Postdoctoral Position in the Philosophy of Set Theory
The Hyperuniverse: Laboratory of the Infinite

The KGRC (Kurt Goedel Research Center, University of Vienna) will serve as host for the Hyperuniverse Project, an investigation into set-theoretic truth that combines philosophical principles with the methods of modern set theory. This 33-month project, supported by the John Templeton Foundation and with projected starting date January 1, 2013, includes funding for two full-time postdoctoral positions, one in mathematical set theory and the other in the philosophy of set theory. The postdoctoral researchers are expected to collaborate with senior visitors, the project leader, Sy-David Friedman, and others at the KGRC for the duration of the project. Applications are currently sought for the philosophy of set theory position, with a submission deadline of September 1, 2012.

A description of the project can be found below. The philosophy of set theory postdoc is expected to hold a PhD in philosophy and to be familiar with the modern mathematical developments of set theory, including the theory of forcing and large cardinals.

Project postdocs will receive an annual gross salary of about 47,000 euros (which includes health insurance) plus an annual travel allowance of 2,000 euros. Applications should contain a CV (including a list of senior scholars who can be contacted for recommendations), a publication list and a description of prior research. They should be sent by the closing date to <sdf@logic.univie.ac.at>. For further information, please contact the Project Leader at this e-mail address.

JTF Full Proposal
The Hyperuniverse: Laboratory of the Infinite

The Research Question

The universe V of all sets is the mathematical realisation of the Infinity concept. Thus our understanding of Infinity is largely based on an understanding of what is "true" in V. The traditional axioms for set theory (those of the system ZFC, Zermelo-Fraenkel set theory with the axiom of choice) are widely accepted as being true; what further statements can we justify as being true and therefore adopt as new axioms?

The *Hyperuniverse Program* is a new approach to set-theoretic truth that shares many, but not all, features of Gödel's philosophical program for the discovery of new axioms. Gödel's view was that candidates for new axioms should conform to motivating principles that are more evident and persuasive than the axioms themselves. A special emphasis is placed on the "maximum iterative concept of set", however Gödel is open to other sources for new axioms and conjectures that "there may exist, besides the ordinary axioms [...], other (hitherto unknown) axioms of set theory which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts" ([13]). He also suggests ([13]) that some "maximum property" of the system of sets may be devised that, although not directly suggested by the concept of set, may qualify as a suitable new axiom for set theory. And Gödel invokes "success" as a criterion for judging candidates for new axioms, thereby bringing considerations of a purely mathematical character into the discussion.

Thus the research task addressed in this project is to devise a strategy for arriving at justifiable truths in set theory by employing contemporary mathematical techniques guided by strict philosophical criteria. This investigation will lead to a deeper understanding of the concept of Infinity as it enhances what we accept as true about the set-theoretic universe.

The Research Itself

In the Hyperuniverse Program one seeks to arrive at new axioms of set theory, but unlike Gödel, no Platonistic view of the universe of sets as a fixed reality is invoked. Instead "truth in V" is taken to be a manner of speaking that conveys certain epistemic attitudes, leading to different "pictures" of V which reflect a set-theorist's conception of set-theoretic truth. In the Hyperuniverse Program one follows Gödel's suggestion of using concepts of set theory and logic as the basis for motivating principles for the choice of preferred universes, and then develops mathematically precise criteria for the choice of preferred universes based upon these motivating principles. The goal is to find ultimate answers to questions that are not resolved by the ZFC axioms alone. In contrast to Gödel's Platonistic program, this calls for an active strategy for discovering these answers by exploring a range of possible universes and selecting those which are preferred, ultimately based on unbiased and well-justified criteria. Statements that are true across the preferred universes are then taken to be true in V.

A precise and detailed description of how the Hyperuniverse Program works is described later in the proposal. As we will see, the concept of "maximality", suggested by Gödel, plays a fundamental role as a motivating principle in this program. By virtue of the fact that the set-theoretic universe is determined by the nature of the ordinal numbers and of the power set operation, the compelling realisations of this principle are the mathematical criteria of ordinal maximality and power set maximality, discussed in detail below. The examination of these criteria has already generated a paradigm-shifting[1] and rich mathematical investigation, which will be taken much further with the present project.

[1] A major example of such a paradigm shift is the re-examination of the role of large infinities in the foundations of set theory; see the discussion below.

Although the bulk of our work on this project is of a mathematical nature, for example in the complex examination of the consequences and mutual compatibility of different realisations of the principle of maximality, philosophical issues play an essential role in guiding our work. Following Gödel, justifiable criteria for preferred universes must arise from motivating principles that are more persuasive than the axioms to which they themselves give rise. At present two such motivating principles have been invoked in the Hyperuniverse Program: *maximality* and *omniscience*. The former is the principle that the universe should be "as large as possible", the latter states that in the universe one should be able to define the class of statements that can hold in alternative universes. These are to be viewed as "ideal attributes" for the universe V of all sets. The program does not presume that these are the only motivating principles which serve as ideal attributes for V and makes an open-ended appeal to the field of philosophy for further principles of this nature. There is an extensive philosophical literature on the role of maximality in set theory ([1, 3, 13, 14, 15, 16, 17, 20, 21, 23]) and this literature should be fully addressed when adopting maximality as a motivating principle in the Hyperuniverse Program. Are there other "ideal attributes" of the universe of sets which could or should serve as such motivating principles? Is "omniscience" a philosophically justifiable such attribute? How is one to resolve possible conflicts between different "ideal attributes"? Is a synthesis (as proposed below to resolve conflicts between forms of maximality and omniscience) philosophically justifiable, and if so, what form should such a synthesis take?

Another philosophical issue that needs to be addressed concerns the meaning of "truth in V". In formulating the Hyperuniverse Program the expression "true in V" is not used to reflect an ontological state of affairs concerning the universe of all sets as a reality to which existence can be ascribed independently of set-theoretic practice. Instead "true in V" is meant as a *way of speaking* that only conveys information about set-theorists' epistemic attitudes, as a description of the status that certain statements have or are expected to have in the minds of set-theorists. Sentences "true in V" are meant to be sentences that are or should be regarded by set-theorists as definitive. Within the Hyperuniverse Program two sorts of statements qualify for this status. The first are those set-theoretic statements that, due to their role in the practice of set theory and more generally of mathematics, should not be contradicted by any further set-theoretic statement that may be considered as definitive. Let us call these statements "*de facto* set theoretic truths". The axioms of ZFC are examples of such truths. But secondly, within the Hyperuniverse Program, one is ready to regard as true in V statements that, beyond not contradicting *de facto* set-theoretic truth, obey a condition for truth explicitly established at the outset (i.e., they hold in all preferred universes of sets). Let us call these "*de jure*" set-theoretic truths. In the Hyperuniverse Program, formulating *de jure* set theoretic truths is an autonomously regulated process. No "external" constraint is imposed while engaged in this process, in particular there is no independently existing well-determined reality to which one must be faithful. Instead, in searching for *de jure* set-theoretic truths one is only expected to follow justifiable procedures. In short, the search for *de jure* set-theoretic truth, which lies at the core of the Hyperuniverse Program, may be understood as the *active* response of a non-Platonistically minded mathematician, who believes that it makes sense to search for new truths in V beyond *de facto* set-theoretic truths. (This contrasts with any form of skepticism concerning such a search, be this motivated by the assumption that such a search is hopeless or by the confidence, possibly grounded on Platonism, that the well-determined features of V will somehow manifest themselves without any effort of our own.) Equivalently, one may characterize the Hyperuniverse Program as a *dynamic* approach to set-theoretic truth, free from external constraints (although internally regulated), in contrast to any *static* Platonistic view that truth concerning sets is *restricted* to a fixed state of affairs to which one must be "faithful".

Thus another key task of a philosophical nature in this project is to examine the nature and legitimacy of these two kinds of truth, *de facto truth* and *de jure truth*, taking into account the existing literature on truth in set theory.

Although I am not a philosopher, I have had the opportunity to collaborate with Tatiana Arrigoni, a philosopher of mathematics of considerable expertise (see [1, 2]) who has given the Hyperuniverse Program a clear vision and a realistic plan for the further development of its philosophical aspects (see our joint paper [4], to appear in the Bulletin of Symbolic Logic). For the present project I will again collaborate with a well-informed philosopher of mathematics (as one of the two project postdocs). Note that the relationship between the mathematical and philosophical parts of the project is more than a superficial one: Exploring a mathematical criterion for the choice of preferred universes is of limited value without a philosophical justification for it, and conversely, one cannot base such criteria purely on philosophical considerations without knowing if they meet the essential mathematical requirement of being consistent (true in some universe), typically a challenging question of considerable depth.

Thus the Hyperuniverse Program provides a promising new strategy for discovering new set-theoretic truths, i.e. new truths about Infinity. It has already forced a paradigm-shifting re-examination of the roles of large cardinals and determinacy axioms in set theory and led to a rich collection of interesting and challenging questions of a purely mathematical nature. I'll now describe the Hyperuniverse Program in more detail, revealing the exact nature of the research that we plan to do.

The Hyperuniverse

In set theory we have many methods for creating new universes (i.e., well-founded models of ZFC) from old ones: set-forcing, class-forcing, hyperclass-forcing, ..., model-theoretic methods. This fact leads to the concept of *multiverse*, consisting of the different universes that one can obtain (perhaps from an initial universe) via these methods. Woodin [23] first isolated this term in the form of the *set-generic multiverse*, in which only the method of set-forcing is permitted. Earlier work of mine ([11]) explored aspects of the *class-generic multiverse*, obtained by closing under class-forcing. These two notions of multiverse are rather different: the former preserves large cardinals notions and does not lead beyond set forcing, whereas the latter can destroy large cardinals and lead to universes that are not directly obtainable via class forcing.

For the Hyperuniverse Program the multiverse must be as rich as possible, i.e., closed under all conceivable methods for creating new universes. It is not obvious how to obtain such a multiverse, because by working with universes that contain all of the ordinals, quantification over outer models which are not set-generic ceases to be first-order. Another point is that for it to be usable in the Hyperuniverse Program, the multiverse cannot be described in vague terms, but must be given a precise mathematical formulation. This is a consequence of the basic assumption of the program that it is desirable to search for preferred universes on the basis of justified criteria for choosing certain universes over others; to make these choices it is necessary to have a precise formulation of the spectrum of possible universes within which these choices are to be made.

The above requirements are met by identifying the multiverse with the hyperuniverse, i.e., the collection of all countable transitive models of ZFC. Indeed, this is precisely formulated and results in a multiverse which is closed under all known universe-creation methods. We lose nothing by restricting ourselves to well-founded (or equivalently transitive) universes, as the elements of the hyperuniverse are intended to provide possible pictures of V, which is indeed well-founded.

The Hyperuniverse Program

What happened to V? By the Löwenheim-Skolem theorem, when we explore the universes within the hyperuniverse we see the full range of possible first-order properties that the full universe V of all sets may satisfy. Naturally, our picture of V is reflected by one of the pictures given by the preferred universes of the hyperuniverse. For this reason, first-order properties shared by all preferred universes must be true in V.

Thus we have arrived at a clear strategy for discovering first-order properties of the universe of all sets: We have a context *closed under arbitrary universe-creation methods* in which we can explore the different possible pictures of V, and then by imposing *justifiable preferences* for certain universes over others we can discover common first-order properties of these preferred universes which can be regarded as being true in V. This is the Hyperuniverse Program.

Criteria for Preferred Universes

Which universes should we prefer? I.e., what criteria should we use for choosing certains universes over others? There are two sources of such criteria.

The first type of criteria are those which arise directly from set-theoretic practice. These are criteria which prefer universes in which the difficulties in a specific area of set theory are more easily resolved. Some examples are: CH, $V = L$, PD and forcing axioms (MA, BPFA, BMM, ...). But these criteria are unsatisfying in many respects. They are local, in the sense that they only reflect the needs of a specific area of set theory. As interests in set theory change, so will these criteria. Because these criteria do not reflect a broad point of view, and are not stable over time, they are impossible to justify.

The second type of criteria are those that arise directly from an unbiased look at the hyperuniverse. These criteria are formulated without reference to set-theoretic practice. In particular, the technical notions of modern set theory, such as forcing, large cardinals, determinacy, combinatorial principles, . . . which reflect set-theoretic practice do not appear in them. Some examples are: maximality properties of universes (provided they do not mention technical notions like forcing), reflection principles (which are in fact certain types of maximality principles), omniscience and absoluteness principles (clarifed below). The hyperuniverse program takes the perspective that it is the criteria of this second type that can serve as justifiable criteria for the choice of preferred universes.

A possible risk in applying criteria of the second type is that they may lead to the adoption of first-order statements which contradict set-theoretic practice. As an example, consider the criterion of minimality, which says that the preferred universes of the hyperuniverse should be as small as possible. This leads to the preferred choice of just one universe, the minimal model of ZFC, and therefore to the statement that set-models of ZFC do not exist, in obvious conflict with set-theoretic practice. The same applies to a weaker notion of minimality, embodied by the axiom $V = L$. Although this does allow for the existence of set-models of ZFC, it does not allow for the existence of inner models of ZFC with measurable cardinals, another conflict with set-theoretic practice (see a further discussion of this point below). Criteria of the second type which lead to first-order statements in conflict with set-theoretic practice must be rejected; in Gödel's sense they are not "successful".

Examples

Justifiable criteria for the choice of preferred universes have so far been based either on *maximality* or *omniscience*. *Maximality* is the idea, advocated by Gödel and subsequently by others, that the universe should be as "large as possible". *Omniscience* is instead the idea that the universe should be able to "see" as much as possible of the full range of alternative universes. These principles can be realised as criteria of the second type in various ways. I begin with maximality.

Of course one cannot have "structural" maximality in the sense that a preferred universe contain all ordinals or all real numbers. This is simply because there is no tallest countable transitive model of ZFC and over any such model we can add new reals to obtain another such model. Instead the known maximality criteria make use of logic. Recall that this was predicted by Gödel, who referred to the use of fundamental concepts of logic in the search for new axioms. Let v be a variable that ranges over the elements of the hyperuniverse. Maximality criteria express the idea that if a set-theoretic statement with certain parameters holds externally, i.e., in some universe containing v, then it already holds internally, i.e., in some “subuniverse” of v. Different criteria arise depending on what one takes as parameters and what one takes for the concept of “subuniverse”.

When contemplating mathematical realisations of the maximality concept it is unavoidable to take into account the fundamental fact that the set-theoretic universe is determined by the class of ordinal numbers together with the power set operation. This inextricably leads to two types of maximality, one with regard to the “height” of the universe, ordinal maximality, and the other with regard to its “width”, power set maximality. As we will see, these forms of maximality not only constitute fundamental principles, but the latter form also gives birth to a powerful and paradigm-shaping mathematical theory.

In the case of *ordinal (or vertical) maximality*, we maximise with respect to the ordinals, having fixed the power-set operation. More precisely, let us say that a universe w is a *lengthening* of v if v is a (proper) rank initial segment of w. Then *ordinal maximality* says that v has a lengthening w such that for all first-order formulas φ and subsets A of v belonging to w, if $\varphi(A)$ holds in w then $\varphi(A \cap v_\alpha)$ holds in v_β for some pair of ordinals $\alpha < \beta$ in v. This is also known as a high-order *reflection principle* and is of the type already considered by Gödel. It leads to the existence of “small” large cardinals, i.e., large cardinal notions consistent with $V = L$ such as inaccessibles, weak compacts, ω-Erdős cardinals, $\ldots$. Ordinal maximality (reflection) is a well-established idea and is in perfect accord with set-theoretic practice. There is little controversy in the set theory community about its validity.

In analogy with ordinal maximality, *power set (or horizontal) maximality* expresses the idea that preferred universes are maximal with respect to the power set operation, having fixed the ordinals. More precisely: If a parameter-free sentence holds in some outer model of v (i.e., in some universe w containing v with the same ordinals as v) then it holds in some inner model of v (i.e. in some universe v_0 contained in v with the same ordinals as v). This is equivalent to my *inner model hypothesis (IMH)*, which states that by passing to an outer model of v we do not change *internal consistency*, i.e., we do not increase the set of parameter-free sentences which can hold in some inner model.

Power-set maximality is relatively new ([12], 2006), has striking consequences and has been the source of a rich mathematical investigation. It has also forced a paradigm shift with regard to set theorists' views of the role of large cardinals in set theory. The consistency of the IMH is established using the profound coding theorem of Jensen, together with another deep result, the consistency of projective determinacy (PD). When combining power set maximality with ordinal maximality, one is forced to dig even deeper and develop new versions of Jensen coding, a rich and ongoing study. Another exciting development triggered by this work is the study of strong absoluteness, including a stronger version of the IMH, which introduces “absolute” parameters into power set maximality. A pressing issue to be explored in this project is whether this form of maximality is consistent; if so, this yields a dramatic new solution to Cantor's continuum problem (indeed, CH is false in a strong sense).

The fact that these ideas, which emanate directly from natural realisations of the motiviating principle of maximality, have such a profound impact on set theory points to their fundamental importance.

As hinted at above, power set maximality has triggered a paradigm shift in our assessment of the status of large infinities (and of the principle of determinacy). The IMH refutes the existence of inaccessible cardinals as well as projective determinacy. This has led to a re-examination of the roles of large cardinals and determinacy in set-theoretic practice. Below is a brief discussion of this re-examination.

Aside 1: The role of large cardinals in set theory

Large cardinals arise in set theory in a number of ways: One starts with a model M of ZFC which contains large cardinals and then via forcing produces an outer model $M[G]$ in which some important statement holds. Notice that in the resulting model, large cardinals may fail to exist; they only exist in an inner model. And of course we did not have to assume that M was the full universe V; it was sufficient for M to be any transitive model with large cardinals. An important part of large cardinal theory consists of Jensen's program of building L-like inner models which realise them. Again, the emphasis here is on inner models for large cardinals, not on their existence in V. Large cardinals are also of great importance as they provide measures of the consistency strengths of statements. A typical consistency lower bound result is obtained by starting with a statement of interest and then constructing an inner model with a large cardinal. Once again, one sees that set-theoretic practice is concerned with inner models of large cardinals, and not with their existence in the full universe V. The conclusion is that set-theoretic practice, although it demands the existence of inner models with large cardinals, does not demand their existence in V.

Aside 2: The status of PD (projective determinacy) in set theory

It is commonly said that since Borel and analytic sets are regular (in the sense that they are measurable and have the Baire and perfect set properties) and PD extends this fact to all projective sets, that PD can be justified as being "true" based on this natural extrapolation. But there is a problem with this argument: Consider Shoenfield absoluteness, the absoluteness of Σ^1_2 statements with respect to arbitrary outer models. This is provable in ZFC even if one allows arbitrary real parameters. One is then naturally led to conjecture Σ^1_n absoluteness with arbitrary real parameters. But Σ^1_3 absoluteness with arbitrary real parameters is provably false ([10]). With arbitrary real parameters a consistent principle can only be obtained by making an artificial restriction to set-generic outer models; as soon as one relaxes this to class-genericity, the principle becomes inconsistent. So if one is so easily led to inconsistency when extrapolating from Σ^1_2 to Σ^1_3 absoluteness, how can one feel confident about the extrapolation from Σ^1_1 measurability to Σ^1_2 measurability? More reasonable would be the extrapolation *without parameters*. Indeed, parameter-free Σ^1_3 absoluteness, unlike the version with arbitrary real parameters, is consistent (and indeed follows from the IMH). Thus a natural conclusion with regard to PD is the following: The regularity of projective sets is a reasonable extrapolation from the regularity of Borel and analytic sets, *provided one does not allow real parameters*. Similarly, although PD cannot be justified based on extrapolation, it is plausible that parameter-free PD or even OD (ordinal-definable) determinacy without real parameters can be so justified.

In light of the above two Asides, let us consider the question of the compatibility of the IMH (power-set maximality) with set-theortic practice. If one accepts that the role of large cardinals in set theory is via inner models and that the importance of PD is captured by its parameter-free version then this compatibility appears to be restored: The IMH is consistent both with inner models of large cardinals and with parameter-free PD (indeed with OD-determinacy without real parameters).

However there does remain the problem that the IMH contradicts ordinal maximality, as the latter gives rise to inaccessible cardinals. Fortunately there is a clear and precise scenario for dealing with this difficulty (see the "conjectured synthesis" below).

Omniscience

The *omniscience principle* is stated as follows: Let Φ be the set of sentences with arbitrary parameters which can hold in some outer model of v. Then Φ is first-order definable in v.

I first saw this kind of statement in work of Mack Stanley [19], where he shows that there are omniscient universes (my terminology, not his), assuming the consistency of sufficiently many Ramsey cardinals, less than the consistency of a measurable cardinal. I find omniscience, like maximality principles, to be a justifiable motivating principle for the choice of preferred universes.

A Conjectured Synthesis

Now where do we stand with regard to possible justified criteria for preferred universes? So far, we have introduced three examples: ordinal-maximality, power-set maximality and omniscience. The mathematical work of the Hyperuniverse Program is to investigate the mutual compatibility of principles of these types, possibly in conjunction with other set-theoretic statements that may in the future emanate from justified criteria. This is therefore a rich and dynamic process, consisting of much more than a focus on a single question, but rather on a broad spectrum of questions, each making special mathematical demands and guided by the philosophical demands of the program.

It would be ideal if we could combine all of our justifiable criteria into a single consistent criterion, i.e., a criterion that is satisfied by at least one element of the hyperuniverse. I conjecture that such a convincing synthesis is possible for the forms of maximality and omniscience so far introduced. As mentioned above, we cannot simply combine power-set maximality with ordinal-maximality and omniscience, as even the first two of these principles contradict each other. Instead we propose the following sample conjecture:

Conjecture. Let IMH* be the IMH restricted to ordinal-maximal and omniscient universes (i.e., the statement that if a sentence holds in an ordinal-maximal and omniscient outer model of v then it holds in an inner model of v). Then the conjunction of IMH*, ordinal-maximality and omniscience is consistent.

A key aim of the mathematical part of this project is to prove this conjecture. If verified, it constitutes a major success for the Hyperuniverse Program, as it shows that one can arrive at principles with dramatic consequences which can be taken to be true in V, based on justified principles for choosing preferred universes which obey them. Other programs for arriving at new axioms of set theory ([7, 22]) do not share the unbiased approach of the Hyperuniverse Program, and therefore success with the Hyperuniverse Program constitutes progress in the foundations of set theory of a ground-breaking nature.

The power of the IMH* can be seen as follows: By virtue of power set maximality, it follows from the IMH* that for some real number R, there is no inner model with a measurable cardinal containing R. This is a consequence of a version of Jensen's coding theorem in the context of Ramsey cardinals, an adaptation of the work in [8], which in addition takes into account higher-order forms of stationarity in a strong theory of classes. Now applying the Dodd-Jensen work on covering over the Dodd-Jensen core model relative to R, one obtains nearly all of the combinatorial and definability-theoretic consequences of the nonexistence of an inner model with a measurable: the singular cardinal hypothesis is true, there are no precipitous ideals, projective determinacy and the proper forcing axiom are false and much more. In addition, omniscience gives us the existence of $\#$'s for reals, so it follows for example that any two non-Borel analytic sets of reals are Borel isomorphic. Further sophisticated work involving uses of Jensen's $\square$ principle in the Dodd-Jensen core model implies that although there are reals contained in no inner model with a measurable, one does have inner models with rather large cardinals, such as measurable cardinals of arbitrarily high Mitchell order. The IMH* is also consistent with the axiom of determinacy for all ordinal-definable sets of reals.

The mathematical work of this project is not confined to the form of the Synthesis Conjecture stated above. Indeed, there is a wide range of possible "Synthesis Conjectures", depending upon what one accepts as criteria for the choice of preferred universes on justifiable philosophical grounds. As mentioned earlier, omniscience is rather new and there may be philosophically convincing arguments that it should either be strengthened or weakened. In either case, the new mathematics required to establish the resulting forms of the Synthesis Conjecture will be demanding. As the Hyperuniverse Program as a whole is also rather new, it is too early to say if philosophically convincing arguments will lead to criteria that imply the existence of large cardinals stronger than Ramsey cardinals; if so, this will bring in major new mathematical challenges and conjectures. Probably the most exciting possibility was briefly mentioned above: perhaps it will be possible to consistently introduce "absolute" parameters into the IMH. This will provide a solution to the continuum problem, the oldest and most fundamental open question in set theory. For example, the IMH for universes which preserve ω_1 and ω_2 forces the continuum to have size at least ω_2. But is it consistent? There are current mathematical developments and conjectures which point in this direction, concerning the possibility of coding cardinal-preserving sets into generic sets. A big challenge in this project is to make such an argument work.

Methodology

This is a complex topic, which in an interactive way combines philosophical considerations with mathematical considerations within the field of set theory (specifically, the study of large cardinals and generalisations of Jensen coding). For its investigation I will require the aid of two postdocs, one from philosophy and the other from set theory, and one doctoral student. Also 12 senior visitors from these fields will be invited for two-week visits to enhance the work of the project.

The main tasks of the philosophy postdoc will be to provide the philosophical basis for the mathematical work of this project. Specifically, this concerns the following issues: (1) Justification of the epistemological approach of the Hyperuniverse Program. As stated above, this program is based on an anti-Platonist perspective whereby one actively develops criteria for the preference of preferred pictures of V. This approach must be further analysed and justified in light of the extensive literature on realism in mathematics. (2) An examination of the concepts of *de facto* versus *de jure* truth in set theory. What can justifiably be declared as a *de facto* truth? How can one defeat the skeptic who insists that the search for *de jure* truth is doomed to fail? (3) An examination of maximality as a motivating principle for preferred universes of set theory. There is an extensive literature on this topic, from Gödel to Maddy, with many other contributions as well. (4) What are the justifiably realisations of the maximality principle as mathematical criteria? This is one of the most difficult tasks for the philosopher in this project, as there are many versions of ordinal and power set maximality to consider and they are sometimes in conflict with each other. (5) Does ominiscience qualify as a motivating principle, and if so what mathematical form should it take? The precise mathematical formulation of omniscience stated above is not the only possibility. (6) What is the philosophical justification for the synthesis of a priori conflicting criteria for preferred universes? If justified, what form should such a synthesis take? The precise mathematical formulation of the Synthesis Conjecture above is not the only possibility. (7) Or is there an inherent bifurcation in the choice of criteria for preferred universes? This will be a dramatic conclusion, as it will mean that the Hyperuniverse Program is not one program but several, pursued along different paths leading to different conclusions about set-theoretic truth.

The above list is not meant to be exhaustive, as it may be that in the course of the project, new philosophical questions will arise as a result of the mathematics being developed. Moreover, although my colleague Tatiana Arrigoni has put the initial formulation of the Hyperuniverse Program on a solid philosophical footing (see our joint draft [4]), I am not a philosopher and therefore will defer to the philosopher on the project to augment the above list as he sees fit. In any case, it will be important to not only reflect upon our mathematical work but also to take into account important contributions in the existing literature, such as writings by Arrigoni ([1]), Benacerraf, Bernays, Boolos, Parsons and Putnam (all in [5]), Gödel ([13]), Hauser ([14]), Jensen ([15]), Koellner ([16]), Maddy ([17]), Wang ([21]) and Woodin ([23]), as well as the views of the 12 two-week senior visitors to the project.

The main tasks of the set theory postdoc and of the doctoral student will be to aid me in verifying various forms of maximality, omniscience and the Synthesis Conjecture. In some cases the methods for establishing these results are already available using my coding methods (see [11]), known facts about determinacy or Mack Stanley's methods in [19]. But in many cases, challenging new coding methods will be required, such as in my [8]. The first step, intended for the doctoral student, will be to handle ordinal maximality and omniscience, in the absence of power-set maximality. This will not demand the more challenging methods, reserved for the set theory postdoc. Most challenging of all will be the introduction of parameters into power set maximality, aimed at establishing the consistency of the strong IMH (see [12]). Whereas the IMH alone has already proved itself to be a powerful principle with dramatic and paradigm-shifting consequences, the strong IMH, if proved consistent, may provide the first philosophically justifiable solution to Cantor's continuum problem.

Yet another task, more of mathematical than of philosophical interest, will be to understand the consistency strength of these principles. The IMH is known to have strength between measurable cardinals of high order and Woodin cardinals; its exact strength is not known and can only be resolved via a profound investigation via core model theory. The strong IMH is known to require at least a strong cardinal, but more than that is not currently known.

The philosophical and mathematical work on this project are heavily dependent upon each other. To illustrate this I mention two simple examples. Some philosophers have taken the existence of large infinities to be a consequence of the maximality of the set-theoretic universe and have on this basis asserted their existence as being a "true" statement of set theory. But this claim is easily refuted by the mathematical analysis of maximality as provided by the IMH. Conversely, some mathematicians have considered reflection principles that give rise to very large cardinals (beyond what is given by ordinal-maximality, which is all that Gödel had in mind). It is even possible to formulate a version of power set maximality relative to universes with such cardinals. But as pointed out by some philosophers (Koellner, for example), such forms of reflection are not justifiable on philosophical grounds. The point I am making is that the work of mathematicians and philosophers on the foundations of set theory is interactive, and each group needs the other to avoid making mistakes. In the present project, the relevant forms of the Synthesis Conjecture depend on the known philosophically-justifiable criteria for preferred universes, and conversely, the choice of criteria must take into account what is mathematically possible in terms of consistency. It will be interesting to see how the project develops as its mathematical and philosophical sides influence each other.

It is important to embed this work into the broader context of the foundations of set theory and this can best be achieved by inviting a number of senior experts in this field to visit and offer input into the development of the project. Scholars I currently have in mind are the philosophers Arrigoni, Koellner, Maddy and Martin as well as the set-theorists Fuchino, Hamkins, Löwe, Magidor, Sakai, Stanley, Welch and Woodin.

An approximate timetable for our work is as follows:

Year 1. Philosophically: We'll compare the approach of the Hyperuniverse Program with approaches and ideas of Gödel [13], *Woodin* [23], *Shelah* [18] *and others. A special emphasis will be given to the justification of an epistemological approach to the foundations of set theory and to the disctinction between "de facto" and "de jure" truth, in light of the relevant literature on these topics. Mathematically: We'll prove the existence of universes satisfying different forms of ordinal maximality and omniscience.*

Year 2. Philosophically: We'll examine maximality as a motivating principle for the choice of preferred universes, taking into account its implications of maximality for the non-existence of large cardinals. We'll also analyse omniscience together with other candidates for such motivating principles. Mathematically: We'll prove the consistency of power set maximality for ordinal-maximal universes, first without and then with omniscience.

Year 3. Philosophically: We'll develop arguments in favour of the synthesis of potentially conflicting criteria for preferred universes, in light of the motivating principles examined in Year 2 and the mathematical work of the project. We'll also harvest the implications of the mathematical work in Years 1 and 2 for the nature of set-theoretic truth and develop arguments to justify the use of maximality principles to resolve Cantor's continuum problem (CH). Mathematically: We'll explore maximality with absolute parameters with the aim of arriving at a compelling and consistent principle that resolves CH.

Audience

In recent years, leading set-theorists have taken a keen interest in philosophy, as philosophical ideas are needed to sort out the huge number of possibilities for extensions of the standard ZFC axioms. Conversely, philosophers of set theory have become increasingly sophisticated in mathematical techniques and are now able to make valuable suggestions about the direction of future research in set theory. Thus there is now a sizeable and talented community of people working on the frontier between set theory and philosophy and this project is primarily aimed at this community. Beyond this, questions in the foundations of set theory are of wide appeal throughout both set theory and mathematics in general, as well as to philosophers who take an analytical approach to their subject.

Publication Plans

We expect about 20 research articles to result from this project and will submit them all to leading journals in logic, mathematics and philosophy, such as the Journal of Symbolic Logic, the Bulletin of Symbolic Logic and the Annals of Mathematical Logic. About half of these papers will be of a purely mathematical nature, verifying instances of the Synthesis Conjecture and relating these results to other set-theoretic statements, such as the existence of large cardinals. The remainder of the papers will focus on the philosophical implications of the mathematical work we have done for truth in set theory. A few of these papers, written by the philosophy postdoc, possibly in collaboration with me, will be of a purely philosophical nature, discussing the legitimacy of the Hyperuniverse Program and its formulation in terms of *de facto* and *de jure* truth. Typically, articles appear about two years after submission, which means that we expect about seven articles to appear before the end of the project, another seven within one year after the end of the project and the remainder within two years after the end of the project. In addition, we will produce a Proceedings Volume for the project which will collect preliminary versions of these articles together into a single volume; this volume will appear within one year after the end of the project.

Relation to Prior Work

There is a strong literature on the foundations of set theory; some of the leading articles on this topic appear in the bibliography below. This literature has however been tied to a "static" view of set-theoretic truth, in which attempts are made to philosophically justify familar axioms that arise in set theory as partial descriptions of a fixed Platonistic universe V of sets. What is exciting about the Hyperuniverse Program is that it provides a "dynamic" approach to truth about V, by actively searching for principles to compare universes for the purpose of selecting those which are "preferred". Thus as opposed to a fixed Platonistic conception of the universe of sets, the program provides an epistemic approach which enables one to uncover new axioms as the result of justifiable criteria for preferring certain pictures of V over others. This dynamic approach has already forced a re-examination of the role of large infinites and determinacy principles in set theory and raises many interesting new mathematical challenges.

As explained above, the Hyperuniverse Program is a unique approach to set-theoretic truth that combines serious philosophy with challenging mathematics. In recent years, an increasing number of set-theorists have taken an interest in the philosophy of their subject, and conversely, an increasing number of philosophers have informed themselves about the mathematical issues in modern set theory. The Hyperuniverse Program aims to fully exploit this multidisciplinary interest and due to its breadth promises to remain an important and active program in the foundations of set theory for the long-term future.

Final Conference

The theme of the (non-JTF funded) final conference to be held at the end of the project will be "Truth and Infinity". It will take place in June 2015 at the Kurt Gödel Research Center in Vienna. The intended audience will be mathematicians and philosophers with an interest in the foundations of set theory. All of the participants in the Hyperuniverse project will be invited to speak at this conference, in addition to other leading scholars in this field. Slides for the lectures delivered at the conference will be posted on a website afterwards to be made accessible to the logic and philosophy communities at large.

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