Forcing with random variables

Moritz Müller

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This is the content of a one-semester course in the winter semester 2012 at the Kurt Gödel Research Center at the University of Vienna. The course consists in 14 lectures, each lasting 90 min. It adresses master students of mathematic logic.

Text of course announcement Scott and Solovay observed that Cohen's method of forcing can be interpreted as a method to construct Boolean valued models. Scott proved the independence of CH from a higher order theory of the reals by interpreting first-order variables of the language over random reals, i.e. real valued random variables. The course starts with a review of Scott's construction and then treats Krajíček's recent book where such forcing with random variables is developed as a method to construct Boolean valued models for weak theories of arithmetic.

1 Part 1: Scott forcing

This part presents the material from Scott's article [4].

1.1 The language $L^2_{\mathbf{ar}}$, Syntax

1.2 The language $L^2_{\mathbf{ar}}$, Semantics

1.3 The calculus $\mathfrak{S}$

1.4 Notation for probability spaces

1.5 The algebra $\mathcal{B}$

1.6 The Boolean model

1.7 Validity

1.8 Interpretation of second order variables

1.9 $\mathfrak{S}$ preserves validity

1.10 Violating CH

2 Part 2: Nonstandard models of arithmetic

The first two sections give a elementary discussion of coding and nonstandard models of arithmetic. The thrid gives a generalized ultrapower construction (to construct submodels of the usual ultrapower construction) and gives a version of the definable ultraproduct from [2].

2.1 Arithmetic as set theory

2.2 Nonstandard models

2.3 Generalized ultrapowers

3 Part 3: Krajíček forcing

Sections 3.1 – 3.4 are from [3, Part I: Basics]. Section 3.5 is a simple example of results as in [3, Chapter 24]. The rest of the material deviates somewhat from the formalism in the source [3]. Section 3.7 is from [1]. Section 3.6 is from [3, Chapters 5, 6, 9]. Sections 3.8 – 3.10 are from [3, Chapters 10–12]. Section 3.11 is from [3, Chapter 18].

3.1 The algebra $\mathcal{B}$

3.2 The Boolean model $K(F)$

3.3 Preservation, truth values and a metric

3.4 Propositional approximation of truth values

3.4.1 Getting rid of $\bigwedge / \bigvee$

3.4.2 Getting rid of ε

3.5 Witnessing and independence from S^1_2

3.6 The models $K(F, G)$

3.7 Hastad's switching lemma

3.8 The tree model

3.9 Quantifier elimination in the tree model

3.10 Witnessing and independence from V^0_1

3.11 Outlook: the road to proof lower bounds

References

[1] P. Beame. A switching lemma primer. Technical Report UW-CSE-95-07-01, University of Washington, 1994.

[2] P. Hájek, P. Pudlák. Metamathematics of First-order Arithmetic. Perspectives in Mathematical Logic, Volume 3, Springer, 1998.

[3] J. Krajíček. Forcing with random variables and proof complexity. London Mathematical Society Lecture Note Series 382, Cambridge University Press, 2011.

[4] D. Scott. A proof of the independence of the continuum hypothesis. Mathematical Systems Theory 1(2): 89-111, 1967.