Winter 2015, Introduction to mathematical logic

Series 7

Exercise 1 (First proof of the ultrafilter theorem) Let X be a set.

1. Assume $\mathcal{Y} \subseteq \mathcal{P}(X)$ has the *finite intersection property*, that is, $\mathcal{Y} \neq \emptyset$ and $\bigcap \mathcal{Z} \neq \emptyset$ for all nonempty, finite $\mathcal{Z} \subseteq \mathcal{Y}$. Show that the set of supersets of intersections of finitely many sets from $\mathcal{Y}$ is a filter on X.

2. If $\mathcal{F}$ is a filter on X and $\mathcal{Y} \subseteq \mathcal{P}(X)$ finite, then there exists a filter $\mathcal{F}'$ on X such that for every $Y \in \mathcal{Y}$ either $Y \in \mathcal{F}'$ or $X \setminus Y \in \mathcal{F}'$.

3. Consider a first-order language $\mathcal{L}$ containing a binary function symbol $\cap$, constants $0, 1$ and unary relation symbols A, B. Write a closed universal φ such that a structure $\mathcal{M}$ with universe $\mathcal{P}(X)$ interpreting $0, 1$ by $\emptyset, X$ and $\cap$ by intersection, is a model of φ if and only if $B^{\mathcal{M}}$ is a filter containing $A^{\mathcal{M}}$.

4. Let $\mathcal{Y} \subseteq \mathcal{P}(X)$ have the finite intersection property. Consider the $(\mathcal{L} \setminus \{B\})$-structure $\mathcal{M}_0$ with universe $\mathcal{P}(X)$, $0^{\mathcal{M}_0} = \emptyset, 1^{\mathcal{M}_0} = X$, $\cap^{\mathcal{M}_0}$ is intersection and $A^{\mathcal{M}_0} = \mathcal{Y}$. Show that its algebraic diagram (using constant $\underline{Y}$ for $Y \in \mathcal{P}(X)$) is consistent with
$$\{\varphi\} \cup \{B(\underline{Y}) \vee B(\underline{X \setminus Y}) \mid Y \in \mathcal{P}(X)\}.$$
Conclude that there exists an ultrafilter containing $\mathcal{Y}$.

Exercise 2 (Second proof of the ultrafilter theorem) Let X be a set and let $\mathcal{Y} \subseteq \mathcal{P}(X)$ have the finite intersection property. Use Zorn's lemma to show that $\mathcal{Y}$ is contained in some ultrafilter on X.

Exercise 3 Let $\mathcal{L}$ be a first-order language containing $+, -, 0$ and consider groups as $\mathcal{L}$-structures (see Exercise 2 (c), Series 3). A group is *torsion-free* if it satisfies $\forall x(n \cdot x = 0 \rightarrow x = 0)$ for all $n \geq 1$. Here, the term $n \cdot x$ is recursively defined as $((n-1) \cdot x + x)$ understanding $0 \cdot x$ as 0.

Let $P \subseteq \mathbb{N}$ be the set of primes. For $p \in P$ write $\mathcal{Z}_p$ for the group $\mathbb{Z}/p\mathbb{Z}$ as an $\mathcal{L}$-structure. Let $\mathcal{F}$ be an ultrafilter on P containing the Fréchet filter. Show that the ultraproduct $\prod_{p \in P} \mathcal{Z}_p / \mathcal{F}$ is torsion-free. Conclude that there does not exist a single closed $\mathcal{L}$-formula φ such that for every group $\mathcal{G}$:
$$\mathcal{G} \models \varphi \iff \mathcal{G} \text{ is torsion-free.}$$