Winter 2015, Introduction to mathematical logic

Series 5

Exercise 1 Let $\mathcal{I} = (I, <^{\mathcal{I}})$ be a partial order such that any two elements have a common upper bound, that is,

$$(I, <^{\mathcal{I}}) \models \forall x \forall y \exists z ((x < z \lor x = z) \land (y < z \lor y = z)).$$

Let $\mathcal{L}$ be a language and $(\mathcal{A}_i)_{i \in I}$ a family of structures for $\mathcal{L}$ such that $\mathcal{A}_i$ is a substructure of $\mathcal{A}_j$ whenever $i <^{\mathcal{I}} j$. Let $\mathcal{A}$ be the $\mathcal{L}$-structure with universe $\bigcup_{i \in I} A_i$ (where A_i is the uiverse of $\mathcal{A}_i$) that interprets relation symbols R, functions symbols f and constants c of $\mathcal{L}$ as follows:

$$R^{\mathcal{A}} := \bigcup_{i \in I} R^{\mathcal{A}_i}, \; f^{\mathcal{A}} := \bigcup_{i \in I} f^{\mathcal{A}_i}, \; c^{\mathcal{A}} := c^{\mathcal{A}_{i_0}} \tag{1}$$

where i_0 is some arbitrary fixed element of I.

(a) Show that $\mathcal{A}$ is well-defined and does not depend on the choice of i_0. It is often denoted $\bigcup_{i \in I} \mathcal{A}_i$ and called *the union of* $(\mathcal{A}_i)_{i \in I}$.

(b) An $\forall\exists$-formula is one of the form $\forall x_1 \cdots \forall x_r \exists y_1 \cdots \exists y_s \; \psi$ for $r, s \in \mathbb{N}$ and ψ quantifier-free. Show that if a closed $\forall\exists$-formula holds true in all $\mathcal{A}_i$, then it holds true in $\mathcal{A}$.

Exercise 2 Let $\mathcal{L}$ be a language.

(a) Let $\mathcal{M}$ be a structure for $\mathcal{L}$. Let $X \subseteq M$ be nonempty. Show that there exists a substructure $\langle X \rangle^{\mathcal{M}}$ of $\mathcal{M}$ whose universe contains X and is "the smallest such substructure", i.e., $\langle X \rangle^{\mathcal{M}}$ is a substructure of every substructure of $\mathcal{M}$ whose universe contains X. Substructures of the form $\langle X \rangle^{\mathcal{M}}$ with $X \subseteq M$ nonempty and finite are called *finitely generated*.

(b) Give a precise formulation of the following statement and prove it: "each structure is the union of its finitely generated substructures".

Exercise 3 Let $\mathcal{C}$ be the field of complex numbers.

(a) Show that a closed $\forall\exists$-formula that holds in all finite fields, also holds in $\mathcal{C}$.

Hint: First apply Series 4, Exercise 3 (b). Then apply (b) of the previous exercise.

(b) A map $f : \mathbb{C}^r \to \mathbb{C}^s$ is *polynomial* if there are polynomials $p_1, \dots p_s \in \mathbb{C}[X_1, \dots, X_r]$ such that

$$f(c_1, \dots c_r) = \big(p_1(c_1, \dots c_r), \dots, p_s(c_1, \dots c_r)\big)$$

Show that every injective polynomial map from $\mathbb{C}^r$ to $\mathbb{C}^r$ is surjective.

Hint: For each $d \in \mathbb{N}$ write a closed formula expressing that injective polynomial maps "of degree $\leq d$" are surjective.

Exercise 4 Let $\mathcal{L}$ be a language and $\mathcal{M}, \mathcal{N}$ structures for $\mathcal{L}$. An *algebraic embedding of* $\mathcal{M}$ *into* $\mathcal{N}$ is an isomorphism of $\mathcal{M}$ onto a substructure of $\mathcal{N}$.

The *algebraic diagram of* $\mathcal{M}$ is the the set $Alg(\mathcal{M})$ of all closed $\mathcal{L}(M)$-literals true in $\mathcal{M}$ (a *literal* is a formula which is either atomic or the negation of an atomic formula).

Let Γ be an $\mathcal{L}$-theory.

(a) Show that there is an algebraic embedding of $\mathcal{M}$ into some model of Γ if and only if $\Gamma \cup Alg(\mathcal{M})$ is consistent.

(b) Show that there is an algebraic embedding of $\mathcal{M}$ into some model of Γ if and only if $\mathcal{M}$ satisfies every closed universal $\mathcal{L}$-formula which is implied by Γ.

(c) If every finitely generated substructure of $\mathcal{M}$ is algebraically embeddable into some model of Γ, then so is $\mathcal{M}$.