Exercise 1  Let $\mathcal{L}$ be a first-order language, $\varphi$ a closed $\mathcal{L}$-formula and $\Gamma$ be an $\mathcal{L}$-theory. Let $I$ be a nonempty index set, and for each $i \in I$ let $J_i$ be a nonempty index set. For $i \in I$ and $j \in J_i$ let $\psi_{ij}$ be a closed $\mathcal{L}$-formula. Assume $\Gamma$ implies the “infinitary formula”

$$\varphi \leftrightarrow \bigvee_{i \in I} \bigwedge_{j \in J_i} \psi_{ij},$$

that is, assume a model $\mathcal{M}$ of $\Gamma$ is a model of $\varphi$ if and only if there is $i \in I$ such that for all $j \in J_i$ we have $\mathcal{M} \models \psi_{ij}$.

Prove that there exists a finite subset $I^0 \subseteq I$ and for each $i \in I^0$ finite subsets $J^0_i \subseteq J_i$ such that

$$\Gamma \models \psi \leftrightarrow \bigvee_{i \in I^0} \bigwedge_{j \in J^0_i} \psi_{ij}.$$

*Hint:* prove this first for the case that $I$ is a singleton.

Exercise 2 (Beth’s Definability Theorem)  Let $\mathcal{L}$ be a first-order language and $P$ a new unary relation symbol. Let $\Gamma$ be an $(\mathcal{L} \cup \{P\})$-theory. Assume that $\Gamma$ implicitly defines $P$: for all models $\mathcal{M}, \mathcal{N}$ of $\Gamma$ with $\mathcal{M} \models \mathcal{L} = \mathcal{N} \models \mathcal{L}$ we have $P^\mathcal{M} = P^\mathcal{N}$ (and hence $\mathcal{M} = \mathcal{N}$).

Prove that $\Gamma$ explicitly defines $P$: there exists an $\mathcal{L}$-formula $\varphi = \varphi(x)$ with at most one free variable $x$ such that

$$\Gamma \models \forall x(\neg P x \leftrightarrow \varphi(x)).$$

*Hint: Let $\Gamma'$ be $\Gamma$ with $P$ replaced by a copy $P'$. Let $c$ be a new constant. Show $(\Gamma \cup \{Pc\}) \cup (\Gamma' \cup \{\neg P'c\})$ is inconsistent and apply Joint Consistency.