $i <^\mathcal{I} j$. Let $\mathcal{A}$ be the $\mathcal{L}$-structure with universe $\bigcup_{i \in I} \mathcal{A}_i$ (where $\mathcal{A}_i$ is the universe of $\mathcal{A}_i$) that interprets relation symbols $R$, functions symbols $f$ and constants $c$ of $\mathcal{L}$ as follows:

$$R^\mathcal{A} := \bigcup_{i \in I} R^{\mathcal{A}_i}, \; f^\mathcal{A} := \bigcup_{i \in I} f^{\mathcal{A}_i}, \; c^\mathcal{A} := c^{\mathcal{A}_{i_0}}$$  \hspace{1cm} (1)

where $i_0$ is some arbitrary fixed element of $I$.

(a) Show that $\mathcal{A}$ is well-defined and does not depend on the choice of $i_0$. It is often denoted $\bigcup_{i \in I} \mathcal{A}_i$ and called the union of $(\mathcal{A}_i)_{i \in I}$.

(b) An $\forall\exists$-formula is one of the form $\forall x_1 \cdots \forall x_r \exists y_1 \cdots \exists y_s \psi$ for $r, s \in \mathbb{N}$ and $\psi$ quantifier-free. Show that if a closed $\forall\exists$-formula holds true in all $\mathcal{A}_i$, then it holds true in $\mathcal{A}$.

Exercise 2  Let $\mathcal{L}$ be a language.

(a) Let $\mathcal{M}$ be a structure for $\mathcal{L}$. Let $X \subseteq M$ be nonempty. Show that there exists a substructure $\langle X \rangle^\mathcal{M}$ of $\mathcal{M}$ whose universe contains $X$ and is “the smallest such substructure”, i.e., $\langle X \rangle^\mathcal{M}$ is a substructure of every substructure of $\mathcal{M}$ whose universe contains $X$. Substructures of the form $\langle X \rangle^\mathcal{M}$ with $X \subseteq M$ nonempty and finite are called finitely generated.

(b) Give a precise formulation of the following statement and prove it: “each structure is the union of its finitely generated substructures”.

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Exercise 3  Let $\mathcal{C}$ be the field of complex numbers.

(a) Show that a closed $\forall \exists$-formula that holds in all finite fields, also holds in $\mathcal{C}$.

*Hint:* First apply Series 4, Exercise 3 (b). Then apply (b) of the previous exercise.

(b) A map $f : \mathbb{C}^r \to \mathbb{C}^s$ is *polynomial* if there are polynomials $p_1, \ldots, p_s \in \mathbb{C}[X_1, \ldots, X_r]$ such that

$$f(c_1, \ldots, c_r) = (p_1(c_1, \ldots, c_r), \ldots, p_s(c_1, \ldots, c_r))$$

Show that every injective polynomial map from $\mathbb{C}^r$ to $\mathbb{C}^r$ is surjective.

*Hint:* For each $d \in \mathbb{N}$ write a closed formula expressing that injective polynomial maps “of degree $\leq d$” are surjective.

Exercise 4  Let $\mathcal{L}$ be a language and $\mathcal{M}, \mathcal{N}$ structures for $\mathcal{L}$. An *algebraic embedding of $\mathcal{M}$ into $\mathcal{N}$* is an isomorphism of $\mathcal{M}$ onto a substructure of $\mathcal{N}$.

The *algebraic diagram of $\mathcal{M}$* is the set $\text{Alg}(\mathcal{M})$ of all closed $\mathcal{L}(\mathcal{M})$-literals true in $\mathcal{M}$ (a *literal* is a formula which is either atomic or the negation of an atomic formula).

Let $\Gamma$ be an $\mathcal{L}$-theory.

(a) Show that there is an algebraic embedding of $\mathcal{M}$ into some model of $\Gamma$ if and only if $\Gamma \cup \text{Alg}(\mathcal{M})$ is consistent.

(b) Show that there is an algebraic embedding of $\mathcal{M}$ into some model of $\Gamma$ if and only if $\mathcal{M}$ satisfies every closed universal $\mathcal{L}$-formula which is implied by $\Gamma$.

(c) If every finitely generated substructure of $\mathcal{M}$ is algebraically embeddable into some model of $\Gamma$, then so is $\mathcal{M}$. 