Section 1. **Background for the Theory.**

*α*-Recursion Theory, or Recursion Theory on \( \Sigma_1 \)-admissible ordinals, is certainly a very successful generalization of Recursion Theory on \( \omega \). It has demonstrated that constructions exploiting the \( \Sigma_2 \) or \( \Sigma_3 \) admissibility of \( \omega \) can be replaced by finer constructions which rely only on \( \Sigma_1 \) admissibility.

The minimal degree construction however has resisted generalization to arbitrary admissible ordinals, though important progress was made by Richard Shore ([14]) who treated the \( \Sigma_2 \)-admissible case. It is instructive to examine his argument, as the attempt to adapt it to all admissible ordinals led to the development of \( \beta \)-Recursion Theory.

Shore's construction can be outlined as follows: If \( \alpha \) is \( \Sigma_2 \) admissible, then the structure \( \langle L_\alpha, \epsilon, C \rangle \) is admissible where \( C \) is a complete \( \alpha \)-recursively enumerable set. A minimal \( \alpha \)-degree can then be constructed by applying the \( \alpha \)-finite injury method to this structure, much in the way Sacks and Simpson ([13]) first used it in their solution to Post's Problem.

What if \( \alpha \) is only \( \Sigma_1 \) admissible? Then the structure \( \langle L_\alpha, \epsilon, C \rangle \) is no longer admissible. Nonetheless, is it still possible to apply the \( \alpha \)-finite injury method to this structure, yielding a minimal \( \alpha \)-degree?

This leads to the study of Recursion Theory on possibly inadmissible structures \( \langle L_\beta[A], \epsilon, A \rangle \), or \( \beta \)-Recursion Theory. There are two goals for this theory:
1) To produce new constructions of recursively enumerable sets which are not dependent on any admissibility assumption.

2) To clarify the concepts and techniques used in Admissible Recursion Theory. In both cases we hope to provide new data in the search for the axioms needed to do Recursion Theory. In this paper we report on the progress that has been made in these directions.

Section 2. Basic Notions.

The correct general setting for \(\beta\)-Recursion Theory is Jensen's \(S\)-Hierarchy for \(L\). For limit ordinals \(\beta, S_\beta\) has all of the important properties shared by limit levels of Gödel's \(L\)-Hierarchy. We proceed to define the \(S\)-Hierarchy and list these properties, referring the reader to [1] for a more thorough treatment. For ordinals \(\beta\) such that \(\omega^\omega\) divides \(\beta\), we have \(S_\beta = L_\beta\), so in this case one may work with the usual \(L\)-Hierarchy.

A function \(f: V^n \to V\) is \underline{rudimentary} if it can be generated from the following schemata:

\begin{enumerate}
  \item \(f(x_1, \ldots, x_n) = x_i\)
  \item \(f(\<x\>) = \{x_i, x_j\} \quad \text{for } 1 \leq i, j \leq n\)
  \item \(f(\<x\>) = x_i - x_j\)
  \item \(f(\<x\>) = h(g_1(\<x\>), \ldots, g_k(\<x\>))\)
  \item \(f(y, \<x\>) = \bigcup_{z \in y} g(z, \<x\>)\)
\end{enumerate}

\(R \subseteq V^n\) is \underline{rudimentary} if for some rudimentary function \(f, \<x\> \in R \iff f(\<x\>) = \emptyset\).

If \(X\) is transitive, the \underline{rudimentary closure} of \(X\) = the closure of \(X\) under the rudimentary functions. Also define \(\text{rud}(X) = \text{rudimentary closure of } X \cup \{X\}\).

Lemma. There is a rudimentary function \(S\) such that for transitive \(X\), \(S(X)\) is transitive, \(X \cup \{X\} \subseteq S(X)\) and \(\text{rud}(X) = \bigcup_{n \in \omega} S^n(X)\).
The S-Hierarchy is now defined by:

\[ S_0 = \emptyset, \]
\[ S_{\alpha+1} = S(S_\alpha), \]
\[ S_\lambda = \bigcup_{\alpha < \lambda} S_\alpha. \]

Properties of the S-Hierarchy

1) \( \beta \) limit \( \rightarrow S_\beta \) is closed under rudimentary functions, \( \text{On}(S_\beta) = \beta. \)

2) \( S_\beta \cap \bigotimes(S_\beta) = \{X \subseteq S_\beta | X \text{ is first-order definable over } <S_\beta, \varepsilon>\}. \)

3) \( L_\beta = S_\beta \iff \omega^\omega \text{ divides } \beta, \bigcup_{\beta} S_\beta = L. \)

4) Suppose \( \beta \) is a limit ordinal and \( \text{Tr}_n = \{<i, x_1, \ldots, x_n> | \text{ the } i\text{th } \Sigma_n \text{ formula } \varphi \text{ is } n\text{-ary and } <S_\beta, \varepsilon> \models \varphi(x_1, \ldots, x_n)\}. \) Then \( \text{Tr}_n \) is \( \Sigma_n \) over \( <S_\beta, \varepsilon> \), uniformly in \( \beta. \)

5) There is a well-ordering \( < \) of \( L \) such that for limit \( \beta, <S_\beta \times S_\beta > \text{ is } \Sigma_1 \over <S_\beta, \varepsilon>, \) uniformly in \( \beta. \)

The above properties are sufficient to safely adapt the definitions of \( \alpha \)-Recursion Theory to an arbitrary \( <S_\beta, \varepsilon>, \) \( \beta \) limit.

For \( A \subseteq S_\beta \), we define:

- \( A \) is \( \beta \)-recursively enumerable (\( \beta \)-r.e.) \( \iff \) \( A \) is \( \Sigma_1 \)-Definable over \( <S_\beta, \varepsilon> \).
- \( A \) is \( \beta \)-recursive (\( \beta \)-rec) \( \iff \) \( A, \overline{A} = S_\beta - A \) are both \( \beta \)-r.e.
- \( A \) is \( \beta \)-finite \( \iff \) \( A \subseteq S_\beta \).

By virtue of property 4) above, there is a universal \( \beta \)-r.e. set \( W(e, x) \) such that any \( \beta \)-r.e. set is of the form \( W_e = \{x | W(e, x)\} \) for some \( e. \)

The reducibilities of \( \beta \)-Recursion Theory are also derived from the admissible case. For any \( A \subseteq S_\beta \), let \( A^* = \{<x, y> | x \subseteq A, y \subseteq \overline{A}, x, y \beta \text{-finite}\} \) and \( A^*_f = \{<x, y> | x \subseteq A, y \subseteq \overline{A}, x, y \text{ finite}\}. \) Then:

- \( A \) is finitely \( \beta \)-reducible to \( B \) \( \iff \) (for some \( \beta \)-r.e. \( R), \)

\[ (A \leq_f B) \quad x \in A^*_f \iff \exists y \in B^*_f[<x, y> \in R]) \]
A is weakly $\beta$-reducible to B $\iff A^*_1 \leq_f^\beta B^*$

(A $\leq_{w\beta} B$)

A is $\beta$-reducible to B $\iff A^* \leq_f^\beta B^*$

(A $\leq_{\beta} B$)

Both finitely $\beta$-reducible and $\beta$-reducible imply weakly $\beta$-reducible (but not conversely in general) and these two notions are in general incomparable.

$\beta$-reducibility is the fundamental reducibility for $\beta$-Recursion Theory.

$\beta$-reducibility is reflexive and transitive and thus we may define the $\beta$-degree of $A = \{B \mid A \leq_{\beta} B, B \leq_{\beta} A\}$; the $\beta$-degree are partially-ordered by $\leq_{\beta}$. There is a smallest $\beta$-degree 0 and a largest $\beta$-degree of a $\beta$-r.e. set, denoted $0'$.

Tameness and Regularity

$\beta$-r.e. sets are constructed in stages. As computations from a set $A$ are determined by pairs $<x, y>$, where $x$ and $y$ are $\beta$-finite and satisfy $x \subseteq A$, $y \subseteq \overline{A}$, it is convenient to know that such pairs are satisfied by some stage of the construction; i.e., if $A^\sigma$ denotes the amount of $A$ enumerated by stage $\sigma$, we would like to know that:

(*) $x$ $\beta$-finite, $x \subseteq A \implies$ For some $\sigma$, $x \subseteq A^\sigma$.

In this case, the enumeration $\{A^\sigma\}_{\sigma < \beta}$ is said to be tame. More precisely, if $\exists y \varphi(x, y)$ ($\varphi$ a $\Delta_0$ formula) defines $A$ over $<S_\beta, \epsilon>$, then it gives rise to the enumeration $\{A^\sigma\}$ where $x \in A^\sigma \iff \exists y \in S_\sigma \varphi(x, y) \land x \in S_\sigma$. $A$ is tamely-r.e. if $A$ has such an enumeration with the property (*) above. This is equivalent to the assertion that $\{\sigma$-finite $x \mid x \subseteq A\}$ is $\beta$-r.e. From this it follows that $\deg A = 0 \iff A, \overline{A}$ are both tamely-r.e.

**Theorem 1 ([3]).** Assume that $\beta$ is inadmissible. Let $W$ be a universal $\beta$-r.e. set. Then there is a $\beta$-recursive set $A$ such that $0 < \beta A < \beta W$ and every $\beta$-recursive or tamely-r.e. set is $\beta$-reducible to $A$. 
It follows that in the inadmissible case there are $\beta$-recursive sets of non-zero $\beta$-degree. In particular, t.r.e. $\not\leq_{\text{r.e.}}$ and $\leq_{\text{w} \beta} \not\leq_{\beta}$. Theorem 1 provides a weak solution to Post's Problem in $\beta$-Recursion Theory (the question of the existence of $\beta$-r.e. degrees between 0 and 0'). The solution is weak because it does not provide incomparable $\beta$-r.e. degrees; moreover, in this case we have $W \leq_{\text{w} \beta} A \leq_{\text{w} \beta} 0$. Denoting $\deg(A)$ by $0^{1/2}$, we are led to the following picture of the $\beta$-r.e. degrees:

In general, however, nonzero tamely-r.e. degrees will not exist (though $0^{1/2}$ provides an example of a nonzero $\beta$-recursive degree). Simple questions regarding these degrees remain unsettled; in particular, it is not known if the t.r.e. degrees or the $\beta$-recursive degrees form an initial segment of the $\beta$-r.e. degrees. It follows from [10], though, that the t.r.e. degrees are always contained in the $\beta$-recursive degrees when $\beta$ is inadmissible.

$A \subseteq_{\beta} S$ is regular if $A \cap x$ is $\beta$-finite whenever $x$ is $\beta$-finite. It is a theorem of Sacks ([12]) in the case that $\beta$ is admissible that every $\beta$-r.e. degree has a regular $\beta$-r.e. representative. Regular $\beta$-r.e. sets are more rare for inadmissible $\beta$; in fact, for some $\beta$'s, every regular $\beta$-r.e. set (even every regular set) has degree 0 (see [3]). However, a slight extension of t.r.e.-ness is enough to guarantee the existence of regular, $\beta$-r.e. representatives:
Theorem 2 ([3],[10]). If \( \{ x \mid x \text{ is } \beta\text{-finite}, x \cap A \neq \beta \} \) is t. r. e., then \( A \) has the same \( \beta \)-degree as a regular \( \beta \)-r.e. set.

However, for some \( \beta \)'s there are t. r. e. sets which do not lie in the same \( \beta \)-degree as some regular set. Thus, there appears to be no simple characterization of the regular \( \beta \)-r.e. degrees.

The reader is referred to [10] for proofs of the above facts as well as further information concerning the \( \beta \)-r.e., t. r. e., and regular \( \beta \)-r.e. degrees.

In the further development of the theory, the limit ordinals fall into two classes determined by their degree of admissibility. This split into cases was first revealed in Jensen's proof of \( \Sigma_2 \)-Uniformization for \( S^\beta \) and is determined by the values of certain key parameters which we now proceed to define.

A relation on \( S^\beta \) is \( \Sigma_n \) if it can be defined over \( <S^\beta, \epsilon> \) by a formula consisting of \( n \) alternating unbounded quantifiers (beginning with an existential) followed by a limited formula. A function is \( \Sigma_n \) if its graph is.

The first type of parameter that we define measures the extent to which \( \beta \) is not a cardinal. The \( \Sigma_n \)-projectum of \( \beta \), \( p^\beta_n \), is the least ordinal \( \gamma \) such that there is a \( \Sigma_n \) injection of \( \beta \) into \( \gamma \). Jensen shows ([9]) that this is the same as the least \( \gamma \) such that some \( \Sigma_n \) subset of \( \gamma \) is not \( \beta \)-finite. As there is always a \( \Sigma_1 \) bijection between \( \beta \) and \( S^\beta \) (see [1]), we can in fact inject \( S^\beta \) into \( p^\beta_n \) via a \( \Sigma_n \) function.

Our second set of parameters describes the extent to which \( \beta \) is singular.

The \( \Sigma_n \)-cofinality of \( \beta \), \( \Sigma_n \text{cf} \beta \), is the least \( \gamma \) such that some \( \Sigma_n \) function with domain \( \gamma \) has range unbounded in \( \beta \). If \( \beta \) is \( \Sigma_{n-1} \)-admissible, then this is the same as the least \( \gamma \) such that some \( \Sigma_n \) function with domain \( \gamma \) is not \( \beta \)-finite (though this equivalence is not true for all \( \beta \)).
In case $n = 1$, $\rho_1^\beta$ and $\Sigma_1 \text{cf} \beta$ are alternatively written $\beta^*$ and $\text{Rcf} \beta$, respectively. ($\text{Rcf}$ abbreviates "Recursive Cofinality"). We shall be mostly concerned with $\beta^*$, $\text{Rcf} \beta$, $\rho_2^\beta$ and $\Sigma_2 \text{cf} \beta$. Note that if $\beta$ is admissible and $A$ is a regular $\beta$-r.e. set of degree $0'$, then $\rho_2^\beta$ and $\Sigma_2 \text{cf} \beta$ are just the $\Sigma_1$-projectum and $\Sigma_1$ cofinality of the relativized structure $\langle L_{\beta'}, \epsilon, A \rangle$.

In case $\text{Rcf} \beta \geq \beta^*$ we say that $\beta$ is weakly admissible. In this case, many of the arguments from admissibility theory apply. The reason for this is that many priority arguments use $\beta^*$ to index a listing of requirements and the above assumption allows one to perform $\Sigma_1$ inductions of length $\beta^*$.

$\Sigma_2$-Uniformization is also easy in this case. If $\beta$ is admissible and $\Sigma_2 \text{cf} \beta \geq \rho_2^\beta$, then we say that $\beta$ is weakly $\Sigma_2$-admissible. In this case, one can carry out the construction of minimal $\beta$-degrees, minimal pairs of $\beta$-r.e. degrees and major subsets of $\beta$-r.e. sets.

If $\text{Rcf} \beta < \beta^*$ we say that $\beta$ is strongly inadmissible. In this case, the arguments of admissibility theory do not apply and new techniques are needed.

This is the difficult case of $\Sigma_2$-Uniformization. If $\beta$ is admissible and $\Sigma_2 \text{cf} \beta < \rho_2^\beta$, then we say that $\beta$ is strongly $\Sigma_2$-inadmissible. The constructions of minimal $\beta$-degrees, minimal pairs of $\beta$-r.e. degrees and major subsets of $\beta$-r.e. sets are all very difficult for such $\beta$ and have only been accomplished in very special cases. However, the techniques of $\beta$-Recursion Theory are now beginning to apply themselves to this case (see Section 5).

Section 3. Weak Admissibility

As mentioned before, the methods of $\alpha$-Recursion Theory apply in this case. In particular, the method of Shore blocking (see [17]) was used in [3] to prove:
Theorem 3. If $\beta$ is weakly admissible, then there are regular t. r. e. sets $A, B$ such that $A \not\leq_w^{w_\beta} B$, $B \not\leq_w^{w_\beta} A$.

W. Maass (in [10]) has found a technique for transferring many results from $\alpha$-Recursion Theory to arbitrary weakly admissible ordinals. He associates to each weakly admissible $\beta$ an admissible structure $\mathcal{U}$ such that $\mathcal{U}$-r. e. degrees embed into the $\beta$-r. e. degrees. In this way, known results about the admissible structure $\mathcal{U}$ have consequences about the $\beta$-r. e. degrees. We now describe his construction in more detail.

Let $\kappa = \Sigma_1 \text{cf} \beta$. As $\kappa \geq \beta^+$, there certainly is a $\Sigma_1$ injection of $\beta$ into $\kappa$. In fact, more is true: there is a $\Sigma_1$ bijection of $\beta$ onto $\kappa$ (see [3], p. 15).

Let $f: \beta \to \kappa$ be such a bijection. Let $<e, x, \sigma> \in \mathcal{T} \leftrightarrow x \in W_e^\sigma$

and $T = f[\widetilde{T}]$. Then $T \subseteq \kappa$, $T$ is $\beta$-recursive and $\mathcal{U} = \langle L_\kappa, \epsilon, T \rangle$ is admissible. Moreover, if $A \subseteq \kappa$, then $A$ is $\beta$-r. e. if and only if $A$ is $\mathcal{U}$-r. e. (as $\Sigma_1$ over $\mathcal{U}$). Define $\leq_{\mathcal{U}}$ analogously to $\leq_\beta$. Then these two reducibilities do not necessarily agree on subsets of $\kappa$. $A \subseteq \kappa$ is $\beta$-immune if

$x$ $\beta$-finite, $x \subseteq A \rightarrow x \in L_\kappa$

and

$x$ $\beta$-finite, $x \subseteq \kappa - A \rightarrow x \in L_\kappa$.

Then $\leq_{\mathcal{U}}$ and $\leq_\beta$ do agree on $\beta$-immune sets. Maass shows that every $\mathcal{U}$-r. e. degree has a $\beta$-immune $\mathcal{U}$-r. e. representative. This gives an embedding $E$ of the $\mathcal{U}$-r. e. degrees 1-1 into the $\beta$-r. e. degrees.

Theorem 4 (Maass) ($\beta$ weakly admissible). The range of $E$ is the t. r. e. degrees = the recursive degrees. $E$ (complete $\mathcal{U}$-r. e. set) = $0^{1/2}$.

An application of admissibility theory to $\mathcal{U}$ ([15] and [16]) yields:
Corollary. Any nonzero t.r.e degree is the join of two lesser t.r.e. degrees. If one t.r.e. degree is below another, then there is a t.r.e. degree in between.

Section 4. Strong Inadmissibility

This is the most challenging case for $\beta$-Recursion Theory, for the lack of admissibility is now so strong that many of the ideas from the admissible case become useless. The alternative is to employ deeper techniques from the Fine Structure of $L$ as developed initially by Gödel [8] and more extensively by Jensen [9]. All of these techniques emanate from two basic lemmas due to Gödel:

Lemma. For each limit ordinal $\beta$, there is a partial function $h: \omega \times S_\beta \to S_\beta$ which is $\Sigma_1$ over $S_\beta$ such that for any $\Sigma_1$ formula $\varphi(x, p)$,

$$<S_\beta, \epsilon> \models \exists x \varphi(x, p) \iff \exists i \epsilon \omega \varphi(h(i, p), p).$$

Proof. Recall the canonical $\Sigma_1$ well-ordering $<$ of $S_\beta$. Then if the $i$th $\Sigma_1$ formula is $\exists y \psi(x, p, y)$, define $h'(i, p) \approx$ least (in the sense of $<$) pair $<x, y>$ such that $\psi(x, p, y)$. Then $h(i, p) = \text{first component of } h'(i, p)$.

The $h$ above is called the canonical $\Sigma_1$ skolem function for $S_\beta$.

Transitive Collapse Lemma. If $X \triangleleft S_\beta$ (i.e., $X \subseteq S_\beta$ and any $\Sigma_1$ formula with parameters from $X$ and a solution in $S_\beta$ has a solution in $X$) then $<X, \epsilon>$ is isomorphic to a unique $<S_\gamma, \epsilon>$.

Using these two lemmas, we can now illustrate in a simple example how Fine Structure technique can be used to generalize to arbitrary $\beta$ a result whose "recursion-theoretic" proof only succeeds for admissible $\beta$. 
Proposition (Jensen). Suppose $A \subseteq \gamma < \beta^*$ and $A$ is $\beta$-r.e. Then $A$ is $\beta$-finite.

Proof Number 1, $\beta$ admissible. If $A$ is not $\beta$-finite, then it has a 1-1 $\beta$-recursive listing $f: \beta \rightarrow A$. But then $\beta^* \leq \sup A \leq \gamma < \beta^*$, contradiction.

Proof Number 2, $\beta$ arbitrary. Let $p \in S_\beta$ be a parameter defining $A$ as a set $\Sigma_1$ over $S_\beta$. Form $X = \text{Range } h$ on $\omega \times (\gamma \cup \{p\})$, where $h$ is from the Lemma. Then $X \subseteq S_\beta$, so apply the Transitive Collapse Lemma to get $j_3X \cong S_\delta$. Let $g = j \circ h$.

Now $g$ is $\Sigma_1$ over $S_\delta$ (simply transfer the $\Sigma_1$ definition for $h$ over $X$ to $S_\delta$). Then so is $g^{-1}$. But if $f$ uniformizes $g^{-1}$, $f \Sigma_1$ over $S_\delta$, we see that $f$ injects $S_\delta$ into $\omega \times (\gamma \cup \{p\})$; hence into $\gamma$. Since $\gamma < \beta^*$, we have proved that $\delta < \beta$.

But $A$ is $\Sigma_1$ definable over $S_\delta$, so $A \in S_\beta^\delta$. 

Further ideas of Jensen, in particular an effectivized version of his $\mathbf{\square}$ principle, were used in [4] to establish:

**Theorem 5.** If $\beta^*$ is regular with respect to $\beta$-recursive functions, then there are $\beta$-r.e. sets $A, B \subseteq \beta^*$ such that $A \not\leq^w \beta B, B \not\leq^w \beta A$.

This is the best solution to Post's Problem so far known in the strongly inadmissible case. This covers the case where $S_\beta \models "\beta^* is a successor cardinal"$.

**Open Problem.** Does the conclusion of Theorem 5 hold for arbitrary strongly inadmissible $\beta$?

Forcing can be used to achieve a stronger and more model-theoretic incomparability than that in Theorem 5. The following result will appear in [5]:

**Theorem 6.** Assume $\beta^*$ is regular with respect to $\beta$-recursive functions and $S_\beta \models "\beta^* is the largest cardinal. "$ Then there are $\beta$-r.e. sets $A, B \subseteq \beta^*$
such that

\[ A \text{ is not } \Delta_1 \text{ over } <S_\beta[B],\epsilon>, \]
\[ B \text{ is not } \Delta_1 \text{ over } <S_\beta[A],\epsilon>. \]

\( S_\beta[A] \) is the \( \beta \)-th level of the \( S[A] \)-hierarchy. This hierarchy is defined exactly as the \( S \)-hierarchy except the function \( f(x) = A \cap x \) is added to the schemes for the rudimentary functions.)

We conclude this section by sketching the proof of a theorem which illustrates the use of Skolem Hulls and \( \Box \) in \( \beta \)-Recursion Theory.

**Theorem 7.** There are \( \beta \)-r.e. sets \( A, B \) such that \( A \not\leq_{\beta} B, B \not\leq_{\beta} A. \)

The proof of this theorem is not uniform in the sense that it divides into cases depending on the nature of \( \beta \). Thus, the sets \( A, B \) will be defined relative to the choice of a parameter \( p \in S_\beta \).

**Open Problem.** Can Theorem 7 be made uniform in that the sets \( A, B \) have parameter-free \( \Sigma_1 \) definitions independent of \( \beta \)?

We believe that the answer is "yes." In fact, we

**Conjecture.** There are integers \( m, n \) such that for all limit ordinals \( \beta, \)

\[ W_\beta^m \text{ is not } \Delta_1 \text{ over } <S_\beta[W_\beta^m],\epsilon>, \]
\[ W_\beta^n \text{ is not } \Delta_1 \text{ over } <S_\beta[W_\beta^n],\epsilon>, \]

where \( W_\beta^m = \text{ the } n \text{ th parameter-free } \beta\text{-r.e. set.} \)

Before giving our proof sketch of Theorem 7, we make some preliminary definitions and remarks. In view of Theorem 3, it suffices to prove Theorem 7 in the strongly inadmissible case. Choose \( \beta \)-recursive functions

\[ f_0: S_\beta \xrightarrow{1-1} \beta^* \text{ and } g_0: Rcf(\beta) \rightarrow \beta \text{ such that } g_0 \text{ is order-preserving, } \text{Range } g_0 \]

unbounded in \( \beta \). Let \( p_0' \in S_\beta \) be such that both \( f_0 \) and \( g_0 \) are \( \Sigma_1 \) over \( S_\beta \) in the parameter \( p_0' \). Let \( p_0 = <p_0',\beta^*> \).
Let \( h(i, p) \) be the canonical \( \Sigma^1_1 \) Skolem function for \( S_\beta \) and for \( \gamma < \beta^+ \) define \( H(\gamma) = \{ h(i, <\gamma', p_0>) \mid i \in \omega, \gamma' < \gamma \} \). Thus \( H(\gamma) \) is the "\( \Sigma^1_1 \) Skolem Hull of \( \gamma \cup \{ p_0 \} \)". Note that \( \bigcup_{\gamma < \beta^+} H(\gamma) = S_\beta \). In our construction, \( H(\gamma) \) consists of those reduction procedures \( e \) of priority \( \gamma \). (Thus the construction is redundant in that each reduction procedure is assigned a final segment of different priorities.) Of special importance are those \( \gamma < \beta^+ \) such that \( \gamma \notin H(\gamma) \).

**Claim.** Let \( \kappa < \beta^+ \) be a \( \beta \)-cardinal (i.e., \( S_\beta \models \" \kappa \text{ is a cardinal}\" \)). Let \( \kappa^+ \) be the next \( \beta \)-cardinal. Then \( \{ \gamma < \kappa^+ \mid \gamma \notin H(\gamma) \} \) is closed, unbounded in \( \kappa^+ \).

**Proof.** See [4], Page 24.

Let \( \mathcal{A} = \{ \gamma < \beta^+ \mid \gamma \notin H(\gamma) \text{ and } \gamma \text{ is not a } \beta \text{-cardinal} \} \). The form of \( \mathcal{A} \) that we need (which more resembles \( \mathcal{A} \) in fact) reads as follows:

\[ \mathcal{A}^{\beta^+} : \text{There is a sequence } <D_{\gamma} \mid \gamma < \beta^+>, \Sigma^1_1 \text{-definable without parameter } \]

over \( L_{\beta^+} \), such that

1) \( D_{\gamma} \subseteq \text{Power Set of } \gamma \)

2) \( D_{\gamma} \in L_{\gamma^+} \) (where \( \gamma^+ = \text{least } \beta \text{-cardinal} > \gamma \))

3) If \( A \subseteq \beta^+ \) is \( \beta \)-r.e. with parameter \( p_0 \), then

\[ \gamma \in \mathcal{A} \implies A \cap \gamma \in D_{\gamma} \]

**Proof.** See [4], Page 25. \( D_{\gamma} = \{ x \subseteq \gamma \mid x \in L_{\gamma} \} \) where \( \hat{\gamma} = \text{least } \delta \text{ such that } \gamma \text{ is not a } \delta \text{-cardinal} \).

We are now ready to outline the construction. We wish to satisfy the requirements:

\[ R^A_e : B \neq \{ x \mid \exists \text{ finite } \gamma \subseteq A <x, \gamma > \in W_e \} \]

\[ R^B_e : A \neq \{ x \mid \exists \text{ finite } \gamma \subseteq B <x, \gamma > \in W_e \} \]

It is easy to see that satisfying these requirements for each \( e \in S_\beta \) guarantees \( A \notin f_B, B \notin f_A \). Here, \( W_e = e^{\text{th }} \beta \)-r.e. set. Our method for attempting to satisfy \( R^A_e \) is to put some \( x \) into \( B \) at stage \( \sigma \) if some finite \( \gamma \subseteq A^\sigma \) satisfies \( <x, \gamma > \in W_e^\sigma \). (\( A^\sigma = \text{part of } A \text{ enumerated by stage } \sigma \), similarly for \( W_e^\sigma \).)
If we can succeed in guaranteeing \( y \subseteq A^\sigma \) for all \( \sigma' > \sigma \), \( R_e^A \) will be satisfied.

These attempts at the above requirements conflict with each other. The solution is to order the requirements in a list, the requirements lower on the list having higher priority. In this construction, each requirement \( R_e^A \) (or \( R_e^B \)) is assigned all of the priorities \( \gamma < \beta^* \) such that \( e \in H(\gamma) \).

The construction proceeds in \( R \text{cf} \beta \) steps. Recall the function 
\[ g_0 : R \text{cf} \beta \to \beta. \]
At each stage \( \sigma \) we will use an approximation to the set \( A \) and the function \( H \). Let \( H^\sigma(\gamma) = \{ h^\sigma(i, <\gamma', \delta, p_0>) | i \in \omega, \gamma' < \gamma \} \) where \( h^\sigma \) is the canonical \( \Sigma_1 \) skolem function for \( S \).

Stage \( \sigma \). Do the following for each \( \gamma \in A^\sigma \): Form all pairs \( <R_e^A, z> \), \( <R_e^B, z> \) where \( z \in D_\gamma \), \( e \in H^\sigma(\gamma) \). Order such pairs in a list and choose the least \( \beta \)-finite bijection \( j \) between \( H^\sigma(\gamma) \) and \( \gamma \). Given that earlier pairs have been considered, attack \( <R_e^A, z> \) as follows: See if there is an \( x \notin H^\sigma(\gamma) \), \( x > \gamma \), \( x \) not being restrained from entering \( B \) by \( \gamma \), and a finite \( y \subseteq A^\sigma \) such that \( <x, y> \in W^\sigma_e \) and \( y \cap j^{-1}[x] = \emptyset \). Then for the least such pair \( <x, y> \), put \( x \) into \( B \) and have \( \gamma \) restrain the members of \( y \) from entering \( A \). The pairs \( <R_e^B, z> \) are handled similarly. This ends the construction.

The idea, then, is that the members of \( D_\gamma \) (via the bijection \( j: H^\sigma(\gamma) \to \gamma \)) provide "guesses" at \( A \cap H^\sigma(\gamma), B \cap H^\sigma(\gamma) \). Of course, since the parameter \( p_0 \) defines the entire construction, \( \Box \beta^* \) implies that \( j[A \cap H^\sigma(\gamma)], j[B \cap H^\sigma(\gamma)] \in D_\gamma \) if \( \gamma \in A \). So for \( \gamma \in A \), one of the "guesses" is correct. Then these guesses are each used to search for an \( x \) and \( y \) which attempt to satisfy \( R_e^A \) (or \( R_e^B \)).
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Now each pair \(<R^A_x, z>, \ e \in H(y), \ z \in D_γ \> is attacked at most once at each stage of the construction; thus, any \(x\) put into \(A\) or \(B\) by \(γ\) and any \(y\) restrained from intersecting \(A\) or \(B\) by \(γ\) must belong to \(H(γ + 1)\).

Lemma. Suppose \(y \in H(γ + 1) - H(γ)\) and \(y \in \mathscr{A}\). Then \(y \not\in H(γ)\).

Proof. Otherwise, let \(y \in H(γ')\), \(γ' < γ\), \(σ'\) least. Assume that \(γ'\) and \(γ\) have the same \(β\)-cardinality \(κ\). Let \(δ_0\) be the least \(δ < κ^+\) so that for some \(τ < σ', \ y \in H(δ) - H(γ')\). Then \(δ \in H(γ' + 1)\) and \(δ \geq γ\). But as \(κ ≤ γ'\), \(δ \notin H(γ' + 1)\) and so \(y \not\in H(γ' + 1)\). This contradicts \(y \in \mathscr{A}\). —

Now any \(y\) restrained by \(γ\) at stage \(σ\) must belong to \(H(γ + 1) - H(γ)\). Thus, by the Lemma, if \(γ \in \mathscr{A}\), we have \(y \not\in H(γ)\). But then as each \(γ' < γ\) only puts members of \(H(γ)\) into \(A\) or \(B\), no member of \(y\) can ever be put into \(A\) or \(B\). If in addition the attempt associated with \(y\) used a correct guess \(z\) for \(A \cap H(γ)\) (or \(B \cap H(γ)\)), then this attempt will succeed and the corresponding requirement \(R^A_x\) (or \(R^B_x\)) will be satisfied.

Lastly, note that no \(γ \in \mathscr{A}\) can ever be put into \(A\) or \(B\), by construction.

Thus we may argue for \(B \not\in \mathfrak{A}\) as follows (\(A \leq \mathfrak{B}\) \(B\) is similar): If \(\overline{B} = \{x| \exists \text{ finite } y \subseteq \overline{A}, \ <x, y> \in \mathbb{W}_e\}\), then choose \(γ \in \mathscr{A}\) and \(σ\) so that if \(γ' = \text{ least member of } \mathscr{A} \text{ greater than } γ\),

1) \(\exists \text{ finite } y \subseteq A^σ, \ <γ', y> \in \mathbb{W}^σ_e\),
2) \(y \cap (A \cap H^γ(γ)) = \emptyset\),
3) \(e \in H^γ(γ)\).

Such a \(γ\) and \(σ\) exist since \(\mathscr{A} \subseteq \overline{B}\). Then there must be an attempt made at this stage or an earlier stage for the pair \(<R_x^A, j[A \cap H^γ(γ)]\>\). By earlier Remarks, this attempt will succeed. End of proof sketch.
Section 5. **Minimal \( \alpha \)-Degrees Revisited**

We return now to the original problem which motivated our study. Have we learned anything new concerning minimal \( \alpha \)-degrees through the study of \( \beta \)-Recursion Theory? The following result gives an affirmative answer:

**Theorem 8.** If \( \alpha^* = \alpha \) and \( \rho \alpha \) is a successor \( \alpha \)-cardinal, then there is a minimal \( \alpha \)-degree which is \( \alpha \)-r.e. in \( 0' \).

The proof, which applies the techniques of \( \beta \)-Recursion Theory to the structure \( <L, \epsilon, C> \) (C a complete \( \alpha \)-r.e. set), will appear in [6].

**References**

3. Friedman, Sy D. \( \beta \)-Recursion Theory, to appear.
5. Friedman, Sy D. Forcing and the Fine Structure of \( L \), in preparation.
6. Friedman, Sy D. On Minimal \( \alpha \)-Degrees, in preparation.


