Winter 2015, Introduction to mathematical logic

Series 6

Exercise 1 Let $\mathcal{L}$ be a first-order language, φ a closed $\mathcal{L}$-formula and Γ be an $\mathcal{L}$-theory. Let I be a nonempty index set, and for each $i \in I$ let J_i be a nonempty index set. For $i \in I$ and $j \in J_i$ let ψ_{ij} be a closed $\mathcal{L}$-formula. Assume Γ implies the "infinitary formula"

$$\varphi \leftrightarrow \bigvee_{i \in I} \bigwedge_{j \in J_i} \psi_{ij},$$

that is, assume a model $\mathcal{M}$ of Γ is a model of φ if and only if there is $i \in I$ such that for all $j \in J_i$ we have $\mathcal{M} \models \psi_{ij}$.

Prove that there exists a finite subset $I^0 \subseteq I$ and for each $i \in I^0$ finite subsets $J_i^0 \subseteq J_i$ such that

$$\Gamma \models \psi \leftrightarrow \bigvee_{i \in I^0} \bigwedge_{j \in J_i^0} \psi_{ij}$$

Hint: prove this first for the case that I is a singleton.

Exercise 2 (Beth's Definability Theorem) Let $\mathcal{L}$ be a first-order language and P a new unary relation symbol. Let Γ be an $(\mathcal{L} \cup \{P\})$-theory. Assume that Γ *implicitly defines* P: for all models $\mathcal{M}, \mathcal{N}$ of Γ with $\mathcal{M} \upharpoonright \mathcal{L} = \mathcal{N} \upharpoonright \mathcal{L}$ we have $P^{\mathcal{M}} = P^{\mathcal{N}}$ (and hence $\mathcal{M} = \mathcal{N}$).

Prove that Γ *explicitly defines* P: there exists an $\mathcal{L}$-formula $\varphi = \varphi(x)$ with at most one free variable x such that

$$\Gamma \models \forall x (Px \leftrightarrow \varphi(x)).$$

Hint: Let Γ' be Γ with P replaced by a copy P'. Let c be a new constant. Show $(\Gamma \cup \{Pc\}) \cup (\Gamma' \cup \{\neg P'c\})$ is inconsistent and apply Joint Consistency.