Winter 2015, Introduction to mathematical logic

Series 8

Exercise 1 (The random graph) Let $\mathcal{L}$ be the first-order language containing as non-logical symbols just one binary relation symbol E. A *graph* is an $\mathcal{L}$-structure satisfying

$$\varphi_0 := (\forall x \forall y (Exy \to Eyx) \wedge \forall x \, \neg Exx).$$

An *extension axiom* is a closed $\mathcal{L}$-formula of the form

$$\forall x_1 \cdots \forall x_n \exists y \Big(\bigwedge_{1 \leq i < j \leq n} \neg x_i = x_j \to \bigwedge_{1 \leq i \leq n} \neg x_i = y \wedge \bigwedge_{i \in I} E x_i y \wedge \bigwedge_{i \notin I} \neg E x_i y \Big)$$

where $n \in \mathbb{N}, I \subseteq [n]$.

1. Show that there is an countable graph satisfying all extension axioms. Such a graph is called a *random graph*.
2. Show that any two random graphs are isomorphic.
3. Conclude that the random graph satisfies exactly those closed $\mathcal{L}$-formulas ψ such that there exist a finite set of extension axioms Φ such that $(\bigwedge \Phi \wedge \varphi_0 \to \psi)$ is valid.

Exercise 2 (Nonstandard analysis) Let $\mathcal{R} = (\mathbb{R}, +, \cdot, 0, 1, \leq)$ und $\mathcal{K}$ an elementary extension of $\mathcal{R}$. Call an element $a \in K$ *finite* if there is $r \in \mathbb{R}$ such that $-r \leq^{\mathcal{K}} a \leq^{\mathcal{K}} r$, and *infinitesimal* if $-r \leq^{\mathcal{K}} a \leq^{\mathcal{K}} r$ for all $r \in \mathbb{R}, r > 0$. Let I be the set of infinitesimal elements of $\mathcal{K}$, and F the set of finite elements of $\mathcal{K}$.

1. Show that there exists $\mathcal{K}$ such that I and $K \setminus F$ are infinite.
2. Show that F is closed under $+^{\mathcal{K}}, \cdot^{\mathcal{K}}, -^{\mathcal{K}}$; for $a \in K$ here $-^{\mathcal{K}} a$ is the inverse of a with respect to $+^{\mathcal{K}}$. Hence, with the operations from $\mathcal{K}$, F forms a ring $\mathcal{F}$ (commutative, with unity).
3. Show that I is a maximal ideal in $\mathcal{F}$.

4. Define $st : F \to \mathbb{R}$ by

$$st(a) := \inf\{r \in \mathbb{R} \mid a \leq^{\mathcal{K}} r\}.$$

   Show st is a ring homomorphism with kernel I and

$$\mathcal{F}/I \cong (\mathbb{R}, +, \cdot, 0, 1).$$

Exercise 3 (Space of types) Let $\mathcal{L}$ be an first-order language, and Γ an $\mathcal{L}$-theory and $n \in \mathbb{N}$. Let $S_n(\Gamma)$ be the set of all complete n-types containing Γ.

1. For an n-formula φ let $[\varphi] := \{p \in S_n(\Gamma) \mid \varphi \in p\}$. Show that these sets form the basis of a topology on $S_n(\Gamma)$.

   From now on, we consider $S_n(\Gamma)$ as a topological space.

2. Show $S_n(\Gamma)$ is compact and Hausdorff.
3. Show that the clopen sets are precisely those of the form $[\varphi]$.
4. Show that the closed sets are precisely those of the form $\{p \in S_n(\Gamma) \mid \Sigma \subseteq p\}$ for sets of n-formulas Σ.
5. Show that the isolated points in $S_n(\Gamma)$ are precisely the atomic types over Γ.
6. Assume now that $\mathcal{L}$ is countable and Γ is complete. Show that Γ has a countable atomic model if and only if for all $n \in \mathbb{N}$, the set of isolated points is dense in $S_n(\Gamma)$.