Forcing when there are Large Cardinals

Summary:

1. What are large cardinals?

2. Forcings which preserve large cardinals (failure of GCH at a measurable)

3. Forcings which destroy large cardinals, but do something interesting (Singular Cardinal Hypothesis)

4. Some open questions
What are large cardinals?

\( \kappa \) is \textit{inaccessible} iff:
- \( \kappa > \aleph_0 \)
- \( \kappa \) is regular
- \( \lambda < \kappa \to 2^\lambda < \kappa \)

\( \kappa \) inaccessible implies \( V_\kappa \) is a model of ZFC

\( \kappa \) is \textit{measurable} iff:
- \( \kappa > \aleph_0 \)
- \( \exists \) nonprincipal, \( \kappa \)-complete ultrafilter on \( \kappa \)
What are large cardinals?

**Embeddings:**

\[ V = \text{universe of all sets} \]
\[ M \text{ an inner model (transitive class satisfying ZFC, containing Ord)} \]

\[ j : V \to M \] is an embedding iff:
\[ j \text{ is not the identity} \]
\[ j \text{ preserves the truth of formulas with parameters} \]

**Critical point** of \( j \) is the least \( \kappa, j(\kappa) \neq \kappa \)

Idea: \( \kappa \) is “large” iff \( \kappa \) is the critical point of an embedding \( j : V \to M \) where \( M \) is “large”
What are large cardinals?

Suppose that $\kappa$ is the critical point of $j : V \to M$

$\kappa$ is $\lambda$-hypermeasurable iff $H(\lambda) \subseteq M$

$\kappa$ is $\lambda$-supercompact iff $M^\lambda \subseteq M$

Fact: Measurable $= \kappa^+$-hypermeasurable $= \kappa$-supercompact.

Kunen: No $j : V \to M$ witnesses $\lambda$-hypermeasurability for all $\lambda$, i.e., $M$ cannot equal $V$

However: $\kappa$ could be $\lambda$-hypermeasurable for all $\lambda$ (i.e., the critical point of embeddings with arbitrary degrees of hypermeasurability)
Forcings that preserve large cardinals

Question: Suppose \( \kappa \) is a large cardinal and \( G \) is \( P \)-generic over \( V \). Is \( \kappa \) still a large cardinal in \( V[G] \)?

*Lifting method (Silver):*

Given \( j : V \to M \) and \( G \) which is \( P \)-generic over \( V \)

Let \( P^* \) be \( j(P) \)

Goal: Find a \( G^* \) which is \( P^* \)-generic over \( M \) such that \( j[G] \subseteq G^* \)

Then \( j : V \to M \) lifts to \( j^* : V[G] \to M[G^*] \), defined by

\[ j^*(\sigma^G) = j(\sigma)^{G^*} \]

If \( G^* \) belongs to \( V[G] \) then \( \kappa \) is still measurable (and maybe more) in \( V[G] \)
Forcings that preserve large cardinals

An example: Making GCH fail at a measurable cardinal

**Theorem**

Suppose that $\kappa$ is $\kappa^{++}$-hypermeasurable. Then in a forcing extension, $\kappa$ is still measurable and $2^\kappa = \kappa^{++}$.

Theorem is due to Woodin; the proof below is due to Katie Thompson and myself.

Step 1. Choose a forcing to make GCH fail at kappa.

Obvious choice: $\text{Cohen}(\kappa, \kappa^{++})$
Add $\kappa^{++}$-many $\kappa$-Cohen sets
Conditions are partial functions of size $< \kappa$ from $\kappa \times \kappa^{++}$ to 2

Better choice: $\text{Sacks}(\kappa, \kappa^{++})$
Add $\kappa^{++}$-many $\kappa$-Sacks subsets of $\kappa$ (defined later)
Forcings that preserve large cardinals

Step 2: Prepare below $\kappa$

Here is the problem (illustrated using just $\kappa$-Cohen forcing):
Suppose that $C \subseteq \kappa$ is $\kappa$-Cohen generic
Want to lift $j : V \to M$ to $j^* : V[C] \to M[C^*]$
Need to find $C^*$ which is $j(\kappa)$-Cohen generic over $M$ and “extends” $C$, i.e., such that $C = C^* \cap \kappa$
Impossible! $C$ does not belong to $M$!
Need the forcing to add $C^*$ to be defined not in $M$ but in a model that already has $C$

Solution: Force not just at $\kappa$, but at all inaccessible $\alpha \leq \kappa$, via an iteration

$$P = P(\alpha_0) \ast P(\alpha_1) \ast \cdots \ast P(\kappa)$$

where $P(\alpha)$ denotes $\alpha$-Cohen forcing.
Let $C(\alpha_0) \ast C(\alpha_1) \ast \cdots \ast C(\kappa)$ denote the $P$-generic
Forcings that preserve large cardinals

Now we want to lift \( j : V \to M \) to
\[ j^* : V[C(\alpha_0) \ast C(\alpha_1) \ast \cdots \ast C(\kappa)] \to M[C^*(\alpha_0) \ast C^*(\alpha_1) \ast \cdots \ast C^*(\kappa) \ast C^*(\beta_0) \ast C^*(\beta_1) \ast \cdots \ast C^*(j(\kappa))] \]
where the \( \beta_i \)'s are the inaccessibles of \( M \) between \( \kappa \) and \( j(\kappa) \).

To find the \( C^* \)'s:
Set \( C^*(\alpha) = C(\alpha) \) for \( \alpha < \kappa \)
Set \( C^*(\kappa) = C(\kappa) \)
Take \( \langle C^*(\beta) \mid \kappa < \beta < j(\kappa) \rangle \) to be any generic (they exist)

Last lift: Take \( C^*(j(\kappa)) \) to be any generic for \( j(\kappa) \)-Cohen forcing of
\[ M[C^*(\alpha_0) \ast C^*(\alpha_1) \ast \cdots \ast C^*(\kappa) \ast C^*(\beta_0) \ast C^*(\beta_1) \ast \cdots \] containing the condition \( C(\kappa) = C^*(\kappa) \) (such generics exist).
Forcings that preserve large cardinals

Step 3: Make this work with $\kappa$-Cohen forcing replaced by some forcing that kills the GCH at $\kappa$

Here is the problem:

For inaccessible $\alpha \leq \kappa$ replace $\alpha$-Cohen by Cohen($\alpha$, $\alpha^{++}$)

All goes well until the last lift: we can choose $C^*(\gamma)$ for all $M$-inaccessible $\gamma < j(\kappa)$ and lift $j : V \rightarrow M$ to $j' : V[C(\alpha_0) \ast C(\alpha_1) \ast \cdots] \rightarrow$ $M[C^*(\alpha_0) \ast C^*(\alpha_1) \ast \cdots \ast C^*(\kappa) \ast C^*(\beta_0) \ast C^*(\beta_1) \ast \cdots]$

We then need to find a generic for the Cohen($j(\kappa)$, $j(\kappa^{++})$)-forcing of $M[C^*(\alpha_0) \ast C^*(\alpha_1) \ast \cdots \ast C^*(\kappa) \ast C^*(\beta_0) \ast C^*(\beta_1) \ast \cdots]$ which contains $j'[C(\kappa)]$. But Cohen($j(\kappa)$, $j(\kappa^{++})$) is a very big forcing (it may have no generic; we may have to force one!) and $j'[C(\kappa)]$ is a very complicated set of conditions in this forcing (it is not easy to force a generic that contains it!)
Forcings that preserve large cardinals

Here is the solution: Use Sacks($\kappa, \kappa^{++}$) instead of Cohen($\kappa, \kappa^{++}$). Then we don’t have to build a generic $S^*(j(\kappa))$ for Sacks($j(\kappa), j(\kappa^{++})$) because $j'[S(\kappa)]$ builds one for us!

Illustrate with $\kappa$-Sacks: A condition is a perfect $\kappa$-tree with a closed unbounded set of splitting levels. If $G$ is generic then the intersection of the $\kappa$-trees in $G$ gives us a function $g : \kappa \to 2$.

**Lemma**

*(Tuning Fork Lemma)* Suppose that $j : V' \to M'$ has critical point $\kappa$, $g$ is $\kappa$-Sacks generic over $V'$, $M'$ is included in $V'[g]$ and $g$ belongs to $M'$. Then in $V'[g]$ there are exactly two generics $h_0, h_1$ for the $j(\kappa)$-Sacks of $M'$ extending $g$; moreover $h_0(\kappa) = 0$ and $h_1(\kappa) = 1$.

A similar result holds for Sacks($\kappa, \kappa^{++}$), thereby solving the problem of the “last lift”.
Forcings that preserve large cardinals

Some other applications:

(with Magidor) Assume GCH, let $\kappa$ be measurable and let $\alpha$ be any cardinal at most $\kappa^{++}$. Then there is a cofinality-preserving forcing extension in which there are exactly $\alpha$-many normal measures on $\kappa$.

(with Dobrinen) Assume GCH and let $\kappa$ be $\lambda$-hypermeasurable where $\lambda$ is weakly compact and greater than $\kappa$. Then there is a forcing extension in which $\kappa$ is still measurable and the tree property holds at $\kappa^{++}$.

(with Zdomskyy) Assume GCH and let $\kappa$ be $\kappa^{++}$-hypermeasurable. Then there is a cofinality-preserving forcing extension in which $\kappa$ is still measurable and the symmetric group on $\kappa$ has cofinality $\kappa^{++}$.\)
Forcings which use large cardinals: The SCH

Singular cardinal hypothesis (SCH):
If $2^{\text{cof}(\kappa)} < \kappa$ then $\kappa^{\text{cof}(\kappa)} = \kappa^+$
SCH $\Rightarrow$ GCH holds at singular strong limit cardinals

**Theorem**

(Prikry) Suppose that $\kappa$ is measurable and the GCH fails at $\kappa$. Then in a forcing extension, $\kappa$ is still a strong limit cardinal where the GCH fails, but now $\kappa$ has cofinality $\omega$. In particular, the SCH fails in this forcing extension.

Prikry forcing: A forcing that preserves cardinals, adds no new bounded subsets of $\kappa$ but adds an $\omega$-sequence cofinal in $\kappa$
Forcings which use large cardinals: The SCH

Conditions in Prikry forcing:
Fix a normal measure $U$ on $\kappa$. A condition is a pair $(s, A)$ where $s$ is a finite subset of $\kappa$ and $A$ belongs to $U$.

Extension in Prikry forcing:
$(t, B)$ extends $(s, A)$ iff
$t$ end-extends $s$
$B$ is a subset of $A$
t $\setminus s$ is contained in $A$

Facts: (a) If $G$ is $P$-generic then $\bigcup \{s \mid (s, A) \in G \text{ for some } A\}$ is an $\omega$-sequence cofinal in $\kappa$.
(b) $P$ is $\kappa^+$-cc: If $(s, A), (t, B)$ are conditions and $s = t$ then $(s, A)$ and $(t, B)$ are compatible.
The main lemma about Prikry forcing is the following. We say that 
$(t, B)$ is a direct extension of $(s, A)$ iff $s = t$ and $B$ is a subset of $A$.

**Lemma (The Prikry property)**

*For $\sigma$ a sentence of the forcing language, every condition has a
direct extension which decides $\sigma$ (i.e., either forces $\sigma$ or $\sim \sigma$).*
Lemma (The Prikry property)

For $\sigma$ a sentence of the forcing language, every condition has a direct extension which decides $\sigma$ (i.e., either forces $\sigma$ or $\sim \sigma$).

Proof. Suppose that $(s, A)$ is a condition and define $h : [A]<\omega \to 2$ as follows:

$h(t) = 1$ iff $(s \cup t, B) \models \sigma$ for some $B$

$h(t) = 0$ otherwise.

As $U$ is normal there is $A^* \in U$ which is homogeneous for $h$: For each $n$ and $t_1, t_2 \in [A^*]^n$, $h(t_1) = h(t_2)$. Then $(s, A^*)$ decides $\sigma$: Otherwise there would be $(s \cup t_1, B_1), (s \cup t_2, B_2)$ extending $(s, A^*)$ which force $\sigma, \sim \sigma$, respectively. We can assume that for some $n$, both $t_1$ and $t_2$ belong to $[A^*]^n$. But then $h(t_1) = 1, h(t_2) = 0$, contradicting homogeneity. $\square$
Corollary: $P$ does not add new bounded subsets of $\kappa$.

Proof. Suppose $(s, A) \models \dot{a}$ is a subset of $\lambda$, where $\lambda$ is less than $\kappa$. Set $(s, A_0) = (s, A)$ and using the Prirky property choose a direct extension $(s, A_1)$ of $(s, A_0)$ which decides “$0 \in \dot{a}$”. Then choose a direct extension $(s, A_2)$ of $(s, A_1)$ which decides “$1 \in \dot{a}$”, etc. After $\lambda$ steps we have a direct extension $(s, A_\lambda)$ of $(s, A)$ which decides which ordinals less than $\lambda$ belong to $\dot{a}$, and therefore forces $\dot{a}$ to belong to the ground model. □

In summary: If $G$ is $P$-generic then $\kappa$ has cofinality $\omega$ in $V[G]$ and $V$, $V[G]$ have the same cardinals and bounded subsets of $\kappa$. In particular, if GCH fails at $\kappa$ in $V$, then in $V[G]$, $\kappa$ is a singular strong limit cardinal where the GCH fails.
Forcings which use large cardinals: The SCH

An improvement: Model where $\aleph_\omega$ is strong limit and the GCH fails at $\aleph_\omega$

Theorem

(Magidor) Suppose that $\kappa$ is measurable. Then there is a forcing extension in which $\kappa$ equals $\aleph_\omega$.

For the proof, mix Prikry forcing with Lévy collapses:
Suppose that $\alpha < \beta$ are regular. Then $\text{Lévy}(\alpha, \beta)$ is a forcing that makes $\beta$ into $\alpha^+$ and otherwise preserves cardinals:

$p \in \text{Lévy}(\alpha, \beta)$ iff $p$ is partial function of size $< \alpha$ from $\alpha \times \beta$ to $\beta$ such that $p(\alpha_0, \beta_0) < \beta_0$ for each $(\alpha_0, \beta_0)$ in the domain of $p$. 
Collapsing Prikry forcing: 1st try

Fix a normal measure $U$ on $\kappa$. A condition is of the form $((\alpha_0, p_0), (\alpha_1, p_1), \ldots, (\alpha_{n-1}, p_{n-1}), A)$ where:

$\alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \kappa$ are inaccessible

$p_i$ belongs to Lévy($\alpha_i, \alpha_{i+1}$) for $i < n - 1$

$p_{n-1}$ belongs to Lévy($\alpha_{n-1}, \kappa$)

$A$ belongs to $U$

To extend: Strengthen the $p_i$’s, increase $n$, shrink $A$ and take the new $\alpha$’s from the old $A$

Problem: This collapses $\kappa$ to $\omega$ (the $p_i$’s are running wild!)

Solution: Control the $p_i$’s on a measure one set
Collapsing Prikry forcing: 2nd try
Let $j : V \rightarrow M$ witness that $\kappa$ is measurable and choose $U$ to be the normal measure $\{A \mid \kappa \in j(A)\}$

Guiding generic: Choose $G$ in $V$ to be generic over $M$ for $\text{Lévy}(\kappa^+, j(\kappa))$ of $M$ (this is possible)

Now define a condition to be of the form
$((\alpha_0, p_0), (\alpha_1, p_1), \ldots, (\alpha_{n-1}, p_{n-1}), A, F)$ where:

$\alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \kappa$ are inaccessible

$p_i$ belongs to $\text{Lévy}(\alpha_i^+, \alpha_{i+1})$ for $i < n - 1$

$p_{n-1}$ belongs to $\text{Lévy}(\alpha_{n-1}^+, \kappa)$

$A$ belongs to $U$

$F$ is a function with domain $A$ such that $F(\alpha)$ belongs to $\text{Lévy}(\alpha^+, \kappa)$ for each inaccessible $\alpha$ in $A$

$j(F)(\kappa)$ belongs to $G$
Forcings which use large cardinals: The SCH

An extension of
\[ p = ((\alpha_0, p_0), (\alpha_1, p_1), \ldots, (\alpha_{n-1}, p_{n-1}), A, F) \]
is of the form
\[ p^* = ((\alpha_0^*, p_0^*), (\alpha_1^*, p_1^*), \ldots, (\alpha_{n^*-1}^*, p_{n^*-1}^*), A^*, F^*) \]
where:
- \( n^* \) is at least \( n \)
- \( \alpha_i^* = \alpha_i \) and \( p_i^* \) extends \( p_i \) for \( i < n \)
- \( p_j^* \) extends \( F(\alpha_j^*) \) for \( j \geq n \)
- \( A^* \) is contained in \( A \)
- \( F^*(\alpha) \) extends \( F(\alpha) \) for each \( \alpha \in A^* \)
- \( p^* \) is a direct extension of \( p \) if in addition \( n^* = n \)

A generic produces a Prikry sequence \( \alpha_0 < \alpha_1 < \cdots \) in \( \kappa \) together with Lévy collapses \( g_0, g_1, \ldots \) where \( g_i \) ensures \( \alpha_{i+1} = \alpha_i^{++} \). So after collapsing \( \alpha_0 \), we see that \( \kappa \) is at most \( \aleph_\omega \). The forcing is \( \kappa^+-\text{cc} \). But why isn’t \( \kappa \) collapsed?
Forcings which use large cardinals: The SCH

The Prikry property: For $\sigma$ a sentence of the forcing language, every condition has a direct extension which decides $\sigma$.

Using this, one gets: Any bounded subset of $\kappa$ belongs to $V[g_0, g_1, \ldots, g_n]$ for some $n$, and therefore $\kappa$ remains a cardinal.

Summary: Prikry Collapse forcing makes $\kappa$ into $\aleph_\omega$ and preserves cardinals above $\kappa$.

Now start with $\kappa$ measurable and GCH failing at $\kappa$.

Then Prikry Collapse forcing makes $\kappa$ into $\aleph_\omega$ with $\aleph_\omega$ strong limit, GCH failing at $\aleph_\omega$ (Strong failure of the SCH).
1. Preserving large cardinals
Consider various cardinal characteristics of the continuum (almost-disjointness number, bounding number, dominating number, splitting number, ...)
How do these behave at a large cardinal?
Is it consistent that a strongly compact cardinal have a unique normal measure?
Is it consistent with a supercompact cardinal for $H(\kappa^+)$ to have a definable wellordering for every uncountable $\kappa$?
Open Questions

2. Using large cardinals
(SCH-type problems): What are the possibilities for the function $n \mapsto 2_n^\aleph_n$ for $n \leq \omega$?

Is it consistent that there is no $\kappa$-Aronszajn tree for any regular cardinal $\kappa > \omega_1$?

Is it consistent to have stationary reflection at the successor of each singular cardinal?

Can the nonstationary ideal on $\omega_1$ be saturated with CH?

Can $\aleph_\omega$ be Jonsson?