Winter 2015, Introduction to mathematical logic

Series 9

Let $\mathcal{L}$ be the language containing constants $0, 1$, a unary function symbol S, binary function symbols $+, \cdot$ and the binary relation symbol $<$.

Bounded formulas, Σ^0_1 - and Π^0_1-formulas are defined for this language as for the language of Peano arithmetic.

The standard model $\mathcal{N}$ for $\mathcal{L}$ has universe $\mathbb{N}$ and interprets the symbols as usual.

Exercise 1 Let $\mathcal{A}$ be an $\mathcal{L}$-structure and $\mathcal{B}$ be a substructure of $\mathcal{A}$ such that for all $a \in A$ and $b \in B$: if $a <^{\mathcal{A}} b$, then $a \in B$.

1. For every bounded formula $\varphi(x_1, \ldots, x_r)$ and all $b_1, \ldots, b_k \in B$:

$$\mathcal{B} \models \varphi(b_1, \ldots, b_r) \quad \Longleftrightarrow \quad \mathcal{A} \models \varphi(b_1, \ldots, b_r)$$

2. For every Σ^0_1-formula $\varphi(x_1, \ldots, x_r)$ and all $b_1, \ldots, b_k \in B$:

$$\mathcal{B} \models \varphi(b_1, \ldots, b_r) \quad \Longrightarrow \quad \mathcal{A} \models \varphi(b_1, \ldots, b_r)$$

Exercise 2 Let Γ be the $\mathcal{L}$-theory consisting of the closed Π^0_1-formulas true in $\mathcal{N}$ plus *bounded induction*, i.e. for each bounded φ the universal closure of

$$(\varphi(x/0) \wedge \forall x(\varphi \to \varphi(x/Sx))) \to \forall x \varphi$$

Assume $\mathcal{A}, \mathcal{B}$ are as in the previous exercise and $\mathcal{A} \models \Gamma$. Show that $\mathcal{B} \models \Gamma$.

Exercise 3 (Parikh's theorem) Let Γ be as in the previous exercise, and assume

$$\Gamma \vdash \forall x \exists y \ \varphi(x, y)$$

for $\varphi(x, y)$ a Σ^0_1-formula with no free variables other than x, y. Show that there exists a term $t(x)$ with no free variables other than x such that

$$\Gamma \vdash \forall x \exists y < t(x) \ \varphi(x, y).$$

Hint: for a new constant c consider the theory consisting of Γ together with all sentences of the form $\forall y < t(c) \neg\varphi(c, y)$.