For uncountable regular $\kappa$, $\text{NS}_\kappa$ denotes the ideal of nonstationary subsets of $\kappa$

**Proposition**

$\text{NS}_\kappa$ is $\Sigma_1$ definable with parameter $\kappa$.

**Proof.** $X \in \text{NS}_\kappa$ iff $X$ is a subset of $\kappa$ and there exists $C$ such that $C$ is a closed unbounded subset of $\kappa$ disjoint from $X$. This is $\Sigma_1$ with parameter $\kappa$. □

We say that $\text{NS}_\kappa$ is $\Delta_1$ definable if it is both $\Sigma_1$ and $\Pi_1$ definable using subsets of $\kappa$ as parameters.
\[ \Delta_1 \text{ Definability of the Nonstationary Ideal} \]

For \( \text{NS}_\kappa \) to be \( \Delta_1 \) definable one needs to “witness stationarity”. Typically this is not possible:

**Theorem**

Assume \( V = L \). Then \( \text{NS}_\kappa \) is not \( \Delta_1 \) definable.

*Proof Sketch.* Suppose that \( \varphi(X) \) is a \( \Sigma_1 \) formula with a variable \( X \) denoting a subset of \( \kappa \).

If \( \varphi(X) \) is true then by condensation, \( \varphi(X \cap \alpha) \) is true for club-many \( \alpha < \kappa \); in fact, for club-many \( \alpha < \kappa \), \( \varphi(X \cap \alpha) \) is true “while \( \alpha \) still looks regular”, i.e. in some \( L_\beta \models \alpha \) regular.

Conversely, if \( \varphi(X) \) is false then for any club \( C \) there is \( \alpha \) in \( C \) such that \( \varphi(X \cap \alpha) \) is false in the largest \( L_\beta \models \alpha \) regular.

So the club filter is “complete” for \( \Sigma_1 \) subsets of \( \mathcal{P}(\kappa) \) and therefore not \( \Delta_1 \). \( \Box \)
Δ₁ Definability of the Nonstationary Ideal

Large cardinals also prevent NS_κ from being Δ₁ definable.

**Theorem**

*Suppose that κ is weakly compact. Then NS_κ is not Δ₁ definable.*

**Proof.** Again let φ(X) be a Σ₁ formula with a variable X denoting a subset of κ.

As before, if φ(X) is true then by condensation, φ(X ∩ α) is true for club-many α < κ.

Conversely, suppose that φ(X) is false. Then φ(X) is false in H(κ⁺) and the latter is a Π₁⁻ statement about V_κ. By weak compactness (= Π₁⁻ reflection), φ(X ∩ α) is false for stationary-many α < κ.

So again the club filter is “complete” for Σ₁ subsets of ℙ(κ) and therefore not Δ₁. □
$\Delta_1$ Definability of the Nonstationary Ideal

However it is indeed possible for $\text{NS}_{\omega_1}$ to be $\Delta_1$ definable.

**Theorem**

*(Mekler-Shelah, proof repaired by Hyttinen-Rautiela)* Assume GCH. Then there is a proper, cardinal-preserving forcing extension satisfying GCH in which $\text{NS}_{\omega_1}$ is $\Delta_1$ definable.

**Idea of Proof.** For $X \subseteq \omega_1$ let $T(X)$ be the tree of countable, closed subsets of $X$ ordered by end-extension. Then $X$ contains a club iff $T(X)$ has a branch of length $\omega_1$.

The idea is to force a tree $T$ (called a canary tree) of size and height $\omega_1$ with no $\omega_1$-branch such that whenever $X$ is stationary, costationary there are embeddings of $T(X)$ and $T(\sim X)$ into $T$. Then conversely, if there are embeddings of both $T(X)$ and $T(\sim X)$ into $T$ it follows that $X$ is both stationary and costationary. So we have:
\( \Delta_1 \) Definability of the Nonstationary Ideal

\( X \) is stationary iff
\( X \) contains a club or there are embeddings of both \( T(X) \) and \( T(\sim X) \) into \( T \)

and therefore \( \text{NS}_{\omega_1} \) is \( \Delta_1 \) definable. □

With some extra work, Hyttinen-Rautila obtained the natural generalisation to \( \text{NS}_{\kappa^+} \) for any regular \( \kappa \):
Let \( \text{Cof}(\kappa) \) denote the class of ordinals of cofinality \( \kappa \) and \( \text{NS}_{\kappa^+} \upharpoonright \text{Cof}(\kappa) \) the ideal of stationary subsets of \( \kappa^+ \cap \text{Cof}(\kappa) \),

\textbf{Theorem}

\textit{(Hyttinen-Rautila)} Assume GCH and \( \kappa \) regular. Then there is a \( \kappa \)-proper, cardinal-preserving forcing extension satisfying GCH in which \( \text{NS}_{\kappa^+} \upharpoonright \text{Cof}(\kappa) \) is \( \Delta_1 \) definable.
**$\Delta_1$ Definability of the Nonstationary Ideal**

With a different strategy the Hyttinen-Rautila result can be improved. For stationary $A \subseteq \kappa^+$ let $\text{NS}_{\kappa^+} \upharpoonright A$ denote the ideal of nonstationary subsets of $A$.

**Theorem**

(SDF-Hyttinen-Kulikov) Assume $\text{GCH}$ and $\kappa$ regular. Then for any costationary $A \subseteq \kappa^+$ there is a cardinal-preserving forcing extension satisfying $\text{GCH}$ which preserves stationary subsets of $A$ in which $\text{NS}_{\kappa^+} \upharpoonright A$ is $\Delta_1$ definable.

The difference now is that only stationary subsets of $A$, and not of $\sim A$, are preserved. Thus the idea of the proof is to witness the stationarity of subsets of $A$ by selectively killing the stationarity of certain “canonically chosen” subsets of $\sim A$ (obtained via a generic $\Box$ sequence).
$\Delta_1$ Definability of the Nonstationary Ideal: Main Result

Obviously the strategy of making $\text{NS}_{\kappa^+} \upharpoonright A \Delta_1$ definable by killing the stationarity of subsets of $\sim A$ is of no use if one wants to obtain the $\Delta_1$ definability of the full unrestricted $\text{NS}_{\kappa^+}$.

So a new idea is needed to show (our main result):

**Theorem**

*(SDF-Wu-Zdomskyy)* Assume $V = L$ and let $\lambda$ be any infinite cardinal. Then there is a cardinal-preserving forcing extension satisfying GCH which preserves stationary subsets of $\lambda^+$ in which $\text{NS}_{\lambda^+}$ is $\Delta_1$ definable.

Thus we can handle the full NS at all successor cardinals.

I’ll give now an outline of the proof.
$\Delta_1$ Definability of the Nonstationary Ideal: Main Result

Let $\kappa$ denote $\lambda^+$. We want to perform an iteration of length $\kappa^+$ which preserves the stationarity of subsets of $\kappa$, preserves cardinals and produces “witnesses” to the stationarity of subsets of $\kappa$. Note that by Löwenheim-Skolem, if a subset of $\mathcal{P}(\kappa)$ is $\Sigma_1$ with a subset of $\kappa$ as parameter then it is $\Sigma_1$ over $H(\kappa^+)$ and therefore our witnesses should be elements of $H(\kappa^+)$. In fact the only parameter we will need is $\kappa$ and our witnesses will be subsets of $\kappa$.

Now suppose that $S$ is a stationary subset of $\kappa$ and we want to “witness” this fact. The approach of SDF-Hytten-Kulikov was to fix a sequence $(S_i \mid i < \kappa^+)$ of “canonical” stationary subsets of $\kappa$ and arrange that for some $\alpha < \kappa^+$, the stationarity of the $S_i$ for $i$ in $[\kappa \cdot \alpha, \kappa \cdot \alpha + \kappa)$ is selectively killed so as to code $S$. But we can’t do this as we want to preserve the stationarity of subsets of $\kappa$. 

§ 1. Definability of the Nonstationary Ideal: Main Result

So instead we choose “canonical” stationary subsets \((S_i \mid i < \kappa^+)\) of \(\kappa^+\) (concentrating on \(\text{Cof}(\kappa)\)) and arrange that for some \(\alpha < \kappa^+\), the stationarity of the \(S_i\) for \(i\) in \([\kappa \cdot \alpha, \kappa \cdot \alpha + \kappa]\) is selectively killed so as to code \(S\).

But now our witnesses are subsets of \(\kappa^+\) instead of \(\kappa\) so we only get a definition of the collection of stationary subsets of \(\kappa\) which is \(\Sigma_1\) over \(H(\kappa^{++})\) with \(\kappa^+\) as parameter.

How do we convert this into a \(\Sigma_1\) definition over \(H(\kappa^+)\) with \(\kappa\) as parameter?

Here we use localisation (David’s trick).
Instead of just the “global property”

\[ S \subseteq \kappa \text{ is stationary iff } S \text{ is coded into the stationarity of the } S_i \subseteq \kappa^+ \text{ for } i \text{ in } [\kappa \cdot \alpha, \kappa \cdot \alpha + \kappa) \text{ for some } \alpha < \kappa^+ \]

we also ensure its “local version”

\[ S \subseteq \kappa \text{ is stationary iff for some } X \subseteq \kappa, \text{ every “suitable” model } M \text{ of size } < \kappa \text{ containing } X \cap \kappa^M \text{ (where } \kappa^M \text{ denotes } (\lambda^+)^M) \text{ satisfies that } S \cap \kappa^M \text{ is coded into the stationarity of the } S_i^M \text{ for } i \text{ in } [\kappa^M \cdot \alpha, \kappa^M \cdot \alpha + \kappa^M) \text{ for some } \alpha < (\kappa^M)^+, \]

where \((S_i^M \mid i < ((\kappa^M)^+)^M)\) is \(M\)'s version of \((S_i \mid i < \kappa^+)\).

The local version implies the global one by Löwenheim-Skolem and moreover yields a definition of stationarity for subsets of \(\kappa\) which is \(\Sigma_1\) over \(H(\kappa^+)\), as needed.
In the local version

\( S \subseteq \kappa \) is stationary iff for some \( X \subseteq \kappa \), every “suitable” model \( M \) of size \( < \kappa \) containing \( X \cap \kappa^M \) satisfies that \( S \cap \kappa^M \) is coded into the stationarity of the \( S^i_M \) for \( i \) in \( [\kappa^M \cdot \alpha, \kappa^M \cdot \alpha + \kappa^M) \) for some \( \alpha < (\kappa^M)^+ \).

we say that \( X \) is a “local witness” (or “locally witnesses”) that \( S \subseteq \kappa \) is stationary.

We produce such a local witness \( X \) in three steps:
\[ \Delta_1 \text{ Definability of the Nonstationary Ideal: Main Result} \]

1. Localise below \( \kappa^+ \), i.e. produce \( Y \subseteq \kappa^+ \) such that every "suitable" model \( M \) of size \( \kappa \) containing \( Y \cap (\kappa^+)^M \) satisfies that \( S \) is coded into the stationarity of the \( S_i^M = S_i \cap (\kappa^+)^M \) for \( i \) in \([\kappa \cdot \alpha, \kappa \cdot \alpha + \kappa)\) for some \( \alpha < \kappa^+ \).

This is easy and does not require forcing.

2. Almost disjoint code \( Y \) into a subset \( X_0 \) of \( \kappa \).

Then \( X_0 \) also localises below \( \kappa^+ \) as in 1.

3. Add the desired \( X \subseteq \kappa \) satisfying \( \text{Even}(X) = X_0 \) by forcing with initial segments of length less than \( \kappa \).

The fact that \( X_0 \) localises below \( \kappa^+ \) is sufficient to argue that this forcing is \( \kappa \)-distributive.
Δ₁ Definability of the Nonstationary Ideal: Main Result

Now I can describe the iteration $P = (P_\xi, \dot{Q}_\xi \mid \xi < \kappa^+)$.

In $\kappa^+$ steps, choose via bookkeeping names for stationary subsets $S$ of $\kappa$, code such $S$ by killing the stationarity of selected canonical stationary subsets $S_i$ of $\kappa^+$ and localise these stationary-kills, thereby producing local witnesses to the stationarity of each stationary subset $S$ of $\kappa$.

The iteration uses supports of size $\kappa$ for killing the stationarity of selectd $S_i$’s and supports of size less than $\kappa$ for the localisation forcings.

There are three things to check about the iteration:
$\Delta_1$ Definability of the Nonstationary Ideal: Main Result

1. The iteration is $\kappa$-distributive.
   We show that $P_\xi$ is $\kappa$-distributive by induction on $\xi \leq \kappa^+$. Of course the induction hypothesis is stronger than this; we need to know that we can build conditions which serve as strong master conditions for each model in a sequence of models of length $\lambda + 1$ built by taking successive Skolem hulls. So the argument is Jensen-style, tracing back to his coding work, and not Shelah-style; even in the case $\kappa = \omega_1$ there is no form of properness available.

2. Any stationary subset of $\kappa$ that arises during the iteration remains forever stationary.
   Again we need to build a strong master condition for each model in a sequence of models built by taking successive Skolem hulls, but now the sequence has arbitrary successor length less than $\kappa$. A $\square_\lambda$ sequence is used to thin out such a sequence to a subsequence of length at most $\lambda + 1$. 
3. A canonical stationary set $S_i \subseteq \kappa^+$ remains stationary unless in the course of the iteration its stationarity is explicitly killed in order to code some stationary $S \subseteq \kappa$.

Of course here we use the fact that the forcings to kill stationarity of selected $S_i$’s (the “upper part”) are $\kappa$-closed and the localisation forcings (the “lower part”) are $\kappa^+$-cc.

3 implies that $\kappa^+$ is preserved.
As the entire iteration has a dense subset of size $\kappa^+$ all cardinals are preserved and GCH holds at cardinals $\geq \kappa$.
GCH holds below $\kappa$ as no bounded subsets of $\kappa$ are added.
Finally, by localisation together with the fact that no $S_i$ “accidentally” loses its stationarity, we have that $S \subseteq \kappa$ is stationary iff $S$ has a local witness, a $\Sigma_1$ property with parameter $\kappa$.
So the Theorem is proved.
In classical Baire Space, the Baire Property for all $\Delta_1 (= \Delta^1_2)$ sets of reals is equivalent to the existence of a Cohen real over $L[x]$ for each real $x$.

In our model where $\text{NS}_\kappa$ is $\Delta_1$ (for a successor $\kappa$) we have the existence of a $\kappa$-Cohen set over $L[x]$ for each $x \subseteq \kappa$.

As Halko-Shelah showed that $\text{NS}_\kappa$ does not have the Baire Property, our result shows that the classical characterisation of the $\Delta_1$ Baire Property does not generalise to successor $\kappa$. 

**Descriptive Set Theory on $\kappa$-Baire Space**
Δ₁ Definability of the Nonstationary Ideal: Further Remarks

\textit{When } \kappa = \omega₁

Wu and I showed that NS\(\omega_1\) can be both precipitous and \(\Delta₁\),
starting with a measurable, extending a result of Magidor.
Woodin showed that NS\(\omega_1\) can be \(\omega_1\)-dense, and therefore both \(\Delta₁\)
and saturated, using \(\omega\) Woodin cardinals.
Hoffelner and I get that NS\(\omega_1\) can be saturated and \(\Delta₁\) (together
with a \(\Sigma^1_4\) wellorder of the reals) using just one Woodin cardinal.

There are many further questions to ask about the \(\Delta₁\) definability
of NS\(\kappa\), regarding inaccessible \(\kappa\), failures of GCH and saturation for
\(\kappa > \omega₁\), but I’ll stop here.

THANKS!