Winter 2015, Introduction to mathematical logic

Series 2

Exercise 1 Let $\mathcal{L}$ be a first-order language and $\mathcal{A}, \mathcal{B}$ be $\mathcal{L}$-structures. We say $\mathcal{A}$ is a *substructure* of $\mathcal{B}$, and write $\mathcal{A} \subseteq \mathcal{B}$, if $A \subseteq B$ and for all $r \geq 1$:

(i) $R^{\mathcal{B}} \cap A^r = R^{\mathcal{A}}$ for all r-ary relation symbols $R \in \mathcal{R}$;

(ii) $f^{\mathcal{B}} \restriction A^r = f^{\mathcal{A}}$ for all r-ary function symbols $f \in \mathcal{F}$;

(iii) $c^{\mathcal{B}} = c^{\mathcal{A}}$ for all constant symbols $c \in \mathcal{C}$.

A formula in $\mathcal{L}$ is *quantifier-free* if the symbols $\exists, \forall$ do not occur in it. A formula in $\mathcal{L}$ is *existential* if it is of the form $\exists x_1 \cdots \exists x_k \psi$ for variables $x_1, \ldots, x_k$ and $\psi \in \mathcal{L}$ quantifier-free.

Let $\mathcal{A}, \mathcal{B}$ be $\mathcal{L}$-structures, and e an *embedding of* $\mathcal{A}$ *into* $\mathcal{B}$, that is, an isomorphism of $\mathcal{A}$ onto a substructure of $\mathcal{B}$. Show that for every existential $\varphi \in \mathcal{L}$ with free variables $x_1, \ldots, x_r$ and all $a_1, \ldots, a_r \in A$:

$$\mathcal{A} \models \varphi(x_1/a_1, \ldots, x_r/a_r) \Longleftrightarrow \mathcal{B} \models \varphi(x_1/e(a_1), \ldots, x_r/e(a_r)).$$

Exercise 2 Let X, Y, Z be sets. Let $\mathcal{P}(X)$ denote the power set of X, and let $X^Y := \{f \mid f : Y \to X\}$. Show:

(a) There is a surjection of X onto Y if and only if there is an injection of Y into X.

(b) There is a bijection from $\mathcal{P}(X)$ onto $X^{\{0,1\}}$.

(c) There is a bijection from $X^{Y \times Z}$ onto $(X^Y)^Z$.

(d) There is no surjection of X onto $\mathcal{P}(X)$.

Hint: Consider $\{x \in X \mid x \notin f(x)\} \in \mathcal{P}(X)$.

Exercise 3 let $\mathcal{L}$ be a first-order language. A formula in $\mathcal{L}$ is in *prenex normal form* if it has the form $Q_1 x_1 \cdots Q_k x_k\ \psi$ where $Q_1, \ldots, Q_k \in \{\exists, \forall\}$ are quantifiers, $x_1, \ldots, x_k$ are variables and $\psi \in \mathcal{L}$ is quantifier-free.

Prove that for every $\varphi \in \mathcal{L}$ there exists $\varphi' \in \mathcal{L}$ in prenex normal form such that φ and φ' are elementarily equivalent.

Hint: induction on φ; use that the formulas $(Qx\varphi \wedge \psi)$ and $Qx(\varphi \wedge \psi(x/y))$ are elementarily equivalent where Q is a quantifier and y is a "new" variable.

Exercise 4 Let $\mathcal{L}$ be a first-order language. Prove or disprove that the following hold for all $\varphi, \psi \in \mathcal{L}_{\wedge,\neg,\forall}$. Recall that $\exists, \vee, \rightarrow$ are explained via abbreviations.

(a) If $x \notin \text{Free}(\psi)$, then $\varphi \rightarrow \psi \vdash \exists x \varphi \rightarrow \psi$.

(b) $\varphi \rightarrow \psi \vdash \exists x \varphi \rightarrow \psi$.

(c) If $\vdash \varphi \vee \psi$, then $\vdash \varphi$ or $\vdash \psi$.

(d) If $\vdash \varphi$ or $\vdash \psi$, then $\vdash \varphi \vee \psi$.

(e) If there is variable y such that $\vdash \varphi(x/y)$, then $\vdash \exists x \varphi$.