Descriptive Set Theory on Generalised Baire Space

Joint work with Khomskii-Kulikov (first part) and with Hyttinen-Kulikov (second part)

We assume $\kappa = \kappa^{<\kappa}$.

$\kappa$-Baire space $= \kappa^{\kappa}$ consists of all $f : \kappa \to \kappa$, with basic open sets given by

$$\{ f : \kappa \to \kappa \mid s \subseteq f \}$$

where $s \in \kappa^{<\kappa}$.

Nowhere dense $= \text{Closure has no interior}$
Meager $= \text{union of } \kappa\text{-many nowhere dense sets}$

Baire measurable $= \text{differs from an open set by a meager set}$

The Baire Category theorem holds (the intersection of $\kappa$-many open dense sets is dense)
Regularity Properties

Baire measurability is just one example of a regularity property.

A forcing $P$ is $\kappa$-treelike iff it is a $\kappa$-closed suborder of the set of subtrees of $2^{\lt \kappa}$, ordered by inclusion.

Some examples of $\kappa$-treelike forcings:

$\kappa$-Cohen $\mathbb{C}_\kappa$. These are subtrees of $2^{\lt \kappa}$ consisting of a stem and all nodes above it.

$\kappa$-Sacks $\mathbb{S}_\kappa$. These are $\kappa$-closed subtrees of $2^{\lt \kappa}$ with the property that every node has a splitting extension and the limit of splitting nodes is a splitting node.

$\kappa$-Silver $\mathbb{V}_\kappa$, for inaccessible $\kappa$. These are $\kappa$-Sacks trees $T$ which are uniform, i.e. if $s$, $t$ are elements of $T$ of the same length then $s \star i$ is in $T$ iff $t \star i$ is in $T$ for $i = 0, 1$. 
Regularity Properties

κ-Miller $\mathbb{M}_\kappa$. These are $\kappa$-closed subtrees of the tree $\kappa^{<\kappa}$ of increasing sequences in $\kappa^{<\kappa}$ with the property that every node can be extended to a club-splitting node and the limit of club-splitting nodes is club-splitting. We also require continuous club-splitting, which means that if $s$ is a limit of club-splitting nodes then the club witnessing club-splitting for $s$ is the intersection of the clubs witnessing club-splitting for the club-splitting proper initial segments of $s$.

κ-Laver $\mathbb{L}_\kappa$. These are $\kappa$-Miller trees with the property that every node beyond some fixed node (the stem) is club-splitting.

κ-Mathias $\mathbb{R}_\kappa$. Conditions are pairs $(s, C)$ where $s$ is a bounded subset of $\kappa$ and $C$ is a club in $\kappa$. $(t, D) \leq (s, C)$ iff $t$ end-extends $s$, $D \subseteq C$ and $t \setminus s \subseteq C$. This is equivalent to a $\kappa$-treelike forcing.
Regularity Properties

The 6 examples above fall into two groups:

$\mathbb{C}_\kappa$, $\mathbb{L}_\kappa$ and $\mathbb{R}_\kappa$ are topological: The $[T]$ for $T \in \mathcal{P}$ form the base for a topology (either $[S] \cap [T]$ is empty or contains some $[U]$). They are $\kappa^+$-cc.

$\mathbb{S}_\kappa$, $\mathbb{M}_\kappa$ and $\mathbb{V}_\kappa$ are not $\kappa^+$-cc but they satisfy a form of fusion (called Axiom A*), sufficient to show that $\kappa^+$ is preserved.

*Remark.* There is no obvious $\kappa$-analogue of Solovay forcing (random real forcing). However:

**Theorem**

*(SDF-Laguzzi)* If $V = L$ and $\kappa$ is inaccessible then there is a $\Delta^1_1$ $\kappa$-treelike forcing $\mathbb{B}_\kappa$ which is $\kappa^+$-cc and $\kappa^\kappa$-bounding.
Regularity Properties

To define “$\mathcal{P}$-measurability” for $\kappa$-treelike forcings $\mathcal{P}$ we proceed as follows.

A set $A$ is:

Strictly $\mathcal{P}$-null if every tree $T \in \mathcal{P}$ has a subtree in $\mathcal{P}$, none of whose $\kappa$-branches belongs to $A$.

$\mathcal{P}$-null (or $\mathcal{P}$-meager) if it is the union of $\kappa$-many strictly $\mathcal{P}$-null sets.

$\mathcal{P}$-measurable (or $\mathcal{P}$-regular) if any tree $T \in \mathcal{P}$ has a subtree $S \in \mathcal{P}$ such that either all $\kappa$-branches through $S$, with a $\mathcal{P}$-null set of exceptions, belong to $A$ or all $\kappa$-branches through $S$, with a $\mathcal{P}$-null set of exceptions, belong to the complement of $A$. 
Regularity Properties

Proposition

(a) If $\mathcal{P}$ is topological then:
(a1) A set is $\mathcal{P}$-measurable iff it differs from a $\mathcal{P}$-open set by a $\mathcal{P}$-null set. (So $\mathcal{C}_\kappa$-measurable is the same as Baire-measurable.)
(a2) Not every $\mathcal{P}$-null set is strictly $\mathcal{P}$-null.
(a3) Borel sets are $\mathcal{P}$-measurable.
(b) If $\mathcal{P}$ satisfies fusion (Axiom A*) then:
(b1) Every $\mathcal{P}$-null set is strictly $\mathcal{P}$-null.
(b2) Borel sets are $\mathcal{P}$-measurable.

Question. As in the case $\kappa = \omega$, are all $\Sigma^1_1$ sets $\mathcal{P}$-measurable?

Answer: NO!
Regularity Properties

**Fact.** The club filter $\{f : \kappa \rightarrow 2 \mid f(i) = 1 \text{ for club-many } i < \kappa\}$ is not $\kappa$-Sacks ($S_\kappa$) measurable.

**Proof.** Otherwise there is a $\kappa$-Sacks tree $T$ such that either for all $f \in [T]$, $f(i) = 1$ for club-many $i < \kappa$ or for all $f \in [T]$, $f(i) = 0$ for stationary-many $i < \kappa$.

But we can easily build $f_0, f_1$ in $[T]$ such that whenever $f_0 | i$ splits in $T$, $f(i) = 0$ and whenever $f_1 | i$ splits in $T$, $f(i) = 1$.

And the set of $i$ where $f_0 | i$ splits forms a club (same for $f_1$).

So $[T]$ has an element $f_0$ which is not in the club filter and an element $f_1$ which is. $\square$
Regularity Properties

Now we can apply the following result to conclude that $\Sigma^1_1$ sets need not be $\mathcal{P}$-measurable for any of our 6 examples. For a pointclass $\Gamma$, let $\Gamma(\mathcal{P})$ denote that sets in $\Gamma$ are $\mathcal{P}$-measurable.

**Theorem**

\[
\begin{align*}
(a) \quad & \Gamma(\mathbb{C}_\kappa) \to \Gamma(\mathbb{V}_\kappa) \to \Gamma(\mathbb{S}_\kappa). \\
(b) \quad & \Gamma(\mathbb{C}_\kappa) \to \Gamma(\mathbb{M}_\kappa) \to \Gamma(\mathbb{S}_\kappa). \\
(c) \quad & \Gamma(\mathbb{R}_\kappa) \to \Gamma(\mathbb{M}_\kappa). \\
(d) \quad & \Gamma(\mathbb{L}_\kappa) \to \Gamma(\mathbb{M}_\kappa).
\end{align*}
\]

In particular $\Gamma(\mathbb{S}_\kappa)$ is the weakest of them all, so as it fails for $\Gamma = \Sigma^1_1$ so do all the others.

**Question.** What about $\Delta^1_1$ ($\neq$ Borel for $\kappa > \omega$)?
Regularity Properties

**Theorem**

*It is consistent to have $\Delta_1^1(\mathcal{P})$ for $\mathcal{P} = \mathbb{C}_\kappa$, $\mathbb{L}_\kappa$ and $\mathbb{R}_\kappa$ simultaneously.*

This is proved by interleaving iterations with $< \kappa$-support of these three forcings for $\kappa^+$ steps.

Note that in the above model we also have $\Delta_1^1(\mathcal{P})$ for $\mathcal{P} = \mathbb{M}_\kappa$, $\mathbb{V}_\kappa$ and $\mathbb{S}_\kappa$, by the previous slide.

*Question.* But can we separate $\Delta_1^1(\mathcal{P})$ for different $\mathcal{P}$?

This looks hard. But we have one result about it:
Regularity Properties

Theorem

There is a model where $\kappa$ is inaccessible and $\Delta_1^1(\forall \kappa)$ holds but $\Delta_1^1(\mathbb{M}_\kappa)$ fails.

This is proved by iterating $\forall \kappa$ for $\kappa^+$ steps over $L$, where $\kappa$ is inaccessible; $\Delta_1^1(\forall \kappa)$ holds in the resulting model.

The main lemma is that $\Delta_1^1(\mathbb{M}_\kappa)$ yields functions from $\kappa$ to $\kappa$ that are unbounded over $L[f]$, for any given $f : \kappa \rightarrow \kappa$.

As the iteration is $\kappa^\kappa$-bounding and therefore does not add functions which are unbounded over the ground model, we conclude that $\Delta_1^1(\mathbb{M}_\kappa)$ fails.

It follows from our earlier implications between regularity properties that in the above model, $\Delta_1^1(\mathbb{C}_\kappa)$, $\Delta_1^1(\mathbb{R}_\kappa)$ and $\Delta_1^1(\mathbb{L}_\kappa)$ all fail, but $\Delta_1^1(\mathbb{S}_\kappa)$ holds.
Regularity Properties

The main difficulty with separating $\Delta^1_1$ regularity properties is the lack of “Solovay-type characterisations”.
In the classical setting we have:

(Solovay) $\Sigma^1_2$ sets are Baire-measurable iff for every real $x$ there is a comeager set of reals Cohen over $L[x]$.
(Shelah) $\Delta^1_2$ sets are Baire-measurable iff for every real $x$ there is a Cohen real over $L[x]$.

In fact, Shelah’s result provably fails for uncountable $\kappa$:

Theorem

(SDF-Wu-Zdomskyy) Suppose that $\kappa$ is regular and uncountable in $L$. Then in a cofinality-preserving forcing extension, for every $x \subseteq \kappa$ there is a $\kappa$-Cohen over $L[x]$ but the CUB filter on $\kappa$ is $\Delta^1_1$. In particular not all $\Delta^1_1$ sets are Baire-measurable.
Borel Reducibility

If $E$ and $F$ are equivalence relations on $\kappa^\kappa$ then we say that $E$ is \textit{Borel reducible to} $F$, written $E \leq_B F$, if there is a Borel function $f$ such that for all $x, y$: $E(x, y)$ iff $F(f(x), f(y))$. The relation $\leq_B$ is reflexive and transitive and we write $\equiv_B$ for the equivalence relation it induces.
Borel Reducibility: Dichotomies

In the classical setting one has two important Dichotomies:

*Silver Dichotomy.* Suppose that $E$ is a Borel equivalence relation on $\omega^\omega$ with uncountably many classes. Then equality is Borel (even continuously) reducible to $E$.

*Harrington-Kechris-Louveau Dichotomy.* Suppose that $E$ is a Borel equivalence relation. Then either $E$ is Borel reducible to equality or $E_0$ is Borel reducible to $E$, where $E_0$ is the equivalence relation of equality mod finite.

In generalised Baire space, the Silver Dichotomy fails in $L$ but consistently holds (after collapsing a Silver indiscernible to become $\omega_2$), and the Harrington-Kechris-Louveau Dichotomy simply fails.
Borel Reducibility: Small Equivalence Relations

**Theorem**

*If $E$ is the orbit equivalence relation of a Borel action of a group of size at most $\kappa$ then $E$ is Borel reducible to $E_0$.***

**Proof.** The key observation is this: Let $F_\kappa$ denote the free group on $\kappa$ generators. Then $F_\alpha$ has cardinality less than $\kappa$ for $\alpha < \kappa$ (this fails when $\kappa$ equals $\omega$). Using this one shows that the shift action of $F_\kappa$ (sending $(g, X)$ in $G \times \mathcal{P}(F_\kappa)$ to $\{g \cdot x \mid x \in X\}$) reduces to $E_0$: Map $X \subseteq F_\kappa$ to the sequence $f(X) = (<_\alpha$-least element of $\{g_\alpha \cdot (X \cap F_\alpha) \mid g_\alpha \in F_\alpha\} \mid \alpha < \kappa$). If $X, Y$ are equivalent under shift then it is easy to check $f(X)E_0f(Y)$; the converse uses Fodor’s theorem. □
Borel Reducibility: Small Equivalence Relations

**Theorem**

Assume $V = L$. Then there is a smooth Borel equivalence relation with classes of size 2 which is not induced by a Borel action of a small group.

*Proof.* Let $X$ be the Borel set of Master Codes for initial segments of $L$ of size $\kappa$ and $\sim X$ its complement. Define a bijection $f : \sim X \to X$ with Borel graph and define $E(x, y)$ iff $y = f(x)$ or $x = f(y)$. Then $E$ is smooth. If it were induced by a Borel action of a group of size at most $\kappa$ then $f$ would be Borel on a non-meager set, which is impossible. □
Borel Reducibility: $E_1$

**Theorem**

$E_1$ is Borel reducible to $E_0$.

*Proof idea:* For limit $\alpha < \kappa$, define $E_1^\alpha$ to be the equivalence relation on $(2^\alpha)^\alpha$ approximating $E_1$ defined by $(x_i)_{i<\alpha} E_1^\alpha (y_i)_{i<\alpha}$ iff for some $\beta < \alpha$, $x_i = y_i$ for all $i > \beta$.

Now define $F((x_i)_{i<\kappa})(\alpha)$ to be 0 if $\alpha$ is not a limit and otherwise to be a code for the $E_1^\alpha$-equivalence class of $(x_i \upharpoonright \alpha)_{i<\alpha}$.

Clearly if $(x_i)_{i<\kappa} E_1 (y_i)_{i<\kappa}$ then $F((x_i)_{i<\kappa})$ and $F((y_i)_{i<\kappa})$ are $E_0$-equivalent.

Conversely, if $(x_i)_{i<\kappa}$ and $(y_i)_{i<\kappa}$ are not $E_1$ equivalent then for club-many $\alpha^* < \kappa$, $(x_i \upharpoonright \alpha^*)_{i<\alpha^*}$ and $(y_i \upharpoonright \alpha^*)_{i<\alpha^*}$ are not $E_1^\alpha^*$-equivalent; it follows that $F((x_i)_{i<\kappa})$ and $F((y_i)_{i<\kappa})$ are not $E_0$-equivalent. $\Box$
(a) Each Borel isomorphism relation is Borel reducible to the $\alpha$-th jump of equality for some $\alpha < \kappa^+$.  
(b) For each $\alpha < \kappa^+$, the $\alpha$-th jump of equality is Borel reducible to equality on $\kappa^\kappa$ modulo a $\mu$-nonstationary set, for any regular $\mu < \kappa$.  
(c) A first-order theory is classifiable and shallow iff the isomorphism relation on its models of size $\kappa$ is Borel.  
(d) (For a suitable successor $\kappa$) A first-order theory is unclassifiable iff equality on $2^\kappa$ modulo a $\mu$-nonstationary set is Borel reducible to the isomorphism relation on its models of size $\kappa$ for some regular $\mu < \kappa$.

Is equality on $\kappa^\kappa$ modulo a $\mu$-nonstationary set Borel reducible to equality on $2^\kappa$ modulo a $\mu$-nonstationary set?  
If so we have:
Borel Reducibility: Isomorphism Relations

If $T_0$ is classifiable and shallow and $T_1$ is unclassifiable then isomorphism on the models of $T_0$ of size $\kappa$ is Borel reducible to isomorphism on the models of $T_1$ of size $\kappa$ (for example when $\kappa$ is the successor of an uncountable regular and GCH holds).