On optimal probabilistic algorithms for SAT*  

Yijia Chen†, Jörg Flum‡, Moritz Müller§

†Shanghai Jiaotong University, China.  
yijia.chen@cs.sjtu.edu.cn
‡Albert-Ludwigs-Universität Freiburg, Germany.  
joerg.flum@math.uni-freiburg.de
§Kurt Gödel Research Center, University of Vienna, Austria.  
moritz.mueller@univie.ac.at

Abstract. Assuming the existence of one-way functions we show that SAT does not have in certain sense optimal probabilistic algorithms.

Introduction

A major aim in the development of algorithms for hard problems is to decrease the running time. In particular one asks for algorithms that are optimal: A deterministic algorithm $A$ deciding a language $L \subseteq \Sigma^*$ is optimal (or polynomially optimal or $p$-optimal) if for any other algorithm $B$ deciding $L$ there is a polynomial $p$ such that

$$t_A(x) \leq p(t_B(x) + |x|)$$

for all $x \in \Sigma^*$. Here $t_A(x)$ denotes the running time of $A$ on input $x$. If (1) is only required for all $x \in L$, then $A$ is said to be an almost optimal algorithm for $L$ (or to be optimal on positive instances of $L$).

Various recent papers address the question whether such optimal algorithms exist for NP-complete or coNP-complete problems (cf. [1]), even though the problem has already been considered in the seventies when Levin [4] observed that there exists an optimal algorithm that finds a witness for every satisfiable propositional formula. Furthermore the relationship between the existence of almost optimal algorithms for a language $L$ and the existence of “optimal” proof systems for $L$ has been studied [3, 5].

Here we present a result (see Theorem 1.1) that can be interpreted as stating that (under the assumption of the existence of one-way functions) there is no optimal probabilistic algorithm for SAT.

1 Probabilistic speed-up

For a propositional formula $\alpha$ we denote by $\|\alpha\|$ the number of literals in it, counting repetitions. Hence, the actual length of any reasonable encoding of $\alpha$ is polynomially related to $\|\alpha\|$.

The main result of this short note reads as follows:

* presented at the conference Logical Approaches to Barriers in Computing and Complexity, Greifswald. Preprint of the Department of Mathematics and Computer Science at the University Greifswald No. 6, 2010. The authors thank the John Templeton Foundation for its support under Grant #13152, The Myriad Aspects of Infinity.
On optimal probabilistic algorithms for \textsc{Sat}

\begin{theorem}
Assume one-way functions exist. Then for every probabilistic algorithm \(A\) deciding \textsc{Sat} there exists a probabilistic algorithm \(B\) deciding \textsc{Sat} such that for all \(d \in \mathbb{N}\) and sufficiently large \(n \in \mathbb{N}\)

\[
\Pr \left[ \text{there is a satisfiable } \alpha \text{ with } \|\alpha\| = n \text{ such that} \right.
\]

\[
A \text{ does not accept } \alpha \text{ in at most } (t_B(\alpha) + \|\alpha\|)d \text{ steps} \geq \frac{1}{5}.
\]

Note that \(t_A(\alpha)\) and \(t_B(\alpha)\) are random variables, and the probability is taken over the coin tosses of \(A\) and \(B\) on \(\alpha\).

Here we say that a probabilistic algorithm \(A\) decides \textsc{Sat} if it decides \textsc{Sat} as a nondeterministic algorithm, that is

\[
\alpha \in \text{SAT} \implies \Pr[\text{\(A\) accepts } \alpha] > 0,
\]

\[
\alpha \notin \text{SAT} \implies \Pr[\text{\(A\) accepts } \alpha] = 0.
\]

In particular, \(A\) can only err on ‘yes’-instances.

Note that in the first condition the error probability is not demanded to be bounded away from 0, say by a constant \(\varepsilon > 0\). As a more usual notion of probabilistic decision, say \(A\) \textit{decides} \textsc{Sat} with one-sided error \(\varepsilon\) if

\[
\alpha \in \text{SAT} \implies \Pr[\text{\(A\) accepts } \alpha] > 1 - \varepsilon,
\]

\[
\alpha \notin \text{SAT} \implies \Pr[\text{\(A\) accepts } \alpha] = 0.
\]

For this concept we get

\begin{corollary}
Assume one-way functions exist and let \(\varepsilon > 0\). Then for every probabilistic algorithm \(A\) deciding \textsc{Sat} with one-sided error \(\varepsilon\) there exists a probabilistic algorithm \(B\) deciding \textsc{Sat} with one-sided error \(\varepsilon\) such that for all \(d \in \mathbb{N}\) and sufficiently large \(n \in \mathbb{N}\)

\[
\Pr \left[ \text{there is a satisfiable } \alpha \text{ with } \|\alpha\| = n \text{ such that} \right.
\]

\[
A \text{ does not accept } \alpha \text{ in at most } (t_B(\alpha) + \|\alpha\|)d \text{ steps} \geq \frac{1}{3}.
\]

This follows from the fact that in the proof of Theorem 2.1.1 we choose the algorithm \(B\) in such way that on any input \(\alpha\) the error probability of \(B\) on \(\alpha\) is not worse than the error probability of \(A\) on \(\alpha\).

\end{corollary}

\section{Witnessing failure}

The proof of Theorem 2.1.1 is based on the following result.

\begin{theorem}
Assume that one-way functions exist. Then there is a probabilistic polynomial time algorithm \(C\) satisfying the following conditions.

\begin{enumerate}
\item On input \(n \in \mathbb{N}\) in unary the algorithm \(C\) outputs with probability one a satisfiable formula \(\beta\) with \(\|\beta\| = n\).
\item For every \(d \in \mathbb{N}\) and every probabilistic algorithm \(A\) deciding \textsc{Sat} and sufficiently large \(n \in \mathbb{N}\)

\[
\Pr[\text{\(A\) does not accept } C(n) \text{ in } n^d \text{ steps}] \geq \frac{1}{3}.
\]
\end{enumerate}

\end{theorem}
In the terminology of fixed-parameter tractability this theorem tells us that the parameterized construction problem associated with the following parameterized decision problem \( p\text{-CounterExample-Sat} \) is in a suitably defined class of randomized nonuniform fixed-parameter tractable problems.

| Instance: | An algorithm \( \mathcal{A} \) deciding SAT and \( d, n \in \mathbb{N} \) in unary. |
| Parameter: | \( \| \mathcal{A} \| + \, d \). |
| Problem: | Does there exist a satisfiable CNF-formula \( \alpha \) with \( \| \alpha \| = n \) such that \( \mathcal{A} \) does not accept \( \alpha \) in \( n^d \) many steps? |

Note that this problem is a promise problem. We can show:

**Theorem 2.2** Assume that one-way functions exist. Then the problem \( p\text{-CounterExample-Sat} \) is nonuniformly fixed-parameter tractable.\(^1\)

This result is an immediate consequence of the following

**Theorem 2.3** Assume that one-way functions exist. For every infinite set \( I \subseteq \mathbb{N} \) the problem

\[
\text{SAT}_I
\]

| Instance: | A CNF-formula \( \alpha \) with \( \| \alpha \| \in I \). |
| Problem: | Is \( \alpha \) satisfiable? |

is not in PTIME.

The decision problem \( p\text{-CounterExample-Sat} \) has the following associated construction problem:

| Instance: | An algorithm \( \mathcal{A} \) deciding SAT and \( d, n \in \mathbb{N} \) in unary. |
| Parameter: | \( \| \mathcal{A} \| + \, d \). |
| Problem: | Construct a satisfiable CNF-formula \( \alpha \) with \( \| \alpha \| = n \) such that \( \mathcal{A} \) does not accept \( \alpha \) in \( n^d \) many steps, if one exists. |

We do not know anything on its (deterministic) complexity; its nonuniform fixed-parameter tractability would rule out the existence of strongly almost optimal algorithms for SAT. By definition, an algorithm \( \mathcal{A} \) deciding SAT is a strongly almost optimal algorithm for SAT if there is a polynomial \( p \) such that for any other algorithm \( \mathcal{B} \) deciding SAT

\[
t_\mathcal{A}(\alpha) \leq p(t_\mathcal{B}(\alpha) + |\alpha|)
\]

for all \( \alpha \in \text{SAT} \). Then the precise statement of the result just mentioned reads as follows:

**Proposition 2.4** Assume that \( P \neq NP \). If the construction problem associated with \( p\text{-CounterExample-Sat} \) is nonuniformly fixed-parameter tractable, then there is no strongly almost optimal algorithms for SAT.

---

\(^1\)This means, there is a \( c \in \mathbb{N} \) such that for every algorithm \( \mathcal{A} \) deciding SAT and every \( d \in \mathbb{N} \) there is an algorithm that decides for every \( n \in \mathbb{N} \) whether \( (\mathcal{A}, d, n) \) is a positive instance of \( p\text{-CounterExample-Sat} \) in time \( O(n^c) \); here the constant hidden in \( O( \) \) may depend on \( \mathcal{A} \) and \( d \).
3 Some Proofs

We now show how to use an algorithm \( C \) as in Theorem 3.2.1 to prove Theorem 2.1.1.

**Proof of Theorem 2.1.1 from Theorem 3.2.1:** Let \( \mathcal{A} \) be an algorithm deciding \( \text{Sat} \). We choose \( a \in \mathbb{N} \) such that for every \( n \geq 2 \) the running time of the algorithm \( C \) (provided by Theorem 3.2.1) on input \( n \) is bounded by \( n^a \). We define the algorithm \( \mathcal{B} \) as follows:

\[
\mathcal{B}(\alpha) \quad // \quad \alpha \in \text{CNF}
\]

1. \( \beta \leftarrow C(\|\alpha\|) \).
2. if \( \alpha = \beta \) then accept and halt.
3. else Simulate \( \mathcal{A} \) on \( \alpha \).

Let \( d \in \mathbb{N} \) be arbitrary. Set \( e := d \cdot (a + 2) + 1 \) and fix a sufficiently large \( n \in \mathbb{N} \). Let \( S_N \) denote the range of \( C(n) \). Furthermore, let \( T_{n,\beta,e} \) denote the set of all strings \( r \in \{0,1\}^n \) that do not determine a (complete) accepting run of \( \mathcal{A} \) on \( \beta \) that consists in at most \( n^e \) many steps. Observe that a (random) run of \( \mathcal{A} \) does not accept \( \beta \) in at most \( n^e \) steps if and only if \( \mathcal{A} \) on \( \beta \) uses \( T_{n,\beta,e} \), that is, its first at most \( n^e \) many coin tosses on input \( \beta \) are described by some \( r \in T_{n,\beta,e} \). Hence by (2) of Theorem 3.2.1 we conclude

\[
(2) \quad \sum_{\beta \in S_n} \left( \Pr[\beta = C(n)] \cdot \Pr_{r \in \{0,1\}^n} [r \in T_{n,\beta,e}] \right) \geq \frac{1}{3}. 
\]

Let \( \alpha \in S_n \) and apply \( \mathcal{B} \) to \( \alpha \). If the execution of \( \beta \leftarrow C(\|\alpha\|) \) in Line 1 yields \( \beta = \alpha \), then the overall running time of the algorithm \( \mathcal{B} \) is bounded by \( O(n^2 + t\mathcal{C}(n)) = O(n^{a+1}) \leq n^{a+2} \) for \( n \) is sufficiently large. If in such a case a run of the algorithm \( \mathcal{A} \) on input \( \alpha \) uses an \( r \in T_{n,\alpha,e} \), then it does not accept \( \alpha \) in time \( n^e = n^{(a+2)d+1} \) and hence not in time \( (t_\mathcal{B}(\alpha) + \|\alpha\|)^d \). Therefore,

\[
\Pr \left[ \text{there is a satisfiable } \alpha \text{ with } \|\alpha\| = n \text{ such that} \right. \\
\mathcal{A} \text{ does not accept } \alpha \text{ in at most } (t_\mathcal{B}(\alpha) + \|\alpha\|)^d \text{ steps} \bigg] \\
\geq 1 - \Pr \left[ \text{for every input } \alpha \in S_n \text{ the algorithm } \mathcal{B} \text{ does not generate } \alpha \right. \\
\text{in Line 3, or } \mathcal{A} \text{ does not use } T_{n,\alpha,e} \bigg] \\
= 1 - \prod_{\alpha \in S_n} \left( (1 - \Pr[\alpha = C(n)]) + \Pr[\alpha = C(n)] \cdot \Pr_{r \in \{0,1\}^n} [r \notin T_{n,\alpha,e}] \right) \\
= 1 - \prod_{\alpha \in S_n} \left( 1 - \Pr[\alpha = C(n)] \cdot \Pr_{r \in \{0,1\}^n} [r \in T_{n,\alpha,e}] \right) \\
\geq 1 - \left( \sum_{\alpha \in S_n} \left( 1 - \Pr[\alpha = C(n)] \cdot \Pr_{r \in \{0,1\}^n} [r \in T_{n,\alpha,e}] \right) \right) \frac{|S_n|}{|S_n|} \\
= 1 - \left( \frac{\sum_{\alpha \in S_n} \Pr[\alpha = C(n)] \cdot \Pr_{r \in \{0,1\}^n} [r \in T_{n,\alpha,e}]}{|S_n|} \right)^{|S_n|} \\
\geq 1 - \left( \frac{1}{3 \cdot |S_n|} \right)^{\frac{|S_n|}{3}} \geq \frac{1}{5}. 
\]

Theorem 3.2.1 immediately follows from the following lemma.
Lemma 3.1 Assume one-way functions exist. Then there is a randomized polynomial time algorithm \( H \) satisfying the following conditions.

(H1) Given \( n \in \mathbb{N} \) in unary the algorithm \( H \) computes with probability one a satisfiable CNF \( \alpha \) of size \( \|\alpha\| = n \).

(H2) For every probabilistic algorithm \( A \) deciding \( \text{Sat} \) and every \( d, p \in \mathbb{N} \) there exists an \( n_{A,d,p} \in \mathbb{N} \) such that for all \( n \geq n_{A,d,p} \)

\[
\Pr[ A \text{ accepts } H(n) \text{ in time } n^d ] \leq \frac{1}{2} + \frac{1}{n^p},
\]

where the probability is taken uniformly over all possible outcomes of the internal coin tosses of the algorithms \( A \) and \( H \).

(H3) The cardinality of the range of (the random variable) \( H(n) \) is superpolynomial in \( n \).

Sketch of proof: We present the construction of the algorithm \( H \). By the assumption that one-way functions exist, we know that there is a pseudorandom generator (e.g. see [2]), that is, there is an algorithm \( G \) such that:

(G1) For every \( s \in \{0,1\}^\ast \) the algorithm \( G \) computes a string \( G(s) \) with \( |G(s)| = |s| + 1 \) in time polynomial in \( |s| \).

(G2) For every probabilistic polynomial time algorithm \( \mathbb{D} \), every \( p \in \mathbb{N} \), and all sufficiently large \( \ell \in \mathbb{N} \) we have

\[
\left| \Pr_{s \in \{0,1\}^\ast} \left[ \mathbb{D}(G(s)) = 1 \right] - \Pr_{r \in \{0,1\}^\ast} \left[ \mathbb{D}(r) = 1 \right] \right| \leq \frac{1}{\ell^p}
\]

(In the above terms, the probability is also taken over the internal coin toss of \( \mathbb{D} \).)

Let the language \( Q \) be the range of \( G \),

\[ Q := \{ G(s) \mid s \in \{0,1\}^\ast \}. \]

\( Q \) is in NP by (G1). Hence, there is a polynomial time reduction \( S \) from \( Q \) to \( \text{Sat} \), which we can assume to be injective. We choose a constant \( c \in \mathbb{N} \) such that \( |S(r)| \leq |r|^c \) for every \( r \in \{0,1\}^\ast \). For every propositional formula \( \beta \) and every \( n \in \mathbb{N} \) with \( n \geq |\beta| \) let \( \beta(n) \) be an equivalent propositional formula with \( |\beta(n)| = n \). We may assume that \( \beta(n) \) is computed in time polynomial in \( n \).

One can check that the following algorithm \( \mathbb{H} \) has the properties claimed in the lemma.

\[
\mathbb{H}(n) \quad // \quad n \in \mathbb{N}
\]

1. \( m \leftarrow \lfloor \sqrt{n-1} \rfloor - 1 \)
2. Choose an \( s \in \{0,1\}^m \) uniformly at random.
3. \( \beta \leftarrow S(G(s)) \)
4. Output \( \beta(n) \)

\[
\]

References


