Capturing the Universe

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Abstract

We describe the universe $V$ of sets using the ideas of mouse iteration from large cardinal theory and forcing. We introduce Mighty Mouse, an absolute mouse of strength far less than one Woodin cardinal, and show that it captures $V$ in the sense that $V$ is a generic extension of one of its definable iterates via a forcing that is definable and whose antichains are sets. A key tool is the Stable Core of [6], an inner model over which $V$ is generic. We show that the Stable Core is contained in an iterate of Mighty Mouse and has the same reals as Mighty Mouse.

Introduction

The universe of sets $V$ is important for the foundations of mathematics as it provides an arena in which virtually all mathematical constructions can be carried out. But what does $V$ look like? Can it be described using the tools that set-theorists have for building universes of set theory?

Gödel [10] provided us with the universe $L$ of constructible sets, an important subuniverse of $V$ with remarkable combinatorial properties and clear internal structure ([11]). Cohen [2] later produced a method for creating new universes from old, the forcing method, which can be used to obtain generic extensions of $L$ which are larger than $L$. Is $V$ simply a generic extension of

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$L$? If so, then $V$ can be described using just the methods of constructibility and forcing.

However further work of Scott [15] revealed that $V$ cannot be a generic extension of $L$ if large cardinals exist. Large cardinals (also called large infinities) are essential to set theory as they are needed to show that important set-theoretic phenomena are consistent with the traditional axioms for set theory.

So to achieve our goal of describing $V$ we need something more, and this is the notion of mouse. To explain mice we take a closer look at the type of large cardinal that Scott considered, a measurable cardinal. Let us say that $U$ is a measure on a set $X$ if $U$ is a collection of subsets of $X$ such that for any subset $Y$ of $X$, either $Y$ or $X \setminus Y$ belongs to $U$ and whenever $U_0$ is a subcollection of $U$ of size less than the size of $X$, the intersection of the sets in $U_0$ belongs to $U$. We say that $U$ is nonprincipal (or nontrivial) if $U$ consists only of infinite sets and we say that $X$ is measurable if $X$ is uncountable and there is a nonprincipal measure on $X$. A cardinal number $\kappa$ is measurable if it is the cardinality of a measurable set, which can be taken to be $\kappa$ itself.

If $U$ is a nonprincipal measure on the uncountable cardinal $\kappa$ then we can form the universe of sets constructible from $U$, denoted by $L[U]$. Silver [16] showed that $L[U]$ is a very nice ”$L$-like” model, sharing many of the properties of Gödel’s $L$. Like $L$, $L[U]$ is not a set, but a class, as it contains all ordinal numbers. However in his analysis of $L[U]$, Silver was led to study smaller versions $m = \bar{L}[\bar{U}]$ of $L[U]$ which are sets and in which $\bar{U}$ has the appearance of a measure in $m$. Further deep work of Dodd and Jensen [3] developed this idea fully into a theory of what we now call mice.

Not every set-sized version of $L[U]$ is a mouse; like $L[U]$ it must be well-founded and also iterable; the latter means the following. If $m = L[U]$ is a set-sized version of $L[U]$, then there is a natural way to form its ultrapower

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1. Using the axiom of choice, every set $X$ can be put into 1-1 correspondence with an ordinal number and the least such ordinal number is called the cardinality of $X$. The cardinal numbers are simply the cardinalities of the sets. We assume the axiom of choice throughout.

2. This means that there are no infinite decreasing sequences through its ordinals, which is the case for $V$ and its subuniverses.
Ult\((m, \bar{U})\) with a natural embedding from \(m\) into Ult\((m, \bar{U})\). For \(m\) to be iterable both \(m\) and Ult\((m, \bar{U})\) must be well-founded. And we can repeat this, forming Ult\((\text{Ult}(m, \bar{U}))\); this too must be well-founded. In fact, we require that well-foundedness is not lost when we continue this process of forming iterated ultrapowers for any ordinal number of stages. Without the iterability requirement we say that \(m\) is a premouse; a mouse is an iterable premouse. Iterability helps ensure uniqueness; for example there is a unique mouse called 0\# which is contained in all other mice. And this least mouse is not generic over \(L\) (a strong form of Scott’s Theorem).

So far we have only discussed mice which are set-sized versions of \(L[U]\), a model with just one measurable cardinal. But it is not difficult to generalise the above discussion to models with more measurable cardinals. A particularly interesting case is a model with a measurable limit of measurable cardinals; the analogue of 0\# for this property is denoted by \(m_1^{\#}\). Just as 0\# is the least mouse with a measurable cardinal, \(m_1^{\#}\) is the least mouse with a measurable limit of measurable cardinals.

The reason for introducing \(m_1^{\#}\) goes beyond mere generalisation. This mouse can be used to capture something significant about the set-theoretic universe. Let Card denote the proper class of all cardinal numbers. In analogy to the enlargement of Gödel’s \(L\) to the universe \(L[U]\) of sets constructible from the measure \(U\), we can also form the universe \(L[\text{Card}]\) of sets constructible from the class Card. Then \(m_1^{\#}\) captures the universe \(L[\text{Card}]\) in a strong sense:

**Theorem 1** \(L[\text{Card}]\) is contained in the intersection with \(V\) of an iterate of \(m_1^{\#}\).

We explain this result as follows. Recall that \(m_1^{\#}\) is the least mouse with a measurable limit of measurable cardinals. For simplicity of notation let \(m\) denote \(m_1^{\#}\). Now form the ultrapower Ult\((m)\) using the measure on the least measurable cardinal of \(m\). Then form the ultrapower Ult\(^2\)(\(m\)) of Ult\((m)\) using the measure on the least measurable cardinal of Ult\((m)\). Iterate in this way for \(\aleph_1\) steps, where \(\aleph_1\) denotes the least uncountable cardinal (of \(V\)). Then \(\aleph_1\) is a measurable cardinal in this iterate. Use the 2nd measure of this iterate for \(\aleph_2\) steps, where \(\aleph_2\) denotes the 2nd uncountable cardinal. Then \(\aleph_1\) and \(\aleph_2\) are the first two measurable cardinals of this new iterate. Keep
iterating until $\aleph_n$ is the $n$-th measurable cardinal of the resulting iterate for each finite $n$. Then iterate the next available measurable cardinal up to $\aleph_{\omega+1}$, the least cardinal greater than the supremum of the $\aleph_n$’s. Continue this indefinitely until one reaches an iterate $m^*$ (which is now a class and no longer a set) where the ordinals (of $V$) that are measurable cardinals of $m^*$ are exactly the uncountable successor cardinals (of $V$). Thus the class of successor cardinals, and therefore the class Card (the closure of the class of successor cardinals) is definable over the intersection with $V$ of $m^*$ and therefore $L[\text{Card}]$ is contained in this intersection. So $m_1^#$ strongly captures $L[\text{Card}]$.

Using stronger mice we can strongly capture larger models. For example, if we use a mouse for a measure $U$ on a cardinal $\kappa$ such that the set of measurable cardinals less than $\kappa$ belongs to $U$ (a measure of Mitchell order 1) then we can strongly capture $L[\text{Reg}]$, where Reg denotes the class of regular cardinals.\footnote{A cardinal $\kappa$ is regular if the union of fewer than $\kappa$ sets of size less than $\kappa$ still has size less than $\kappa$. A successor cardinal (i.e. a successor in the increasing enumeration of cardinals) is regular, but a limit cardinal need not be regular.} We can also strongly capture $L[\text{Cof}]$, where Cof denotes the cofinality function.\footnote{The cofinality of an infinite cardinal $\kappa$ is the least cardinal $\mu$ such that $\kappa$ is the union of a size $\mu$ family of sets, each of size less than $\kappa$.} For this we use a mouse with a measure of Mitchell order $\omega + 1$ (see [8]).

Can a mouse strongly capture the entire universe $V$? Not unless there is a largest mouse, as no iterate of a mouse can include a larger mouse as an element.

However, recall that we have a second method for building universes of set theory, the forcing method. A mouse $m$ captures a universe $M$ if $M$ is a generic extension of a universe strongly captured by $m$.

Our main result is the following.

**Theorem 2** $V$ is a generic extension of (the intersection with $V$ of) an iterate of some mouse. Moreover both the forcing witnessing the genericity and the iteration providing the iterate are definable, and the mouse is absolute (with a fixed definition in any universe containing it). Thus $V$ is captured by a mouse in a strong sense.
The least mouse witnessing Theorem 2 is called *Mighty Mouse*, denoted \( \text{mm} \). Theorem 2 is proved by showing that \( \text{mm} \) strongly captures (a version of) the *Stable Core* of \([6]\). Then we apply the main result of \([6]\), asserting that \( V \) is a generic extension of the Stable Core.

Thus we have achieved our initial goal: \( V \) can be described using the methods of mouse iteration and forcing.

Theorem 2 has some additional consequences (also see Section 4):

1. *The Generic IMH*. The *IMH* (Inner Model Hypothesis) was introduced in \([5]\). In its simplest version, it asserts that if a sentence holds in an outer model of \( V \) then it holds in an inner model of \( V \). The *Generic IMH* is the weaker statement that if a sentence holds in an outer model of \( V \) then it holds in a generic extension of an inner model of \( V \). Theorem 2 implies that the Generic IMH does hold if Mighty Mouse = \( \text{mm} \) exists. For, if \( \varphi \) holds in an outer model \( W \) of \( V \), then using the absoluteness of \( \text{mm} \), \( W \) is a definably-generic extension of the intersection with \( W \) of an iterate of \( \text{mm} \). But this intersection is elementarily equivalent to \( K_{\text{mm}} \), the intersection with \( V \) of the iterate of \( \text{mm} \) obtained by simply iterating its top measure Ord-many times; thus \( K_{\text{mm}} \) is an inner model of \( V \) and \( \varphi \) holds in a generic extension of \( K_{\text{mm}} \). If we further assume that for each set \( x \), \( \text{mm}_x \), Mighty Mouse relativised to \( x \), exists then the Generic IMH with arbitrary set-parameters holds in \( V \).

2. *A unique Multiverse*. Let \( \mathbb{M}(V) \) denote the multiverse obtained from \( V \) by closing \( \{V\} \) under generic extensions, elementary embeddings and their inverses. I.e. \( \mathbb{M}(V) \) is the smallest multiverse satisfying: \( V \) is in \( \mathbb{M}(V) \) and if \( V_0 \) is a forcing extension of \( V_1 \) or elementarily embeds into \( V_1 \) and either \( V_0 \) or \( V_1 \) is in \( \mathbb{M}(V) \) then both \( V_0 \) and \( V_1 \) are. Then if \( V_0 \) and \( V_1 \) both contain Mighty Mouse it follows that \( \mathbb{M}(V_0) \) equals \( \mathbb{M}(V_1) \). This is because both \( V_0 \) and \( V_1 \) are generic extensions of universes into which \( K_{\text{mm}} \) embeds.

3. *The Stable Core is small*. In \([6]\) I asked if the stable core is a good approximation to \( V \) in the senses that weak covering holds relative to it, large cardinals are witnessed by it, it is rigid and \( V \) is generic over it. By Theorem 2 the first two of these properties fails. Rigidity also fails (see
Section 4). The genericity of $V$ over the Stable Core was already established in [6].

4. Maximalilty. In [9] we argued that the maximality of the universe *in height* is captured by the notion of $\#$-generation. The Generic IMH with parameters is a strong statement of maximality in width. Thus by Theorem 2, a consistent and appealing maximality principle for $V$ in both height and width is captured by the existence of $\text{mm}_x$ for each $x$ together with $\#$-generation, whose consistency strength is far below one Woodin cardinal.

This paper is structured as follows. In Section 1 we discuss the theory of mice, introducing the mouse mm. In Section 2 we present a version of the Stable Core well-suited to our purposes and show that $V$ is a generic extension of it. In Section 3 we show that Mighty Mouse absorbs the Stable Core, finishing the proof of Theorem 2. In Section 4 we discuss corollaries and variants of Theorem 2, and prove that the use of Mighty Mouse is optimal for our results.

1. Mice

*Mice* serve as approximations to models with large cardinals, and large cardinals are often best formulated using the notion of *elementary embedding*. In particular, this is the case for the large cardinal notions central to this paper. A function $j : V \rightarrow M$ from the universe $V$ into a subuniverse $M$ is an *elementary embedding* if for any $x_1, \ldots, x_n$ in $V$, a first-order property $\varphi$ is true for $x_1, \ldots, x_n$ in $V$ exactly if it is true for $j(x_1), \ldots, j(x_n)$ in $M$. In particular $x_1 \in x_2$ iff $j(x_1) \in j(x_2)$.

**Theorem 3** (Scott [15]) The following are equivalent.
(a) $\kappa$ is a measurable cardinal.
(b) There is an elementary embedding $j : V \rightarrow M$ with critical point $\kappa$, i.e. such that $j(\alpha) = \alpha$ for ordinals $\alpha < \kappa$ and $j(\kappa) > \kappa$.

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$^6$We make free use in this paper of functions whose domains are proper classes. This is easily formalised using the standard system GB, Gödel-Bernays class theory.

$^6$A first-order property is a property that is expressible using the membership relation $\in$, equality $=$, variables, logical connectives like *and*, *or* and *not* and quantifiers *for all* and *there exists* that range over elements of the universe.
We can strengthen measurability by imposing further requirements on $M$ in the embedding $j : V \to M$ witnessing the measurability of its critical point $\kappa$. By requiring $M$ to contain $H(\alpha) =$ the union of all transitive sets of size less than $\alpha$ for a cardinal $\alpha$ (perhaps depending on the embedding $j : V \to M$) which is larger than $\kappa^+ =$ the least cardinal greater than $\kappa$, we express the idea of $M$ being "close" to $V$.

**Definition 1** A cardinal $\kappa$ is 1-strong (or just strong) if for any cardinal $\alpha$ there is $j : V \to M$ with critical point $\kappa$ such that $j(\kappa)$ is greater than $\alpha$ and $M$ contains $H(\alpha)$. It is $n + 1$-strong (for finite $n > 0$) if we also require that $V$ and $M$ have the same $n$-strongs less than $\alpha$.

If $\kappa$ is a measurable cardinal then $H(\kappa)$ is a model of set theory and we can talk about the existence of large cardinals in this "local" universe $H(\kappa)$. We are interested in a universe in which there is a measurable cardinal $\kappa$ such that for each $n$, the cardinals less than $\kappa$ which are $n$-strong in the truncated universe $H(\kappa)$ are unbounded in $\kappa$. Just as in the case of a measurable cardinal (without the requirement of $n$-strongs in $H(\kappa)$) we can discuss premice which are set-sized versions of a universe with a measurable $\kappa$ and unboundedly many $n$-strongs in $H(\kappa)$ for each $n$. There is also a corresponding notion of iteration and iterability as well as a least iterable premouse (i.e. mouse) of this type. This is *Mighty Mouse*, denoted $mm$.

To help explain iterations of $mm$ we sketch the theory of iteration trees (invented by Martin and Steel [12]). First we make some remarks about $L[E]$-models. These are models obtained by relativising Gödel’s constructible universe to a predicate $E$ built from extenders. Just as $L$ is the union of a hierarchy $(L_\alpha \mid \alpha \in \text{Ord})$ where $L_\alpha$ consists of the first $\alpha$ levels of $L$, so is $L[E]$ the union of a hierarchy $(L_\alpha[E] \mid \alpha \in \text{Ord})$. Here, $E = (E_\alpha \mid \alpha \in \text{Ord})$ is a sequence of extenders, which means that either $E_\alpha$ is empty or is a cofinal elementary embedding $E_\alpha : L_\bar{\alpha}[E] \to L_\alpha[E]$ where $\bar{\alpha}$ is the least cardinal of $L_\alpha[E]$ greater than the critical point of $E_\alpha$. Further conditions are imposed to ensure that $L[E]$ shares many of the nice features of $L$. In addition, an extender $E_\alpha$ can be used to form an ultrapower $\text{Ult}(L_\beta[E], E_\alpha)$ into which $L_\beta[E]$ naturally embeds, assuming $\beta \geq \alpha$ and $\alpha$ is still a cardinal in $L_\beta[E]$. The $E$-sequences of $L_\beta[E]$ and $\text{Ult}(L_\beta[E], E_\alpha)$ agree below $\alpha$ and if $\beta > \alpha$ then although $\alpha$ is not a cardinal of $L_\beta[E]$, it is a cardinal in $\text{Ult}(L_\beta[E], E_\alpha)$.
We iterate $L[E]$ by successively applying extenders to form ultrapowers, just as in the case of $L[U]$, with one important change: At each stage we choose an extender from the model we have reached in the iteration but instead of applying it to that same model, we have the freedom to apply it to a model that appeared earlier in the iteration. This sequence of iterates of the initial $L[E]$ model carries the structure of a tree, known as an iteration tree: At stage $\alpha$, if we form Ult($M_i, E$) where $E$ is taken from the $\alpha$-th model $M_\alpha$ and $M_i$ denotes the model with index $i \leq \alpha$, then we place $\alpha + 1$ as an immediate successor of $i$ in the tree. In the iterations that we will consider in this paper, $i$ will always be chosen as the least $i$ for which we can form Ult($M_i, E$) (i.e. so that the least cardinal greater than the critical point of $E$ is the same in $M_i$ as it is in $M_\alpha$). Iterability means that wellfoundedness is never lost when iterating with iteration trees. In particular, at a limit stage $\lambda$ we must be able to find a branch through the iteration tree with indices cofinal in $\lambda$ which is wellfounded, i.e. so that the direct limit of the models indexed along the branch is wellfounded.

Everything said above also applies to premice which are just set-sized versions of $L[E]$ models. An iterable premouse is a mouse. Thus mm (Mighty Mouse) is an iterable premouse $L_\alpha[E]$ with a measurable cardinal $\kappa$ that is a limit of cardinals which are $n$-strong in $L_\kappa[E]$ for each $n$. In fact mm is the least such mouse in the sense that $\alpha \leq \beta$ and $L_\alpha[E] = L_\alpha[F]$ for any such mouse $L_\beta[F]$. It follows from the existence of (much less than) a Woodin cardinal that mm exists.

2. The Stable Core

In this section we introduce a slight variant of the Stable Core of [6] and establish its basic properties, the most important of which being that $V$ is one of its generic extensions.

Recall that for an infinite cardinal $\alpha$, $H(\alpha)$ is the union of all transitive sets of size less than $\alpha$. A useful fact is that $H(\alpha)$ is $\Sigma_1$-elementary in $V$ for uncountable cardinals $\alpha$ and therefore if $\alpha < \beta$ are both uncountable cardinals then $H(\alpha)$ is $\Sigma_1$-elementary in $H(\beta)$. Note that if $\beta$ is a strong limit cardinal (i.e. an uncountable cardinal such that $2^\gamma < \beta$ for $\gamma < \beta$) then $H(\alpha)$ is an element of $H(\beta)$ for infinite cardinals $\alpha < \beta$ and $H(\beta)$
has cardinality $\beta$. In this case we write $\mathbb{H}(\beta)$ for the amenable\footnote{A structure $(T, A)$ with $T$ transitive is \textit{amenable} if $A \cap t$ belongs to $T$ for each $t$ in $T$. Amenability ensures the existence of a predicate which is universal for predicates which are $\Sigma_n$-definable over $(T, A)$ for each $n$.} structure $(H(\beta), H \upharpoonright \beta)$ where $H \upharpoonright \beta = \{(\alpha, H(\alpha)) \mid \alpha \text{ is an infinite cardinal less than } \beta\}$.

We say that $\alpha$ is $\beta, n$-stable if $\alpha < \beta$ are strong limit cardinals and $\mathbb{H}(\alpha)$ is $\Sigma_n$-elementary in $\mathbb{H}(\beta)$. (For $n = 0$ the latter condition is automatic.) Using the fact that $H(\alpha)$ is $\Sigma_1$-elementary in $H(\beta)$ for uncountable cardinals $\alpha < \beta$, this can be seen to be equivalent to saying that $\alpha < \beta$ are strong limit cardinals and $H(\alpha)$ is $\Sigma_{n+1}$-elementary in $H(\beta)$.

Let Stable$_n(\beta)$ denote the set of $\alpha$ which are $\beta, n$-stable. We say that $\alpha$ is nicely $\beta, n$-stable if $\alpha$ is $\beta, n$-stable and in addition, if $n > 0$, the $\beta, (n-1)$-stables are cofinal in $\beta$. If the $\beta, n$-stables are cofinal in $\beta$, then $\alpha$ is (nicely) $\beta, (n+1)$-stable if $(\mathbb{H}(\alpha), \text{Stable}_n(\alpha))$ is $\Sigma_1$-elementary in $(\mathbb{H}(\beta), \text{Stable}_n(\beta))$. Note that if $\alpha$ is $\beta, (n+1)$-stable then the $\alpha, n$-stables are cofinal in $\alpha$ (but the $\beta, n$-stables need not be cofinal in $\beta$).

The \textit{Stability Predicate} $S$ consists of all triples $(\alpha, \beta, n)$ such that $\alpha$ is nicely $\beta, n$-stable. The predicate $S$ is $\Delta_2$ definable. The \textit{Stable Core} is the structure $(L[S], S)$.

\textbf{Theorem 4} (In Gödel-Bernays class theory) $V$ is generic over the Stable Core. More precisely, for some $(L[S], S)$-definable and Ord-cc forcing $Q$, there is a $G$ which is $Q$-generic over $(L[S], S)$ such that $V = L[G]$ and $(L[G], G)$ is a model of ZFC. $G$ is generic over $V$ for a definable forcing and if there is a satisfaction predicate for $V$, $G$ is definable over $(V, T, <)$ where $T$ is the $V$-amenable predicate $\{(\alpha, n) \mid \alpha \text{ is Ord, } n \text{-stable, } n \in \omega\}$ and $<$ is a wellorder of $V$ of length Ord. The same is true with $(L[S], S)$ replaced by $(M[S], S)$ for any definable inner model $M$.

The proof of Theorem 4 comes in two parts. First we show that $V$ can be written as $L[F]$ where $F$ is a function from the ordinals to 2 which preserves the Stability Predicate $S$ in the sense that whenever $\alpha$ is nicely $\beta, (n+1)$-stable then $\alpha$ is also nicely $\beta, (n+1)$-stable ”relative to $F$”. Then we use
this function $F$ to prove the genericity of $V$ over $(M[S], S)$ for any definable inner model $M$.

A stability-preserving function

Our aim is to produce a function $F$ from the ordinals to 2 which codes $V$ (i.e. which satisfies $V = L[F]$) and which preserves the Stability Predicate. The latter means that if $\alpha < \beta$ are strong limit cardinals, $\alpha$ is nicely $\beta, n$-stable then $\alpha$ is $\beta, n$-stable relative to $F$, i.e. $(\mathbb{H}(\alpha), F \upharpoonright \alpha)$ is $\Sigma_n$-elementary in $(\mathbb{H}(\beta), F \upharpoonright \beta)^8$. (As $\beta, (n - 1)$-stability implies $\beta, (n - 2)$-stability for $n > 1$, it then follows that $\alpha$ is nicely $\beta, n$-stable relative to $F$, i.e. that the $\beta, (n - 1)$-stables relative to $F$ are cofinal in $\beta$.)

Let $C$ denote the class of strong limit cardinals. We define by induction on $\beta \in C$ a collection $P(\beta)$ of functions from $\beta$ to 2.

If $\beta$ is not a limit point of $C$ then $P(\beta)$ consists of all functions $p : \beta \to 2$ such that $p \upharpoonright \alpha$ belongs to $P(\alpha)$ for all $\alpha \in C \cap \beta$. (Such functions exist, assuming that $P(\alpha)$ is nonempty for all $\alpha \in C \cap \beta$, a fact that we will verify later.)

If $\beta$ is a limit point of $C$ then let $P(< \beta)$ be the union of the $P(\alpha)$ for $\alpha$ in $C \cap \beta$, ordered by extension. Assuming extendibility for $P(< \beta)$, i.e. the statement that for $\alpha_0 < \alpha_1 < \beta$ in $C$, each $q_0$ in $P(\alpha_0)$ can be extended to some $q_1$ in $P(\alpha_1)$, this forcing adds a generic function with domain $\beta$, which we denote by $f : \beta \to 2$. For $n > 0$ we say that $p : \beta \to 2$ is $n$-generic for $P(< \beta)$ if $G(p) = \{p \upharpoonright \alpha \mid \alpha \in C \cap \beta\}$ meets every dense subset of $P(< \beta)$ of the form $\{q \in P(< \beta) \mid q \forces \forall x \varphi \text{ or } q \not\forces \varphi(\sigma) \text{ for some } P(< \beta)-\text{name } \sigma\}$, where $\varphi$ is a $\Sigma_{n-1}(\mathbb{H}(\beta), f)$ sentence with parameters from $H(\beta)$. We define $P(\beta)$ to consist of all $p : \beta \to 2$ which are $n$-generic for $P(< \beta)$ whenever $\mathbb{H}(\beta)$ is $n$-admissible, i.e. satisfies $\Sigma_n$-replacement.

Let $P$ be the union of all of the $P(\beta)$’s, ordered by extension.

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8Note that although $H(\alpha)$ is $\Sigma_1$-elementary in $V$ for uncountable cardinals $\alpha$, this may fail relative to $F$; for this reason we cannot assert that $\Sigma_n$-elementarity in $(\mathbb{H}(\beta), F \upharpoonright \beta)$ coincides with $\Sigma_{n+1}$-elementarity in $(H(\beta), F \upharpoonright \beta)$. 

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Proof. Extendibility implies that it is dense to code any set of ordinals into the $P$-generic function $F$, from which it follows that $V$ is contained in $L[F]$. As $F \upharpoonright \alpha$ belongs to $V$ for each $\alpha \in C$ it also follows that $L[F]$ is contained in $V$ and therefore $L[F]$ equals $V$.

To see that $F$ preserves the Stability Predicate, first note that the relation $q \models \varphi$ for $q$ in $P(< \beta)$ and $\Pi_1(\mathbb{H}(\beta), \dot{f})$ sentences $\gamma$ with parameters from $H(\beta)$ is $\Pi_1$ over $\mathbb{H}(\beta)$: $q \models \gamma$ iff for all $r \leq q$ and strong limit $\alpha \leq \operatorname{Dom}(r)$, $(\mathbb{H}(\alpha), r \upharpoonright \alpha) \vDash \gamma$. It then follows by induction on $n \geq 1$ that the relation $q \models \gamma$ for $q$ in $P(< \beta)$ and $\Pi_n(\mathbb{H}(\beta), \dot{f})$ sentences $\gamma$ with parameters from $H(\beta)$ is $\Pi_n$ over $\mathbb{H}(\beta)$.

Now suppose that $n > 0$ and $\alpha$ is nicely $\beta,n$-stable. As $\mathbb{H}(\alpha)$ is $n$-admissible, $F \upharpoonright \alpha$ is $n$-generic for $P(< \alpha)$. It follows that any $\Pi_n(\mathbb{H}(\alpha), \dot{f} \upharpoonright \alpha)$ sentence $\gamma$ with parameters from $H(\alpha)$ which is true in $(\mathbb{H}(\alpha), F \upharpoonright \alpha)$ is forced in $P(< \alpha)$ by some condition $F \upharpoonright \alpha_0, \alpha_0 < \alpha$. Then as $\alpha$ is $\beta,n$-stable, $F \upharpoonright \alpha_0$ also forces $\gamma$ in $P(< \beta)$. It follows that $F \upharpoonright \alpha_0$ forces $\gamma$ in $P(< \beta_0)$ for all $\beta,(n-1)$-stable $\beta_0$ greater than $\alpha$. If $n = 1$ then $\gamma$ holds in $(\mathbb{H}(\beta_0), F \upharpoonright \beta_0)$ for such $\beta_0$ (else $F \upharpoonright \alpha_0$ could not force $\gamma$ in $P(< \beta_0)$) and therefore such $\beta_0$’s are cofinal in $\beta$, $\gamma$ holds in $(\mathbb{H}(\beta), F \upharpoonright \beta)$. If $n > 1$ then such $\beta_0$ are $(n-1)$-admissible and therefore $F \upharpoonright \beta_0$ is $(n-1)$-generic. It follows that $\gamma$ holds in $(\mathbb{H}(\beta_0), F \upharpoonright \beta_0)$ for such $\beta_0$. As $\beta$ is a limit of $\beta,(n-1)$-stables and therefore of $\beta,(n-2)$-stables, we can apply induction to infer that $\mathbb{H}(\beta)$ is $\Sigma_{n-1}$-elementary in $\mathbb{H}(\beta)$ relative to $F$ for such $\beta_0$ and therefore $\gamma$ holds in $(\mathbb{H}(\beta), F)$. So we have shown that in all cases that $\gamma$ holds in $(\mathbb{H}(\beta), F)$, completing the proof that $F$ preserves the Stability Predicate.

To verify replacement relative to $F$, we need only observe that the above implies that for each $n$, if $\alpha$ is $\operatorname{Ord}$, $n$-stable (i.e., $\mathbb{H}(\alpha)$ is $\Sigma_n$ elementary in $(V, \bar{H})$ where $\bar{H} = \{ (\beta, H(\beta)) \mid \beta \text{ strong limit} \}$) then it remains so relative to $F$. $\square$

We now turn to extendibility for $P$. 

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**Lemma 5** Assume Extendibility for $P$. Suppose that $G$ is $P$-generic over $V$ (meeting $V$-definable dense classes) and let $F$ be the union of the functions in $G$. Then $V = L[F]$ and $F$ preserves the Stability Predicate. Moreover, $V$ satisfies replacement with $F$ as an additional predicate.
Lemma 6 Suppose that $\alpha < \beta$ belong to $C$ and $p$ belongs to $P(\alpha)$. Then $p$ has an extension $q$ in $P(\beta)$.

Proof. By induction on $\beta$. The statement is immediate by induction if $\beta$ is not a limit point of $C$.

Suppose that $\beta$ is a limit point of $C$ but $\mathbb{H}(\beta)$ is not 1-admissible. Then there is a closed unbounded subset $D$ of $C \cap \beta$ of ordertype less than $\beta$ whose intersection with each of its limit points $\gamma < \beta$ is $\Delta_1$ definable over $\mathbb{H}(\gamma)$. We can assume that both $\alpha$ and the ordertype of $D$ are less than the minimum of $D$. Now enumerate $D$ as $\beta_0 < \beta_1 < \cdots$ and using the induction hypothesis, successively extend $p$ to $q_0 \subseteq q_1 \subseteq \cdots$ with $q_i$ in $P(\beta_i)$, taking unions at limits. Note that for limit $i$, $q_i$ is indeed a condition because $\mathbb{H}(\beta_i)$ is not 1-admissible. The union of the $q_i$’s is the desired extension of $p$ in $P(\beta)$.

Next suppose that $\mathbb{H}(\beta)$ is $n$-admissible but not $(n+1)$-admissible for some finite $n > 0$:

If $\beta$ is a limit of $\beta, n$-stables then proceed as in the previous case: Choose a closed unbounded subset $D$ of $C \cap \beta$ of ordertype less than $\beta$, consisting of $\beta, n$-stables, whose intersection with each of its limit points $\gamma < \beta$ is $\Delta_{n+1}$ definable over $\mathbb{H}(\gamma)$. Assume that both $\alpha$ and the ordertype of $D$ are less than the minimum of $D$, enumerate $D$ as $\beta_0 < \beta_1 < \cdots$ and using the induction hypothesis, successively extend $p$ to $q_0 \subseteq q_1 \subseteq \cdots$ with $q_i$ in $P(\beta_i)$, taking unions at limits. For limit $i$, $q_i$ is indeed a condition because $\mathbb{H}(\beta_i)$ is $n$-admissible, $q_i$ is $n$-generic for $P(< \beta_i)$. The union of the $q_i$’s is the desired extension of $p$ in $P(\beta)$, as its $n$-genericity of this union follows from the $n$-genericity of the individual $q_i$’s.

If $\beta$ is not a limit of $\beta, n$-stables then $\beta$ must have cofinality $\omega$ (else by the $n$-admissibility of $\mathbb{H}(\beta)$, we could find cofinally many $\beta, n$-stables using the fact that $\beta$ has uncountable cofinality). If $\varphi = \forall x \psi$ with $\psi \Sigma_{n-1}(\mathbb{H}(\beta), \dot{f})$ is a $\Pi_n(\mathbb{H}(\beta), \dot{f})$ sentence and $q$ is a condition in $P(< \beta)$ we say that $q$ decides $\varphi$ if $q$ either forces $\varphi$ or forces $\sim \psi(\sigma)$ for some $P(< \beta)$-name $\sigma$. It suffices to show that any condition $q$ in $P(< \beta)$ can be extended to decide each of fewer than $\beta$-many $\Pi_n$ sentences with parameters from $H(\beta)$, as given this, we can extend $p$ in $\omega$ steps to a condition in $P(\beta)$ which is $n$-generic. To show this,
first note that the $n$-admissibility of $\mathbb{H}(\beta)$ implies that there are cofinally-many $\delta < \beta$ which are limits of $\beta, (n-1)$-stables. Now let $(\varphi_i \mid i < \delta)$, $\delta < \beta$ enumerate the given collection of fewer than $\beta$-many $\Pi_n$ sentences and let $D$ consist of all $\gamma < \beta$ which are limits of $\beta, (n-1)$-stables and large enough so that $H(\gamma)$ contains both $q$ and this enumeration. Extend $q$ successively to elements $q_i$ of $P(\gamma_i)$, where $\gamma_{i+1} \geq \gamma_i$ is the least element of $D$ so that either $q_i$ forces $\varphi_i$ in $P(\beta)$ or $q_{i+1}$ forces $\sim \psi(\sigma_i)$ in $P(\beta_{i+1})$ (where $\varphi = \forall x \psi_1$) for some $P(\beta_{i+1})$-name $\sigma_i$, taking unions at limits. For limit $i$, $H(\gamma_i)$ is not $n$-admissible as the set of $j < i$ such that $q_j$ forces $\varphi_j$ in $P(\beta)$ can be treated as a parameter in $H(\gamma_i)$. And for limit $i$, $\gamma_i$ is the limit of $\gamma_{i+1}, (n-1)$-stables. It follows that $q_i$ is a condition for limit $i$. As $\mathbb{H}(\beta)$ is $n$-admissible, this construction results in a sequence of $q_i$’s of length $\delta$, whose union it the desired extension of $q$ deciding all of the given $\Pi_n$ sentences.

Finally, suppose that $\mathbb{H}(\beta)$ is $n$-admissible for every finite $n$. Choose $D$ to be closed unbounded in $\beta$ so that any $\gamma < \beta$ which is a limit point of $D$ is a limit of $\gamma, n$-stables for every $n$. (Note that we may choose $D$ to be any cofinal $\omega$-sequence if $\beta$ has cofinality $\omega$.) Assume that $\alpha$ is less than the least element of $D$ and enumerate $D$ as $\beta_0 < \beta_1 < \cdots$. Then successively extend $p$ to $q_0 \subseteq q_1 \subseteq \cdots$ with $q_i$ in $P(\beta_i)$, taking unions at limits, and note that for limit $i$, $q_i$ is a condition because its $n$-genericity follows from the fact that $\beta_i$ is a limit of $\beta_i, n$-stables. This also applies to the union of the $q_i$’s, the desired extension $q$. □

**Lemma 7** (In Gödel-Bernays class theory) Suppose that $T = \{(n, \alpha) \mid \alpha$ is $\text{Ord}, n$-stable} exists (equivalently, there is a satisfaction predicate for $V$). Then there is a $P$-generic which is definable over $(V, T, <)$ where $<$ is a wellorder of $V$ of length $\text{Ord}$.

*Proof.* As in the last case of the proof of the previous lemma, let $D$ be closed unbounded in $\text{Ord}$ so that if $\alpha$ is a limit point of $D$ then $\alpha$ is a limit of $\alpha, n$-stables for every $n$. Such a class $D$ is definable over $(V, T)$. Then take $F : \text{Ord} \rightarrow 2$ to be the union of a sequence of conditions $p_0 \subseteq p_1 \subseteq \cdots$ where $p_{i+1}$ is the $<\text{-least}$ extension of $p_i$ in $P(\alpha_{i+1})$ and the $\alpha_i$’s form the increasing enumeration of $D$. Then $F$ is $n$-generic for $P(< \text{Ord}) = P$ for each $n$. □

**Corollary 8** (In Morse-Kelley class theory) There exists a function $F : \text{Ord} \rightarrow 2$ such that $V = L[F]$ and $F$ preserves the Stability Predicate.
$V$ is generic over the Stable Core

Now fix a function $F : \text{Ord} \to 2$ as in the last section, i.e. with the following properties:

2. If $n > 0$, $\alpha$ is $\beta, n$-stable and the $\beta, (n - 1)$-stables are cofinal in $\beta$, then $\alpha$ is $\beta, n$-stable relative to $F$.

We describe a forcing $Q$ definable over $(L[S], S)$ such that for some $Q$-generic $G$, $G$ is definable over $(L[F], F)$ and $F$ is definable over $(L[G], G)$. It follows that $L[G] = L[F] = V$.

The language $\mathcal{L}$ is defined inductively as follows, where $\hat{F}$ is a unary function symbol.

1. For each ordinal $\alpha$, ”$\hat{f}(\alpha) = 0$" and ”$\hat{f}(\alpha) = 1$” are sentences of $\mathcal{L}$.
2. If $\Phi$ is a set of sentences of $\mathcal{L}$ and $\Phi$ belongs to $L[S]$, then $\bigwedge \Phi$ and $\bigvee \Phi$ are sentences of $\mathcal{L}$.

A sentence $\varphi$ of $\mathcal{L}$ is valid if it is true when the symbol $\hat{f}$ is replaced by any function that belongs to a set-generic extension of $L[S]$. This notion is $L[S]$-definable and moreover if $\varphi$ is a sentence of $\mathcal{L}$ and $M$ is any outer model of $L[S]$, then $\varphi$ is valid in $L[S]$ iff it is valid in $M^9$.

Now let $T$ consist of all sentences of $\mathcal{L}$ of the form

$$\bigwedge (\Phi \cap H(\alpha)) \rightarrow \bigwedge (\Phi \cap H(\beta)),$$

where for some $\alpha < \beta$ and $n > 1$ we have:

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$^9$Indeed, if there is a function witnessing the non-validity of $\varphi$ in a set-generic extension of $M$ then we may assume that this generic extension is $M[G]$ where $G$ is generic for a Lévy collapse making $\varphi$ countable; then $L[S][G]$ also has a witness to the non-validity of $\varphi$, by Lévy absoluteness. Conversely, if the non-validity of $\varphi$ is witnessed in a set-generic extension of $L[S]$ then this will happen in $L[S][G]$ where $G$ is Lévy collapse generic over $L[S]$. Choose a condition in the Lévy collapse which forces this and choose $H$ containing this condition which is Lévy collapse generic over $M$; then the non-validity of $\varphi$ is witnessed in $M[H]$, a set-generic extension of $M$. 

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(a) $\Phi$ is $\Sigma_n$ definable over $H(\beta) \cap L[S]$ using parameters from $H(\alpha) \cap L[S]$.

(b) $\alpha$ is $\beta,n$-stable and the $\beta,(n-1)$-stables are cofinal in $\beta$ (in $V$).

Note that (a) implies that $\Phi$ is $\Sigma_n$ definable over $H(\beta)$ (using parameters from the $H(\alpha)$ of $V$). It follows that the sentences in $T$ are true when $\dot{f}$ is interpreted as $F$. Also note that $T$ is $(L[S],S)$ definable, as (b) is expressed by the Stability predicate $S$.

The desired forcing $Q$ consists of all sentences $\varphi$ of $\mathcal{L}$ which are consistent with $T$, in the sense that for no subset $T_0$ of $T$ is the sentence $\wedge T_0 \rightarrow \sim \varphi$ valid. The sentences in $Q$ are ordered by: $\varphi \leq \psi$ iff $T$ implies $\varphi \rightarrow \psi$.

Lemma 9 $Q$ has the Ord-chain condition, i.e., any $(L[S],S)$-definable maximal antichain in $Q$ is a set.

Proof. Suppose that $A$ is an $(L[S],S)$-definable maximal antichain and consider $\Phi = \{ \sim \varphi \mid \varphi \in A \}$. Then $\Phi$ is also $(L[S],S)$-definable. Choose $n$ so that $\Phi$ is $\Sigma_n$-definable over $(L[S],S)$ and choose $\alpha$ to be $n$-stable in Ord and large enough so that $H(\alpha) \cap L[S]$ contains the parameters in the $\Sigma_n$ definition of $\Phi$. Then $T$ together with $\Phi \cap H(\alpha)$ implies $\Phi \cap H(\beta)$ for all $\beta$ greater than $\alpha$ which are $n$-stable in Ord and since there are arbitrarily large such $\beta$, $T$ together with $\Phi \cap H(\alpha)$ implies all of $\Phi$. It follows that $A$ equals $A \cap H(\alpha)$: Otherwise let $\varphi$ belong to $A \setminus H(\alpha)$. As $\sim \varphi$ belongs to $\Phi$ it is implied by $T$ together with $\Phi \cap H(\alpha)$. But as $A$ is an antichain, $T$ together with $\varphi$ implies $\Phi \cap H(\alpha)$ and therefore $T$ together with $\varphi$ implies $\sim \varphi$, contradicting the fact that $\varphi$ belongs to $Q$. □

Now it is easy to see that $V = L[F] = L[G]$ for some $G$ which is $Q$-generic over $(L[S],S)$: Let $G$ consist of all sentences in $Q$ which are true when $\dot{f}$ is interpreted as $F$. It is obvious that $G$ intersects all maximal antichains of $Q$ which are sets in $L[S]$, as if the set $A$ is an antichain missed by $G$ then $\wedge \{ \sim \varphi \mid \varphi \in A \}$ is consistent with $T$ and witnesses the failure of $A$ to be maximal. By Lemma 9 this gives full genericity over $(L[S],S)$.

The above argument was carried out for the ground model $L[S]$. But the same argument can be used for any ground model $M[S]$ provided $M$ is a definable inner model; simply replace $n$ by $n - k - 1$ in (a) above, where $M$ is $\Sigma_k$-definable. This completes the proof of Theorem 4.
3. Strongly capturing the Stable Core

In this section we show that stability can be expressed in terms of strength in an iterate of Mighty Mouse, and therefore the Stable Core is definable over such an iterate.

We say that $\alpha$ is $\beta,0$-strong if $\alpha < \beta$ are each either measurable or the limit of measurables. Let $\text{Strong}_0(\beta)$ denote the set of $\alpha$ which are $\beta,0$-strong. The notions nicely $\beta,0$-strong and $\beta,0$-correct coincide with $\beta,0$-strong.

An extender $E$ with critical point less than a cardinal $\delta$ is $\delta,1$-strong if $H(\delta)$ is contained in its ultrapower. $\alpha$ is $\beta,1$-strong if $\alpha,0$-strong and for each cardinal $\delta$ between $\alpha$ and $\beta$, there is an extender with critical point $\alpha$ which is $\delta,1$-strong. $\alpha$ is nicely $\beta,1$-strong if in addition the $\beta,0$-strongs are cofinal in $\beta$. And $\alpha$ is $\beta,1$-correct if the $\alpha,1$-strongs are the $\beta,1$-strong less than $\alpha$. Let $\text{Strong}_1(\beta)$ denote the set of $\beta,1$-strongs.

For $n > 0$, an extender $E$ with critical point less than $\beta$ is $\beta,n+1$-strong if it is $\beta,n$-strong and $j_E(\text{Strong}_n(\beta)) \cap \beta = \text{Strong}_n(\beta)$. $\alpha$ is $\beta,n+1$-strong if it is $\beta,n$-strong and for each $\beta,n$-correct $\bar{\beta}$ greater than $\alpha$, there is an extender with critical point $\alpha$ which is $\beta,n+1$-strong. $\alpha$ is nicely $\beta,n+1$-strong if in addition the $\beta,n$-strongs are cofinal in $\beta$ and $\alpha$ is $\beta,n+1$-correct if it is $\beta,n$-correct and the $\alpha,n+1$-strongs are the $\beta,n+1$-strongs less than $\alpha$. Let $\text{Strong}_{n+1}(\beta)$ denote the set of $\beta,n+1$-strongs. We say that $\alpha$ is $\beta$-correct if it is $\beta,n$-correct for every $n$.

Our aim is to show that there is an iterate $\text{mm}^*$ of Mighty Mouse such that $\alpha$ is nicely $\beta,n$-stable (in $V$) iff $\alpha$ is either nicely $\beta,n$-strong or a limit of nicely $\beta,n$-strongs in $\text{mm}^*$. This is made possible by the fact that many of the basic properties of $\beta,n$-stability transfer to $\beta,n$-strength, as expressed in the next lemmas.

**Lemma 10** (Stability Lemma) (a) If $\alpha$ is $\beta,n+1$-stable then it is also nicely $\beta,n$-stable.
(b) If $\alpha$ is $\beta,n+1$-stable and $\beta$ is $\gamma,n+1$-stable then $\alpha$ is $\gamma,n+1$-stable.
(c) If $\alpha$ is $\gamma,n+1$-stable and $\beta$ is $\gamma,n$-stable with $\alpha < \beta$, then $\alpha$ is $\beta,n+1$-stable.
(d) If $\alpha$ is $\beta,n+1$-stable then $\alpha$ is a limit of $\alpha,n$-stables.
(e) If $\gamma$ is a limit of $\gamma, n$-stables and $\alpha$ is $\beta, n + 1$-stable whenever $\beta$ is $\gamma, n$-stable and greater than $\alpha$ then $\alpha$ is $\gamma, n + 1$-stable.

The proof of the above lemma is straightforward.

The analogous statement for $\beta, n$-strength is as follows:

**Lemma 11** (Strength Lemma) (a) If $\alpha$ is $\beta, n + 1$-strong then it is also $\beta, n$-strong.

(b) If $\alpha$ is $\beta, n + 1$-strong, $\beta$ is $\gamma, n + 1$-strong and (if $n > 0$) there is a $\gamma, n$-correct greater than $\beta$ then $\alpha$ is $\gamma, n + 1$-strong.

(c) If $\alpha$ is $\gamma, n + 1$-strong, $\beta$ is $\gamma, n$-strong, $\alpha < \beta$ and (if $n > 0$) there is a $\gamma, n - 1$-correct greater than $\beta$ then $\alpha$ is $\beta, n + 1$-strong.

(d) If $\alpha$ is $\beta, n + 1$-strong and (if $n > 0$) there is a $\beta, n$-correct greater than $\alpha$ then $\alpha$ is a limit of $\alpha, n$-strengths.

(e) If $\gamma$ is a limit of $\gamma, n$-corrects and $\alpha$ is $\beta, n + 1$-strong whenever $\beta$ is $\gamma, n$-correct and greater than $\alpha$ then $\alpha$ is $\gamma, n + 1$-strong.

(f) If $\beta$ is the limit of $\beta, 0$-strengths then $\beta$ is the limit of $\beta, n + 1$-corrects. Moreover, if $\beta$ is the limit of $\beta, n$-strengths then $\beta$ is the limit of $\beta, n + 1$-corrects which are in addition $\beta, n$-strong.

**Proof.** By induction on $n$.

First we treat the base case $n = 0$.

(a) This is immediate by the definition of $\beta, 1$-strength.

(b) Suppose that $\alpha$ is less than $\check{\gamma}$ and $\check{\gamma}$ is a cardinal less than $\gamma$; we want to show that there is a $\delta$ an extender with critical point $\alpha$ which is $\check{\gamma}, 1$-strong, i.e. which has $H(\check{\gamma})$ in its ultrapower. If $\check{\gamma}$ is less than $\beta$ then since $\alpha$ is $\beta, 1$-strong we get the desired extender.

If $\check{\gamma}$ equals $\beta$ then let $\delta$ be a cardinal between $\beta$ and $\gamma$; as $\beta$ is $\gamma, 1$-strong we can choose an extender $E$ with critical point $\beta$ which is $\delta, 1$-strong. By elementarity, $\alpha$ is $j_E(\beta)$, $1$-strong in $M_E$. In $M_E$ pick any cardinal $\beta^*$ between $\delta$ and $j_E(\beta)$. As $\alpha$ is $j_E(\beta)$, $1$-strong in $M_E$, there is an extender with critical point $\alpha$ which is $\beta^*, 1$-strong in $M_E$. This extender is also $\beta, 1$-strong.

Finally, suppose that $\check{\gamma}$ is greater than $\beta$. As $\beta$ is $\gamma, 1$-strong there is an extender $E$ with critical point $\beta$ which is $\check{\gamma}, 1$-strong. $\alpha$ is $j_E(\beta)$, $1$-strong.
Choose any $M_E$-cardinal $\beta^*$ between $\gamma$ and $j_E(\beta)$. Then there is an extender with critical point $\alpha$ which is $\beta^*, 1$-strong. This extender is also $\bar{\gamma}, 1$-strong.

(c) Suppose that $\bar{\beta}$ is a cardinal less than $\beta$ and greater than $\alpha$. Then $\bar{\beta}$ is also less than $\gamma$. As $\alpha$ is $\gamma, 1$-strong there is a $\bar{\beta}, 1$-strong extender with critical point $\alpha$.

(d) The hypothesis implies that there is $j : V \rightarrow M$ with critical point $\alpha$ with $\alpha$ measurable in $M$. If the $\alpha, 0$-strongs were bounded in $\alpha$ then in $M$, $\alpha$ could not be $j(\alpha), 0$-strong. But $\alpha$ is measurable in $M$.

(e) Suppose that $\bar{\gamma}$ is a cardinal less than $\gamma$ and greater than $\alpha$. Choose $\beta$ greater than $\bar{\gamma}$ which is $\gamma, 0$-correct ($= \gamma, 0$-strong). As $\alpha$ is $\beta, 1$-strong there is an extender with critical point $\alpha$ which is $\bar{\gamma}, 1$-strong.

(f) Suppose that $\gamma$ is $\beta, 0$-strong and not $\beta, 1$-correct. By (c) for $n = 0$, any $\beta, 1$-strong less than $\gamma$ is $\gamma, 1$-strong. So there must be some least $\alpha$ which is $\gamma, 1$-strong but not $\beta, 1$-strong. As $\beta$ is the limit of $\beta, 0$-strongs we may choose a $\beta, 0$-strong $\gamma^*$ greater than $\gamma$ so that $\alpha$ is not $\gamma^*, 1$-strong. If $\gamma^*$ is the least such, then $\gamma^*$ is $\beta, 1$-correct: If not, choose $\alpha^*$ which is $\gamma^*, 1$-strong but not $\beta, 1$-strong. Then $\alpha^*$ is not less than $\alpha$, as then by the leastness of $\alpha$, $\alpha^*$ would be $\beta, 1$-strong. And $\alpha^*$ cannot equal $\alpha$ since $\alpha$ is not $\gamma^*, 1$-strong. So $\alpha^*$ is greater than $\alpha$ and by the leastness of $\gamma^*$, $\alpha$ is $\alpha^*, 1$-strong. But then by (b) for $n = 0$, $\alpha$ is $\gamma^*, 1$-strong, contradiction.

Also note that the “Moreover” clause of (f) is immediate for the case $n = 0$.

Now suppose that $n > 0$ and the result holds for $n - 1$.

(a) Again by definition of $\beta, n + 1$-strength.

(b) Suppose that $\alpha < \bar{\gamma}$ and $\bar{\gamma}$ is $\gamma, n$-correct. If $\bar{\gamma}$ is less than $\beta$ then by (b) for $n - 1$, $\beta$ is $\gamma, n$-correct and therefore $\bar{\gamma}$ is $\beta, n$-correct. As $\alpha$ is $\beta, n + 1$-strong we get the desired $\bar{\gamma}, n + 1$-strong extender with critical point $\alpha$.

If $\bar{\gamma}$ equals $\beta$ then let $\delta$ be $\gamma, n$-correct and greater than $\beta$; as $\beta$ is $\gamma, n + 1$-strong we can choose an extender $E$ with critical point $\beta$ which is $\delta, n + 1$-strong. Then $\alpha$ is $j_E(\beta), n + 1$-strong in $M_E$ and the $\beta, n$-strongs are the $j_E(\beta), n$-strongs in $M_E$ less than $\beta$. It follows that $\beta$ is $j_E(\beta), n$-correct in
and therefore there is an extender with critical point \( \alpha \) which is \( \beta, n + 1 \)-strong.

Finally, suppose that \( \bar{\gamma} \) is greater than \( \beta \). As \( \beta \) is \( \gamma, n + 1 \)-strong it is also \( \gamma, n \)-strong and therefore by (b) for \( n - 1 \) it is \( \gamma, n \)-correct and therefore \( \bar{\gamma}, n \)-correct. As \( \beta \) is \( \gamma, n + 1 \)-strong there is an extender \( E \) with critical point \( \beta \) which is \( \bar{\gamma}, n + 1 \)-strong. Thus in \( M_E \), \( \bar{\gamma} \) is \( j_E(\bar{\gamma}), n \)-correct. Also by elementarity, \( j_E(\beta) \) is \( j_E(\bar{\gamma}), n \)-correct so we have that \( \bar{\gamma} \) is \( j_E(\beta), n \)-correct in \( M_E \). Also by elementarity, \( \alpha \) is \( j_E(\beta), n + 1 \)-strong in \( M_E \) and therefore there is an extender with critical point \( \alpha \) which is \( \bar{\gamma}, n + 1 \)-strong.

So the \( \beta^*, n \)-strongs are exactly the \( j_E(\beta), n \)-strongs which

(c) By (b) and (c) for \( n - 1 \), \( \beta \) is \( \gamma, n \)-correct. It follows that the \( \beta, n \)-corrects are the \( \gamma, n \)-corrects less than \( \beta \). Therefore as \( \alpha \) is \( \gamma, n + 1 \)-strong, it is also \( \beta, n + 1 \)-strong.

(d) Choose a \( \beta, n \)-correct \( \bar{\beta} \) which is greater than \( \alpha \). As \( \alpha \) is \( \beta, n \)-strong, it is also \( \bar{\beta}, n \)-strong and by (b), (c) for \( n - 1 \) it is \( \beta, n \)-correct and therefore \( \bar{\beta}, n \)-correct. As \( \alpha \) is \( \beta, n + 1 \)-strong there is an extender \( E \) with critical point \( \alpha \) which is \( \bar{\beta}, n + 1 \)-strong. So \( \alpha \) is also \( j_E(\bar{\beta}), n \)-strong. If \( \alpha \) were not the limit of \( \alpha, n \)-strongs then \( \alpha \) could not be \( j_E(\alpha), n \)-strong in \( M_E \), contradicting the fact that it is \( j_E(\bar{\beta}), n \)-strong and \( j_E(\alpha) \) is \( j_E(\bar{\beta}), n \)-correct.

(e) Let \( \bar{\gamma} \) be \( \gamma, n \)-correct and greater than \( \alpha \). Choose \( \beta \) greater than \( \bar{\gamma} \) which is \( \gamma, n \)-correct. Then \( \bar{\gamma} \) is also \( \beta, n \)-correct. As \( \alpha \) is \( \beta, n + 1 \)-strong, there is an extender with critical point \( \alpha \) which is \( \bar{\gamma}, n + 1 \)-strong.

(f) Suppose that \( \gamma \) is \( \beta, 0 \)-strong but not \( \beta, n + 1 \)-correct. By induction we may increase \( \gamma \) to ensure that it is \( \beta, n \)-correct. Let \( \alpha \) be the least \( \gamma, n + 1 \)-strong that is not \( \beta, n + 1 \)-strong. Then \( \alpha \) is not \( \gamma_0, n + 1 \)-strong for some \( \beta, n \)-correct \( \gamma_0 \). If \( \gamma_0 \) is the least such then \( \gamma_0 \) is a successor \( \beta, 0 \)-strong, else by induction it would be a limit of \( \gamma_0, n \)-corrects, contradicting its leastness. Let \( \gamma_1 \) be the least \( \beta, n \)-correct greater than \( \gamma_0 \) (or the least \( \beta, n \)-strong greater than \( \gamma_0 \), if \( \beta \) is the limit of \( \beta, n \)-strongs). Then \( \gamma_1 \) must be \( \beta, n + 1 \)-correct: If not let \( \alpha^* \) be \( \gamma_1, n + 1 \)-strong but not \( \beta, n + 1 \)-strong. Then \( \alpha^* \) is not less than \( \alpha \), as \( \gamma, n + 1 \)-strongs less than \( \alpha \) are \( \beta, n + 1 \)-strong and therefore \( \gamma_1, n + 1 \)-strong, and \( \alpha^* \) is not equal to \( \alpha \) as \( \alpha \) is not \( \gamma_0, n + 1 \)-strong and therefore not \( \gamma_1, n + 1 \)-strong. Also \( \alpha^* \) is not equal to \( \gamma_0 \) as \( \gamma_0 \) is not the limit of \( \gamma_0, 0 \)-strongs and therefore cannot be \( \gamma_1, n + 1 \)-strong by (d) for \( n = 0 \). So \( \alpha \)
is $\alpha^*, n + 1$-strong by the leastness of $\gamma_0$ and as $\alpha^*$ is $\gamma_1, n + 1$-strong and $\gamma_0$ is $\gamma_1, n$-correct, we can apply (b) for $n + 1$ to conclude that $\alpha$ is $\gamma_1, n + 1$-strong, contradiction. □

It is worth noting that in Lemma 11, (f) implies that the extra assumptions (for $n > 0$) about the existence of a $\gamma$-correct (in (b) and (c)) and of a $\beta$-correct (in (d)) can be replaced by the assumptions that $\gamma$ is a limit of strong limits in (b) and (c) and that $\beta$ is a limit of strong limits in (d).

Also we point out an important dissimilarity between stability and strength: If $\alpha$ is $\beta, n + 2$-stable then the $\beta, n$-stables are cofinal in $\beta$. The analogous statement fails for strength. Indeed with such a property we could not express the order of the iteration tree in terms of (strength-) correctness. However there is a variant of stability called pseudostability which we introduce later in this section and which serves as a better analogue of strength.

We also have the following “limit version” of Lemma 11 (f).

Lemma 12 If $\beta$ is the limit of $\beta, 0$-strongs then $\beta$ is the limit of $\beta$-corrects. Moreover, if $\beta$ is the limit of $\beta, n$-strongs then $\beta$ is the limit of $\beta$-corrects which are in addition $\beta, n$-strong.

Proof. For any $\gamma < \beta$ let $\gamma^*_m$ be the least $\beta, m$-correct greater than or equal to $\gamma$ (which is also $\beta, n$-strong if $\beta$ is the limit of $\beta, n$-strongs). If none of the $\gamma^*_m$’s is $\beta$-correct then for each $m$ let $\alpha_m$ be least so that for some $k$, $\alpha_m$ is $\gamma^*_m, k$-strong but not $\beta, k$-strong. Then $\alpha_{m+1}$ is at most $\alpha_m$ for each $m$ so for some $m$, $\alpha_m = \alpha_l$ for all $l > m$. But then $\gamma^*_m$ must be $\beta, k$-correct for each $k$. So we have found a $\beta$-correct greater than $\gamma$ (which is also $\beta, n$-strong if $\beta$ is the limit of $\beta, n$-strongs), as desired. □

The iteration

We define an iteration $((m_\gamma, E_\gamma) \mid \gamma \in \text{Ord})$ of $\text{mm} = \text{Mighty Mouse}$ where $E_\gamma$ is a total extender of $m_\gamma$. We set $m_0 = \text{mm}$ and obtain $m_{\gamma+1} = \text{Ult}(m_\tilde{\gamma}, E_\tilde{\gamma})$ by applying $E_\gamma$ to $m_\tilde{\gamma}$ where $\tilde{\gamma}$ is least so that this ultrapower makes sense (i.e., so that $m_{\tilde{\gamma}}$ and $m_\gamma$ have the same subsets of the critical point of $E_\gamma$), thereby forming an iteration tree. At limit stages $\gamma$, $m_\gamma$ is the direct limit of the $m_\tilde{\gamma}$’s along the unique cofinal branch through the
But before specifying the $E_\gamma$’s we first have to extend and vary the notion of $\beta,n$-stability.

Recall that for strong limits $\alpha < \beta$, $\alpha$ is $\beta,1$-stable if $\mathbb{H}(\alpha)$ is $\Sigma_1$-elementary in $\mathbb{H}(\beta)$. A beth-number is a cardinal of the form $\beth_\alpha$ for some ordinal $\alpha$ where $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$ and $\beth_\lambda = \bigcup_{\alpha<\lambda} \beth_\alpha$ for limit $\lambda$. The uncountable strong limits are the limits of beth-numbers. Now for an arbitrary beth-number $\alpha$ we define $\mathbb{H}(\alpha)$ to be $(H(\alpha), H||\alpha)$ where $H||\alpha$ denotes the set of pairs $(\bar{\alpha}, H(\bar{\alpha}))$ such that $H(\bar{\alpha})$ belongs to $H(\alpha)$. Then $\mathbb{H}(\alpha)$ is $\Sigma_0$-elementary in $\mathbb{H}(\beta)$ for beth-numbers $\beta$ greater than $\alpha$. We say that $\alpha$ is $\beta,1$-stable if $\mathbb{H}(\alpha)$ is $\Sigma_1$-elementary in $\mathbb{H}(\beta)$, extending this notion to arbitrary beth-numbers. It implies that $\alpha$ is strong limit, but can hold if $\beta$ is a successor beth-number.

Regarding $\beta,n+1$-stability for $n > 0$ we introduce pseudostability. Recall that if the $\beta,1$-stables are cofinal in $\beta$ then $\alpha$ is $\beta,2$-stable iff $(\mathbb{H}(\alpha), \text{Stable}_{\leq 1}(\alpha))$ is $\Sigma_1$-elementary in $(\mathbb{H}(\beta), \text{Stable}_{\leq 1}(\beta))$, where $\text{Stable}_{\leq 1}(\alpha)$ refers to the pair of predicates $\text{Stable}_0(\alpha), \text{Stable}_1(\alpha)$. If we drop the assumption that the $\beta,1$-stables are cofinal in $\beta$ then we refer to this notion as $\beta,2$-pseudostability. More generally, if $\alpha < \beta$ are strong limits we say that $\alpha$ is $\beta,n+1$-pseudostable if $(\mathbb{H}(\alpha), \text{PStable}_{\leq n}(\alpha))$ is $\Sigma_1$-elementary in $(\mathbb{H}(\beta), \text{PStable}_{\leq n}(\beta))$, where $\text{PStable}_{\leq n}(\alpha)$ refers to the sequence of predicates $\text{PStable}_0(\alpha), \text{PStable}_1(\alpha), \ldots, \text{PStable}_n(\alpha)$ and $\text{PStable}_i(\alpha)$ is the set of $\alpha,i$-pseudostables. (For $i = 0, 1$, $\text{PStable}_i(\alpha) = \text{Stable}_i(\alpha)$.)

The reason for extending the notion of 1-stability to beth-numbers is that when we apply an extender in our iteration to reduce the strength of some $\alpha$ which is not $\beta,1$-stable, $\beta$ a limit of strong limits, we have to be careful to choose the length of that extender not to be a strong limit, else we will destroy the measurability of that strong limit. Similarly, we introduce $n+1$-pseudostability for $n > 0$ as when we need to apply an extender to reduce the $n+1$-strength of some $\alpha$ which is not $\beta,n+1$-stable, $\beta$ a limit of $\beta,n$-stables, we have to be careful to choose the length of that extender not to be $\beta,n$-stable, else we will destroy the $\beta,n$-strength of that $\beta,n$-stable.
Now we are prepared to define the $E\gamma$'s. Note that as the $E\gamma$'s will be chosen to be total extenders, each $m\gamma$ has a largest measurable $\kappa\gamma$ which carries exactly one normal measure in $m\gamma$. We say that $\beta < \kappa\gamma$ is *worrisome* for $m\gamma$ if at least one of the following holds.

1. $\beta$ is measurable but not strong limit.

2. $\beta$ is a successor beth-number and some $\alpha$ less than $\beta$ is either not $\beta$, 1-stable or not $\kappa\beta$, 1-strong in $m\beta$, yet there is a $\beta$, 1-strong extender in $m\gamma$ with critical point $\alpha$.

3. $\beta$ is strong limit and for some $(n, \alpha)$ with $n > 0$, the $\beta,n - 1$-stables are unbounded in $\beta$, the $\beta,n$-stables are bounded in $\beta$, $\alpha$ is either not $\beta,n + 1$-pseudostable or not $\kappa\beta,n + 1$-strong in $m\beta$, yet there is a $\beta,n + 1$-strong extender in $m\gamma$ with critical point $\alpha$.

If there is a worrisome $\beta$ for $m\gamma$ then we let $\beta\gamma$ be the least such $\beta$. We note that cases 1 and 2 of worrisome are not mutually exclusive. If 2 holds we let $\alpha\gamma$ be the least $\alpha$ witnessing 2 and $E\gamma$ the extender of least index witnessing 2 for $\alpha\gamma$. If 1 holds but 2 does not, then we take $E\gamma$ to be the unique normal measure on $\beta\gamma$. If 3 holds then as in 2 we let $\alpha\gamma$ be the least $\alpha$ witnessing 3 and take $E\gamma$ to be the extender of least index witnessing 3 for $\alpha\gamma$.

If $m\gamma$ is worry-free (i.e. no $\beta < \kappa\gamma$ is worrisome for $m\gamma$) then we set $\beta\gamma = \kappa\gamma$, the largest measurable of $m\gamma$, and take $E\gamma$ to be the unique normal measure on $\beta\gamma$ in $m\gamma$.

This completes the definition of the iteration.

*Remark.* In cases 2 and 3 of worrisome we consider $\kappa\beta, 1$-strength and $\kappa\beta, n + 1$-strength, respectively. This is needed for the argument we give later than successor $\beta,n + 1$-stables are $\beta,n + 1$-strong.

**Lemma 13 (Increasing Indices)** The $\beta\gamma$'s are strictly increasing.

*Proof.* If $\gamma = \bar{\gamma} + 1$ then as $m\bar{\gamma}$ contains all bounded subsets of $\beta\bar{\gamma}$ in $m\gamma$, it follows that $\beta\gamma$ cannot be less than $\beta\bar{\gamma}$ by the leastness of $\beta\bar{\gamma}$. If $\beta\bar{\gamma}$ is worrisome for reason 2 or 3 in $m\bar{\gamma}$ then by the leastness of the index $i\bar{\gamma}$ of
$E_{\bar{\gamma}}, \beta_{\bar{\gamma}}$ is not worrisome for reason 2 or 3 in $m_{\gamma}$; it is also not worrisome for reason 1 in $m_{\gamma}$ because again by the leastness of $i_{\bar{\gamma}}, \beta_{\bar{\gamma}}$ is not measurable in $m_{\gamma}$. If $\beta_{\bar{\gamma}}$ is worrisome for reason 1 but not for reason 2 in $m_{\gamma}$, then as $m_{\gamma}$ contains all subsets of $\beta_{\bar{\gamma}}$ in $m_{\gamma}, \beta_{\bar{\gamma}}$ is not worrisome for reason 2 in $m_{\gamma}$ and as $\beta_{\bar{\gamma}}$ is not measurable in $m_{\gamma}$, it is not worrisome for reason 1 in $m_{\gamma}$ either. So in all cases, $\beta_{\gamma}$ is greater than $\beta_{\bar{\gamma}}$.

For limit $\gamma$, note that the supremum of the indices of the $E_{\gamma}$ for $\bar{\gamma}$ below $\gamma$ in the iteration tree is the same as the supremum of the $\beta_{\bar{\gamma}}$ for such $\bar{\gamma}$. If $\beta_{\gamma}$ were less than this supremum then it would be less than $\beta_{\bar{\gamma}}$ for some $\bar{\gamma}$ below $\gamma$ in the iteration tree, and as all subsets of $\beta_{\bar{\gamma}}$ in $m_{\gamma}$ belong to $m_{\bar{\gamma}}$, this contradicts the leastness of the index of $E_{\gamma}$. \hfill \Box

As $i_{\gamma} = \text{the index of } E_{\gamma}$ in $m_{\gamma}$ lies between $\beta_{\gamma}$ and $\beta_{\gamma+1}$, it follows that the $i_{\gamma}$’s are also strictly increasing.

Now let $m^*_0$ be the union of the $m_{\gamma}|i_{\gamma}, \gamma \in \text{Ord}$. As $m_{\gamma_0}$ agrees with $m_{\gamma_1}$ below $i_{\gamma_0}$ for $\gamma_0 < \gamma_1$, it follows that $m^*_0$ is a weasel (i.e. class-sized mouse). As the $\beta_{\gamma}$’s are strictly increasing, they converge to Ord and therefore:

**Corollary 14** $m^*_0$ is worry-free.

Next we aim to show that nice $n$-strength in $m^*_0$ matches nice $n$-stability in $V$ in the sense advertised earlier. First we treat the case $n = 0$.

For any property $P$, when we say that $\alpha$ is a $P$-successor we mean that $\alpha$ satisfies $P$ and is not the limit of ordinals satisfying $P$ (thus the least $\alpha$ satisfying $P$ qualifies as a $P$-successor).

**Lemma 15** If $\alpha$ is measurable in $m^*_0$ then $\alpha$ is strong limit. Conversely, if $\alpha$ is a successor strong limit then $\alpha$ is measurable in $m^*_0$.

**Proof.** The first statement follows from the fact that $m^*_0$ is worry-free. For the converse, let $\bar{\alpha}$ be the largest strong limit less than $\alpha$ ($\bar{\alpha} = 0$ if $\alpha$ is the least strong limit). Then $E_{\bar{\alpha}+1}$ is the measure on the least measurable of $m_{\bar{\alpha}+1}$ greater than $\bar{\alpha}$. It follows that for the next $\alpha$-many stages of the iteration, this measure is iterated and $\alpha$ is the least measurable greater than $\bar{\alpha}$ in $m_{\alpha}$. As $\alpha$ is strong limit, it is not worrisome for reasons 1 or 2 in $m_{\alpha}$; as $\alpha$ is not the limit of strong limits it is not worrisome for reason 3 in $m_{\alpha}$. Therefore $\alpha$ remains measurable in $m^*_0$. \hfill \Box

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Corollary 16  (Measurability and Strong Limits) $\alpha$ is strong limit iff $\alpha$ is measurable or the limit of measurables in $m_0^\ast$.

Recall once again that for any $\gamma$, $\kappa_\gamma$ denotes the largest measurable of $m_\gamma$. To equate higher levels of stability with strength we need to consider $\kappa_\gamma, n+1$-strength in $m_\gamma$ and characterise the unique cofinal branches through limit levels of the iteration tree.

Lemma 17  Suppose that $\beta$ is the limit of $\beta, n$-stables.
(a) If $\alpha$ is $\beta, n+1$-strong then $\alpha$ is $\beta, n+1$-stable.
(b) If $b$ is a branch cofinal in the iteration tree below $\beta$, $\gamma$ belongs to $b$ and $E_\gamma$ is $\gamma, n + 2$-strong then the critical point of $E_\gamma$ is $\beta, n + 1$-stable.
(c) Let $b$ be a branch cofinal in the iteration tree below $\beta$ and let $\kappa_b$ be the largest measurable of $m_b$, the direct limit of the $m_\gamma$, $\gamma$ in $b$. Then $\beta$ is $\kappa_b, n+1$-correct in $m_b$.
(d) There is a unique branch $b_\beta$ cofinal through the iteration tree below $\beta$ and $b_\beta$ contains all sufficiently large $\beta$-corrects.
(e) If $\alpha$ is a successor $\beta, n + 1$-stable then $\alpha$ is $\beta, n + 1$-strong.

Proof.  By induction on $\beta$ and for fixed $\beta$ by induction on $n$.

(a) If $\alpha$ is $\beta, n + 1$-strong but not $\beta, n + 1$-stable then we can choose a successor $\beta, n$-stable $\bar{\beta}$ between $\alpha$ and $\beta$ so that $\alpha$ is not $\bar{\beta}, n+1$-pseudostable. By induction $\bar{\beta}$ is $\beta, n$-strong and therefore $\beta, n$-correct, so as $\alpha$ is $\beta, n + 1$-strong there is an extender with critical point $\alpha$ which is $\bar{\beta}, n + 1$-strong. This contradicts the fact that $m_0^\ast$ is worry-free.

(b) Suppose that the extender $E = E_\gamma$ is $\gamma, n + 2$-strong, where $\gamma$ belongs to $b$. Let $\mu$ be the critical point of $E$. Then $\mu$ is $\pi(\gamma), n + 1$-strong in $\text{Ult}(m_\gamma, E_\gamma)$ where $\pi$ is the ultrapower map, and since $\mu$ is $\gamma, n + 1$-strong (even $\gamma, n + 2$-strong), $\mu$ is also $\pi(\mu), n + 1$-strong in $\text{Ult}(m_\gamma, E_\gamma)$ and therefore in $m_{\gamma+1}$. By induction applied to (c), $\mu$ is also $\kappa_{\gamma+1}, n + 1$-strong in $m_{\gamma+1}$ and therefore using the embedding from $m_{\gamma+1}$ to $m_\beta$, $\mu$ is $\kappa_b, n + 1$-strong in $m_b$ and therefore by induction on $n$, $\beta, n + 1$-strong, and by (a) is $\beta, n + 1$-stable.

(c) Suppose that $\alpha$ is $\beta, n + 1$-strong. On a final segment of $b$, no $E_\gamma$ is applied which is $\gamma, n + 2$-strong, as by (b) the critical point of such an extender must be $\beta, n + 1$-strong. Thus $\alpha$ is $\kappa_\gamma, n + 1$-strong for sufficiently large $\gamma$ in $b$. 24
else by 2 and 3 of the definition of worrisome, \( \alpha \) could not be \( \beta, n + 1 \)-strong. By choosing such an \( \gamma \) with the critical point of the embedding from \( m_{\gamma} \) to \( m_b \) greater than \( \alpha \), we see that \( \alpha \) is \( \kappa_b, n + 1 \)-strong in \( m_b \).

(d) Suppose that \( b \) is a branch cofinal in the tree below \( \beta \). We show that all sufficiently large \( \beta \)-corrects belong to \( b \).

Suppose that \( \alpha \) is \( \beta \)-correct. If \( \alpha \) does not belong to \( b \), then choose \( \gamma_0 < \gamma_1 + 1 \) in \( b \) so that \( \gamma_0 < \alpha < \gamma_1 + 1 \). Assume that \( \alpha \) has been chosen large enough so that \( \gamma_0 \) is not \( \beta, n + 1 \)-stable. By (b) this implies that the largest \( m \) so that \( E_{\gamma_1} \) is \( \gamma_1, m + 1 \)-strong is at most \( n \).

Then \( \gamma_0 \) is not \( \gamma_1, m + 1 \)-strong in \( \text{Ult}(m_{\gamma_1}, E_{\gamma_1}) \) and therefore not \( \gamma_1, m + 1 \)-strong in \( m_{\gamma_1 + 1} \) or in \( m_0^* \). We argue that \( \gamma_1 \) is \( \beta, m \)-correct: If \( \delta < \gamma_1 \) is \( \gamma_1, m \)-strong then \( \delta \) is \( \pi_1(\gamma_1), m \)-strong in \( \text{Ult}(m_{\gamma_1}, E_{\gamma_1}) \) where \( \pi_1 \) is the ultrapower map and as \( \gamma_0 \) is \( \gamma_1, m \)-strong (even \( \gamma_1, m + 1 \)-strong) in \( m_{\gamma_1} \), \( \pi_1(\gamma_0) \) is \( \pi_1(\gamma_1), m \)-strong in \( \text{Ult}(m_{\gamma_1}, E_{\gamma_1}) \) and therefore \( \delta \) is \( \pi_1(\gamma_0), m \)-strong in \( m_{\gamma_1 + 1} \). By induction it follows that \( \delta \) is \( \kappa_{\gamma_1 + 1}, m \)-strong in \( m_{\gamma_1 + 1} \) and hence \( \kappa_b, m \)-strong in \( m_b \). As \( m \) is at most \( n \), \( \delta \) is \( \beta, m \)-strong in \( m_0^* \), proving that \( \gamma_1 \) is \( \beta, m \)-correct. In particular \( \gamma_0 \) is not \( \beta, m + 1 \)-strong and as \( \alpha \) is \( \beta \)-correct, \( \alpha \) is \( \gamma_1, m \)-correct. Therefore \( \gamma_0 \), which is \( \gamma_1, m + 1 \)-strong in \( m_{\gamma_1} \), is \( \alpha, m + 1 \)-strong. This contradicts the \( \beta \)-correctness of \( \alpha \).

So any cofinal branch contains all sufficiently large \( \beta \)-corrects and the \( \beta \)-corrects are cofinal in \( \beta \), it follows that there is a unique cofinal branch through the iteration tree below \( \beta \).

(e) Suppose that \( \alpha \) is a successor \( \beta, n + 1 \)-stable. Let \( \bar{\alpha} \) be \( \alpha \)-correct and greater than the largest \( \alpha, n + 1 \)-stable (if there are \( \alpha, n + 1 \)-stables). By (d) for \( \alpha, \bar{\alpha} \) can be chosen to lie on the branch \( b_\alpha \) cofinal in \( \alpha \), which is, as the \( \alpha, n + 1 \)-stables are bounded in \( \alpha \), \( \Sigma_{n+1} \) definable over \( \mathbb{H}(\alpha) \). Let \( \bar{\kappa} \) be the least \( \kappa_{\bar{\alpha}}, n + 1 \)-strong in \( m_{\bar{\alpha}} \) greater than or equal to \( \bar{\alpha} \). Then it follows from the \( \beta, n + 1 \)-stability of \( \alpha \), the definability of \( b_\alpha \) and 3 from the definition of worrisome that \( \bar{\kappa} \) is sent to \( \alpha \) by the iteration map \( \pi_{\bar{\alpha}, \alpha} \). Thus by elementarity, \( \alpha \) is \( \kappa_\alpha, n + 1 \)-strong in \( m_\alpha \).

If \( \alpha \) lies below \( \beta \) on the iteration tree and the map \( \pi \) from \( m_\alpha \) to \( m_\beta \) has critical point greater than \( \alpha \), then by elementarity, \( \alpha \) is \( \kappa_\beta, n + 1 \)-strong in \( m_\beta \) and therefore \( \beta, n + 1 \)-strong, as desired. If \( \pi \) had critical point less than \( \alpha \), then since \( \alpha \) is not a limit of \( \alpha, n + 1 \)-stables, \( \alpha \) would not be \( \pi(\alpha), n + 1 \)-strong in \( m_\beta \), but this is contradicted using restrictions of the extender derived from
Now suppose that $\alpha$ is not below $\beta$ in the iteration tree and let $\gamma_0 < \gamma_1$ in $b_\beta$ overlap $\alpha$. Let $m$ be largest so that the extender $E = E_{\gamma_1}$ is $\gamma_1, m + 1$-strong. As in (d), if $m$ is at most $n$ then we reach a contradiction if $\alpha$ is $\beta, m + 1$-correct. But this is implied by the $\beta, m + 1$-stability of $\alpha$. If $m$ is greater than $n$ then by induction, $\alpha$ is $\gamma_1, n + 1$-strong and also $\pi_1(\gamma_1), n + 1$-strong in $\text{Ult}(m_{\gamma_1}, E_{\gamma_1})$, where $\pi_1$ is the ultrapower map. As $\gamma_0$ is $\gamma_1, m + 1$-strong in $m_{\gamma_1}$, it follows that $\pi_1(\gamma_0)$ is $\pi_1(\gamma_1), m + 1$-strong in $\text{Ult}(m_{\gamma_1}, E_{\gamma_1})$ and therefore $\alpha$ is $\pi_1(\gamma_0), m + 1$-strong in $m_{\gamma_1 + 1}$. By induction it follows that $\alpha$ is $\kappa_{\gamma + 1}, m + 1$-strong in $m_{\gamma_1 + 1}$ and applying the embedding from $m_{\gamma_1 + 1}$ to $m_\beta$, we have that $\alpha$ is $\kappa_\beta, m + 1$-strong in $m_\beta$ and therefore as $m$ is greater than $n$, $\beta, n + 1$-strong, as desired. $\square$

We also have:

**Lemma 18** If $\beta$ is the limit of $\beta, n$-stables for each $n$ then there is a unique cofinal branch through the tree below $\beta$, and this branch contains all $\beta$-corrects.

**Proof.** Note that the proof of Lemma 17 (d) shows that if $\beta$ is $\kappa_\beta, n$-correct in $m_\beta$ for each $n$ then all $\beta$-corrects belong to each branch cofinal in the tree below $\beta$. By Lemma 17 (c) this property holds when $\beta$ is the limit of $\beta, n$-stables for each $n$. $\square$

**Corollary 19** $\alpha$ is nicely $\beta, n$-stable iff $\alpha$ is either nicely $\beta, n$-strong in $m_0^*$ or the limit of nicely $\beta, n$-strengths in $m_0^*$.

**Proof.** By induction on $n$. The case $n = 0$ is Corollary 16.

Suppose the Lemma holds for $n$ and we want to verify it for $n + 1$. If $\alpha$ is nicely $\beta, n + 1$-strong in $m_0^*$ then $\alpha$ is $\beta, n + 1$-stable by Lemma 17 and by induction, $\beta$ is the limit of $\beta, n$-stables. Conversely, suppose that $\alpha$ is a successor $\beta, n + 1$-stable and $\beta$ is the limit of $\beta, n$-stables. By Lemma 17, $\alpha$ is $\kappa_\beta, n + 1$-strong in $m_\beta$. As $\beta$ is the limit of $\beta, n$-stables, we can apply Lemma 17 and induction to conclude that $\beta$ is the limit of $\kappa_\beta, n$-strengths. It follows that $\beta$ is $\kappa_\beta, n$-correct in $m_\beta$ and therefore $\alpha$ is $\beta, n + 1$-strong in $m_\beta$ and hence in $m_0^*$. By induction, $\beta$ is the limit of $\beta, n$-strengths, so $\alpha$ is nicely $\beta, n + 1$-strong, as desired. $\square$
This completes the proof of Theorem 2.

Also note the following.

**Corollary 20** For any limit ordinal \( \lambda \leq \text{Ord} \), the iteration tree has a unique branch \( b_\lambda \) cofinal in \( \lambda \). For \( \beta \) a limit of strong limits, \( b_\beta \) is \( \Sigma_1 \) definable over \( \mathbb{H}(\beta) \) relative to the predicate \( \text{Stable}_\beta = \{ (\alpha, n) \mid \alpha \text{ is } \beta, n\text{-stable} \} \).

**Proof.** It is easy to verify that if \( \lambda \) is not a limit of strong limits then the iteration tree is linear on a final segment of \( \lambda \) and therefore has a unique cofinal branch. If \( \lambda \) is a limit of strong limits then by Lemma 17 (e), the \( \lambda \)-corrects form a cofinal subset of any branch cofinal in \( \lambda \). Therefore the unique branch cofinal in \( \lambda \) consists of the tree predecessors of a final segment of the \( \lambda \)-corrects, and this is \( \Sigma_1 \) definable over \( \mathbb{H}(\beta) \) relative to the predicate \( \text{Stable}_\lambda \). \( \square \)

4. Further results

**The Optimality of Mighty Mouse**

For a fixed finite \( n \), let \( \text{mm}^n \) denote the least mouse with a measurable \( \kappa \) which is a limit of \( \kappa, n \)-strongs. We show that \( V \) is not definably generic over a definable iterate of \( \text{mm}^n \), and therefore the use of Mighty Mouse (= \( \text{mm}^{<\omega} \)) is optimal for our main result. The following lemma is related to work of Schindler [14].

**Lemma 21** Suppose that \( (\text{mm}^n_\gamma \mid \gamma \in \text{Ord}) \) is a definable iteration of \( \text{mm}^n \) sending the largest measurable of \( \text{mm}^n \) to \( \text{Ord} \). Then the resulting iteration tree has a definable branch of length \( \text{Ord} \).

**Proof.** Choose a large \( k \), bigger than \( n \) and the level of definability of the iteration. Suppose \( \alpha \) is \( \text{Ord}, k \)-stable and larger than any parameters needed to define the iteration. If \( \alpha \) is overlapped by an extender \( E \) used in the iteration, then \( E|\alpha \) is an \( \alpha, n \)-strong extender belonging to \( M \). But \( M \) is an iterate of \( \text{mm}^n \) so there are no such extenders in \( M \). So it follows that such \( \alpha \)'s generate a definable cofinal branch of length \( \text{Ord} \). \( \square \)

If \( \mathcal{E} = (\text{mm}^n_\gamma \mid \gamma \in \text{Ord}) \) is an iteration of \( \text{mm}^n \) then the *limit* of \( \mathcal{E} \) is the union of those \( m \) which are an initial segment of \( \text{mm}^n_\gamma \) for sufficiently large \( \gamma \) (equivalently, the union of the \( \text{mm}^n_\gamma | i_\gamma \) where \( i_\gamma \) is the index of the extender \( E_\gamma \) used at stage \( \gamma \) of the iteration).
Theorem 22  Suppose that $M$ is the limit of a definable iteration $(mm^n_\gamma | \gamma \in \text{Ord})$ of $mm^n$. Then $V$ is not definably class-generic over $M$.

Proof. Using Lemma 21 let $b$ be a definable branch through the iteration tree of length $\text{Ord}$. Choose $\gamma_0$ in $b$ so that there are no drops along $b$ at stages at least $\gamma_0$, i.e. so that there is a total embedding of $mm^n_{\gamma_0}$ to $m_b = \text{the direct limit of the } m_\alpha, \alpha \in b$. As $b$ is definable there must be some $\gamma_1$ at least $\gamma_0$ such that some ordinal $\kappa$ in $mm^n_{\gamma_1}$ is sent to $\text{Ord}$ by the embedding from $mm^n_{\gamma_1}$ to $m_b$. Then the successive images of $\kappa$ along $b$ form a definable class $I$ of indiscernibles for $M$.

Now if $V$ were class-generic over $M$ via the $M$-definable forcing $P$, then in $M$ we could define the class $X$ of $\alpha$ such that some condition in $P$ forces $\alpha$ to belong to the definable class $I$. But if $\alpha$ belongs to $X$ then $M|\alpha$ is elementary in $M$ and therefore in $M$ we can define a satisfaction predicate for $M$, contradicting Tarski. \(\blacksquare\)

Note that the satisfaction predicate for $(L[S], S)$ is not definable as $V$ is definably-generic over this model.

Corollary 23  Suppose that $S_n$ denotes the Stability Predicate $S$ restricted to $n$, i.e. the class of triples $(\alpha, \beta, k)$ in $S$ with $k < n$. Then the satisfaction predicate for $(L[S_n], S_n)$ is definable.

Proof. Iterate Mighty Mouse (or just $mm^{n-1}$) just to capture $S_n$ and not the full Stability Predicate $S$. The iteration tree has a definable branch of length $\text{Ord}$ and yields a definable proper class of indiscernibles for the iterate. If $\alpha$ is such an indiscernible then $(L_\alpha[S_n], S_n | \alpha)$ is fully elementary in $(L[S_n], S_n)$ and therefore satisfaction for the latter model is definable. \(\blacksquare\)

Theorem 24  Suppose that $V$ is $\text{Ord-cc}$ generic over the inner model $M$ and $K_M = \text{the core model of } M$ is iterable in $V$. In the comparison of $K_M$ with Mighty Mouse, the latter does not drop. In particular, the reals of $M$ contain the reals of Mighty Mouse.

Note that the above applies to $M = \text{the Stable Core}$ and therefore:

Corollary 25  The reals of the Stable Core are the reals of Mighty Mouse.
Proof of Theorem 24. Suppose that \( mm = \) Mighty Mouse drops in the comparison. Then \( K_M \) is the truncation to Ord of an iterate of a mouse \( mm^- \) with a top measurable \( \kappa \) which is sent to Ord in the comparison, where \( mm^- \) is a proper initial segment of an iterate of \( mm \). But then for some \( n, \kappa \) is not the limit of \( \kappa, n \)-strongs in \( mm^- \) and therefore \( K_M \) has only boundedly many Ord, \( n \)-strongs for some \( n \). As in Lemma 21, it follows that there is a definable branch \( b \) through the iteration tree on \( mm^- \) and the stages along \( b \) where the top extender is applied is definable. Since \( V \) is an Ord-cc extension of \( M \), there is an \( M \)-definable unbounded class of stages where the top extender is applied, and in addition this holds below some \( M \)-singular cardinal \( \lambda \) which is a limit of stages where the top extender is applied. But then \( M \) contains a function from \( \lambda \) to \( \lambda \) which dominates all such functions in \( K_M \), contradicting weak covering between \( M \) and \( K_M \). \( \square \)

Using a similar argument we get:

Theorem 26 Suppose that \( M \) is an inner model not containing \( mm^n \) and \( K_M = \) the core model of \( M \) is iterable. Then \( mm^n \) is not Ord-cc generic over \( M \).

Proof. Compare \( K_M \) and \( mm^n \). The coiteration is definable in \( M[mm^n] \), and by Lemma 21 has a cofinal branch which is definable in \( M[mm^n] \). If the latter were an Ord-cc generic extension of \( M \) then as in the previous proof we violate weak covering between \( M \) and \( K_M \). \( \square \)

The enriched Stable Core.

By enriching the Stable Core as in [7], we obtain stronger forms of genericity over it.

Lemma 27 Suppose that \( mw \) is a mouse satisfying ZFC- whose largest cardinal is Woodin and \( \delta \) is an inaccessible cardinal greater than the cardinality of \( mw \). Then \( mw \) has a definable iterate \( mw^* \) whose Woodin cardinal is \( \delta \) such that \( H(\delta) \) is generic over \( mw^*|\delta \) for a forcing which is definable over \( mw^*|\delta \) and \( \delta \)-cc in \( mw^* \).

Proof. We use a variant of the enriched stable core of [7].
Let $\delta_w$ denote the largest cardinal of $m_w$. We say that $i < \text{Ord}(m_w)$ is suitable if $\delta_w$ is the largest cardinal of $m_w|i$ and the $\Sigma_1$ hull of $\delta_w \cup \{\delta_w\}$ in $m_w|i$ is all of $m_w|i$. To each suitable $i$ we associate the club $C_i(m_w)$ consisting of those $\alpha < \delta_w$ such that the intersection with $m_w|\delta_w$ of the $\Sigma_1$ hull of $\alpha \cup \{\delta_w\}$ in $m_w|i$ is all of $m_w|i$. To each suitable $i$ we associate the club $C_i(m_w)$ consisting of those $\alpha < \delta_w$ such that the intersection with $m_w|\delta_w$ of the $\Sigma_1$ hull of $\alpha \cup \{\delta_w\}$ in $m_w|i$ is all of $m_w|i$.

For beth-numbers $\alpha < \beta$ and a predicate $A$ on $\beta$ we say that $\alpha$ is $\beta,A,1$-stable if $(\mathbb{H}(\alpha), A \cap \alpha)$ is $\Sigma_1$-elementary in $(\mathbb{H}(\beta), A \cap \beta)$. And for $\alpha < \beta$ which are each measurable or the limit of measurables in a mouse $m$ and a predicate $A$ on $\beta$ in $m$, we say that $\alpha$ is $\beta,A,1$-strong if for each cardinal $\gamma$ between $\alpha$ and $\beta$ there is an extender $E$ with critical point $\alpha$ such that $H(\gamma)$ is contained in $\text{Ult}(m, E)$ and $j_E(A) \cap \gamma = A \cap \gamma$.

Now we define an iteration $((m_\gamma, E_\gamma) \mid \gamma < \delta)$ as follows. We say that the pair $(i, \beta)$ where $i < \text{Ord}(m_\gamma)$ is suitable (for $m_\gamma$) and $\beta < \delta_\gamma$ (= the largest cardinal of $m_\gamma$) is worrisome if one of the following holds:

1. $\beta$ is measurable in $m_\gamma$ but not strong limit in $V$.

2. $\beta$ is a successor beth-number in $V$ and some $\alpha$ less than $\beta$ is either not $\beta,C_i \cap \beta,1$-stable or not $\delta_\gamma,C_i,1$-strong, yet there is a $\beta,C_i \cap \beta,1$-strong extender with critical point $\alpha$.

Then we choose $(i_\gamma, \beta_\gamma)$ to be the lexicographically-least worrisome pair $(i, \beta)$ and if 2 above holds, we choose $E_\gamma$ to be the extender with least index witnessing it for $(i_\gamma, \beta_\gamma)$. If 1 holds but 2 does not then we apply the measure on $\beta$ in $m_\gamma$.

Now we take $m_w^*$ to be the limit of this iteration. Then $\delta$ is the largest cardinal of $m_w^*$ and is Woodin in $m_w^*$. The measurables and limits of measurables of $m_w^*$ less than $\delta$ are the strong limits of $V$ less than $\delta$. And for suitable $i$ and strong limit $\alpha < \beta < \delta$, $\alpha$ is $\beta,C_i,1$-strong in $m_w^*$ iff $\alpha$ is $\beta,C_i,1$-stable in $V$.

Now as in the proof of Theorem 4 we can argue that $H(\delta)$ is generic over $m_w^*|\delta$ for a definable forcing which is $\delta$-cc in $m_w^*$. \qed
Lemma 28  Suppose that $m^\#_{\text{pcw}}$ is the least mouse with a measurable limit of Woodin cardinals and there is a proper class of inaccessible cardinals. Then there is a definable Ord-iteration of $m^\#_{\text{pcw}}$ with limit model $M$ such that every set in $V$ is set-generic over $M$.

Theorem 29  Suppose that $mm_{\text{pcw}}$ is the least mouse with a measurable cardinal $\kappa$ which is both a limit of $\kappa, n$-stricts for each $n$ and a limit of Woodin cardinals and there is a proper class of inaccessibles. Then there is a definable Ord-iteration of $mm_{\text{pcw}}$ with limit model $M$ such that every set in $V$ is set-generic over $M$ and $V$ is definably class-generic over $M$.

Proof of Lemma 28. This is a simple elaboration on the previous proof. Note that in $m^\#_{\text{pcw}}$ there is no Woodin limit of Woodin cardinals. Now perform an iteration as in the previous proof to send the least Woodin cardinal of $m_{\text{pcw}}$ to the least inaccessible $\delta_0$, follow this with an iteration above $\delta_0$ to send the second Woodin cardinal of this iterate to the second inaccessible and continue until all Woodin cardinals of the iterate have been moved up to inaccessibles, using the discreteness of the set of Woodin cardinals to avoid creating new Woodin cardinals in the iteration. Then apply the top measure of the iterate to create new Woodin cardinals and continue the iteration. After Ord steps the Woodin cardinals of the limit $M$ of the iteration are the successor inaccessibles and every set is set-generic over $M$ because $H(\delta)$ is generic over $M$ for a forcing definable over $M|\delta$ for successor inaccessibles $\delta$.

Proof of Theorem 29. Here we combine the previous argument, moving Woodin cardinals onto inaccessibles, with the Mighty Mouse argument to get $V$ generic over the limit of the iteration. At stage $\gamma$ of the iteration we let $E_\gamma$ witness a worrisome pair $(i, \beta)$ for the purpose of ensuring that every set is set-generic over the limit $M$ of the iteration, if there is such a pair, and if not, then we choose $E_\gamma$ to witness a worrisome $\beta$ for the purpose of ensuring that $V$ is class-generic over $M$. At a definable club of stages there will be no worrisome pairs for the first purpose and therefore we will be able to ensure the equality in $M$ of nice Ord, $n$-stability with nice Ord, $n$-strength for each $n$.
**Lemma 30** Assume Mighty Mouse exists. Then the Stable Core is not rigid.

*Proof.* Let \((\text{mm}_\gamma \mid \gamma \in \text{Ord})\) be an iteration of \(\text{mm} = \text{Mighty Mouse}\) whose limit \(M\) contains the Stable Core as a definable inner model. Then to show that the Stable Core is not rigid it suffices to show that \(M\) is not rigid. The least measure applied in the iteration is \(E_0\), a measure in \(\text{mm}\). Now \(E_0\) can be applied to \(M\) as \(\text{mm}_0\) and \(M\) have the same subsets of the critical point of \(E_0\), and the ultrapower map sends \(M = \text{Lim}(\text{mm}_\gamma \mid \gamma \in \text{Ord})\) to \(\text{Lim}(\text{mm}_\gamma \mid 0 < \gamma \in \text{Ord}) = M\). \(\square\)

**Woodin’s Extender Algebra**

Another approach to making \(V\) generic over a mouse iterate is due to Hugh Woodin and is based on his Extender Algebra (see [17]). Recall that the genericity of \(V\) over the Stable Core is obtained by showing that \(V\) can be written as \(L[F]\) where \(F : \text{Ord} \to 2\) is a function which preserves the stability predicate, i.e. for each \(n\), if \(\alpha\) is nicely \(\beta,n\)-stable then this holds relative to \(F\). This implies that \(F\) is generic for the algebra of infinitary quantifier-free sentences of the Stable Core when it is factored by an ideal which expresses these \(n\)-stability relationships. In Woodin’s approach, one instead begins with a predicate \(F\) such that \(V = L[F]\) and iterates a mouse with Woodin cardinals so that \(F\) is preserved by the extenders of the iterate.

The advantages of the approach taken in the present paper are twofold. First, the genericity of \(V\) over the Stable Core is without any large cardinal assumptions, which means it can be applied to situations in which Mighty Mouse does not exist. Second is the definability of the iterations used. We get that \(V\) is generic over a definable iterate of Mighty Mouse, via a forcing that is definable over the iterate and is Ord-cc. Such a result does not seem to be obtainable without an appeal to the Stability Predicate, as otherwise the iteration is dependent on a wellorder of \(V\) (which can only be definable if \(V = \text{HOD}\)). It is possible to show using the Extender Algebra that \(\text{HOD}\) is definably generic over a definable iterate of Mighty Mouse and \(V\) is definably generic over \(\text{HOD}\); however the latter entails a forcing which is not Ord-cc and we have seen that the Ord-cc is important for the robustness of the core model (for example \(0^\#\) is not Ord-cc generic over any inner model not containing it; this is open for class-genericity in general).
Some open questions

1. Is there a description of the Stable Core as a generic extension of an iterate of Mighty Mouse via a well-understood forcing?

2. Assuming that Mighty Mouse exists, what large cardinals exist in the Stable Core? The core of the Stable Core has arbitrarily large $n$-strong cardinals for each $n$; is this also the case for the Stable Core itself?

3. Suppose that Mighty Mouse does not belong to the model $M$. Is $M$ the Stable Core of one of its generic extensions?

4. Is there an iterate of the least mouse with a measurable limit of Woodins over which $V$ is generic and every set is set-generic? Theorem 29 obtains this from a stronger mouse.

References


