Let M be a structure wit universe $\mathbb{N}$ interpreting some arbitrary language. Let U be a free ultrafilter on $\mathbb{N}$, and $\prod_U M$ the ultrapower modulo U.

Let $A_0, A_1 \dots$ be definable subsets of $\prod_U M$ such that for all $k \in \mathbb{N}$:

$$\textstyle\bigcap_{i \leq k} A_i \neq \emptyset. \tag{1}$$

We claim that

$$\textstyle\bigcap_{i \in \mathbb{N}} A_i \neq \emptyset.$$

Proof. For $i \in \mathbb{N}$ choose a formula $\varphi_i(x, \bar{y}_i)$ in the language of M and a tuple $\bar{\alpha}_i$ of functions from $\mathbb{N}$ to $\mathbb{N}$ such that A_i is defined in $\prod_U M$ by the formula $\varphi_i(x, \bar{\alpha}_i/\sim)$.

Set

$$\varphi'_i(x, \bar{y}'_i) := \textstyle\bigwedge_{j \leq i} \varphi_j(x, \bar{y}_j).$$

where $\bar{y}'_i := \bar{y}_0 \cdots \bar{y}_i$. Write $\bar{\alpha}'_i$ for the tuple $\bar{\alpha}_0 \cdots \bar{\alpha}_i$. The set defined by $\varphi'_i(x, \bar{\alpha}'_i/\sim)$ in $\prod_U M$ is $A'_i := \bigcap_{j \leq i} A_j$, so $A'_i \neq \emptyset$ by (1), that is,

$$\textstyle\prod_U M \models \exists x \varphi'_i(x, \bar{\alpha}'_i/\sim). \tag{2}$$

Since $\bigcap_{i \in \mathbb{N}} A'_i = \bigcap_{i \in \mathbb{N}} A_i$, it suffices to find $\gamma : \mathbb{N} \to \mathbb{N}$ such that

$$\gamma/\sim \in \textstyle\bigcap_{i \in \mathbb{N}} A'_i.$$

For $i \in \mathbb{N}$ set

$$\begin{aligned} Y_i &:= \{j \in \mathbb{N} \mid M \models \exists x \varphi'_i(x, \bar{\alpha}'_i(j))\}, \\ X_i &:= \{j \in \mathbb{N} \mid i \leq j\} \cap Y_i. \end{aligned}$$

By (2) and Los' theorem we have $Y_i \in U$. Since U is free, $\{j \in \mathbb{N} \mid i \leq j\} \in U$. Hence $X_i \in U$. By definition of φ'_i we have $Y_0 \supseteq Y_1 \supseteq \cdots$ and hence

$$X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$$

and $\bigcap_{i \in \mathbb{N}} X_i = \emptyset$. For every $j \in X_0$ there is a unique $i_j \in \mathbb{N}$ such that j is in the difference $X_{i_j} \setminus X_{i_j+1}$. As $j \in X_{i_j} \subseteq Y_{i_j}$ we have $M \models \exists x \varphi'_{i_j}(x, \bar{\alpha}'_{i_j}(j))$. Choose a_j such that $M \models \varphi'_{i_j}(a_j, \bar{\alpha}'_{i_j}(j))$.

Define the function $\gamma : \mathbb{N} \to \mathbb{N}$ setting

$$\gamma(j) := a_j$$

for all $j \in X_0$; on arguments outside X_0 we let γ take some arbitrary value. Then

$$X_{i_j} \subseteq \{k \in \mathbb{N} \mid M \models \varphi'_{i_j}(\gamma(k), \bar{\alpha}_{i_j}(k))\}.$$

Indeed: if $k \in X_{i_j}$, then $i_k \geq i_j$; since $M \models \varphi'_{i_k}(\gamma(k), \bar{\alpha}'_{i_k}(k))$ by definition of γ, the definition of φ'_{i_k} implies $M \models \varphi'_{i_j}(\gamma(k), \bar{\alpha}'_{i_j}(k))$.

Since $X_{i_j} \in U$, Los' theorem implies

$$\textstyle\prod_U M \models \varphi'_{i_j}(\gamma/\sim, \bar{\alpha}'_{i_j}/\sim).$$

We have shown $\gamma/\sim \in A'_{i_j}$ for all $j \in X_0$.

Now let $i \in \mathbb{N}$ be given. We have to show $\gamma/\sim \in A'_i$. We claim that there is $j \in X_0$ such that $i_j \geq i$. Indeed, assuming $i_j < i$ for all $j \in X_0$ implies $j \notin X_{i_j+1} \supseteq X_i$ for all $j \in X_0$, and hence $X_0 \cap X_i = \emptyset$ – a contradiction to $X_0 \cap X_i \in U$. So, there exists $j \in X_0$ such that $i_j \geq i$, and hence $\gamma/\sim \in A'_{i_j} \subseteq A'_i$. □