DEGREE THEORY ON $\aleph_\omega$

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Received 8 January 1982
Revised 21 December 1982

0. Introduction

Singular cardinals are of particular interest in the study of recursion theory on the ordinals. On the one hand many familiar techniques from Turing degree theory such as the minimal degree construction break down completely when applied to many singular cardinals, due to their $\Sigma_2$-inadmissibility. Moreover it is known that even though the Friedberg–Muchnik solution to Post’s problem can be adapted to all admissible ordinals [3] its relativized version will fail for singular cardinals of uncountable cofinality [2].

On the other hand sometimes the singularity of a cardinal can be an aid rather than a handicap in constructing recursively enumerable sets. In Friedman [1] it was shown that if $\kappa$ is a limit cardinal, then the sets $S(\lambda) = \{ \gamma < \kappa \mid \kappa$-cofinality$(\gamma) = \lambda \}$ occupy distinct intermediate $\kappa$-RE degrees as $\lambda$ varies over infinite regular $\kappa$-cardinals. This result solves Post’s problem for the $\kappa$-degrees without a priority argument.

In this paper we focus on the first singular $L$-cardinal, $\aleph_\omega^L$. The first two sections answer two questions left open from Friedman [1] concerning the sets $S(\aleph_\omega^L)$ described above. In Section 1 we use the infinite injury priority method to construct an incomplete $\aleph_\omega^L$-RE degree greater than the $\aleph_\omega^L$-degrees of the sets $S(\aleph_\omega^L)$. Section 2 provides a natural example of a nonzero $\aleph_\omega^L$-RE degree below the $\aleph_\omega^L$-degrees of both $S(\aleph_\omega^L)$ and $S(\aleph_\omega^L)$, for any $n$, $m$. The methods used here are an elaboration of the Gödel collapse methods used in Friedman [1]. Finally in Section 3 the notion of character of an $\aleph_\omega^L$-RE set is defined and studied. Using it an order-preserving embedding of the partial-ordering $(\mathcal{P}(\omega)/\text{Finite}, \subseteq)$ into the $\aleph_\omega^L$-RE degrees is obtained, without the use of a priority argument.

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1. An incomplete upper bound for the sets $S(\mathbb{N}_n^I)$

Let $\alpha$ denote $\aleph_0^L$. Let

$$S = \{(\gamma, \delta) \mid \gamma, \delta \text{ are limit ordinals } < \alpha \text{ and } \alpha\text{-cofinality}(\gamma) = \alpha\text{-cofinality}(\delta)\}.$$

$S$ is $\alpha$-RE as $(\gamma, \delta) \in S$ iff $\exists f \in L_\alpha$ ($f$ is an order-preserving function from an unbounded subset of $\gamma$ onto an unbounded subset of $\delta$). Moreover $S(\mathbb{N}_n^I) \leq_\alpha S$ for each $n$ as $S(\mathbb{N}_n^I) = (S)_{\aleph_1}^\alpha = \{\delta \mid (\mathbb{N}_n^I, \delta) \in S\}$. Unfortunately $S -_\alpha 0'$ as $\{\mathbb{N}_n^I \mid n \in \omega\} = \{\gamma \mid \gamma \text{ is a limit ordinal } < \alpha \text{ and } (S)_\gamma \cap \gamma = \emptyset\}$ has $\alpha$-degree $0'$. Our result in this section is that there is an incomplete $\alpha$-RE thick subset of $S$. A set $A \subseteq S$ is thick if for each $\gamma < \alpha$, $(S)_\gamma - (A)_\gamma$ is bounded in $\alpha$ (where $(S)_\gamma = \{\delta \mid (\gamma, \delta) \in S\}$). Clearly any thick subset of $S$ is an upper bound (in the sense of $\leq_\alpha$) for the sets $S(\mathbb{N}_n^I)$.

**Theorem 1.1.** $S$ has an $\alpha$-RE thick subset $A$ of $\alpha$-degree $< 0'$.

The original Thickness lemma for classical recursion theory was established by Shoenfield (see [4]). It its simplest form it states that if $B \subseteq \omega \times \omega$ is RE, $(B)_n$ is recursive for each $n$ and $C$ is nonrecursive, then $B$ has a thick RE subset $A$ such that $C \leq_T A$. The corresponding result for $\alpha$ is false. For, it is easy to construct an $\alpha$-RE $B \subseteq \alpha \times \alpha$ such that any thick $\alpha$-RE $A \subseteq B$ is high ($A' =_\alpha 0'$). But then $A =_\alpha 0'$ as Shore [5] showed that any incomplete $\alpha$-RE set $A$ is low ($A' =_\alpha 0'$).

There are two key properties of $S$ used in the proof of Theorem 1.1. Let $\alpha_n = \aleph_0^L$. The first fact is that for any $n$, $S \cap (\alpha_n \times \alpha_n)$ has incomplete $\alpha_n$-RE degree. For any $\gamma < \alpha$ let $(S)_{<\gamma} = \{(\gamma', \delta) \in S \mid \gamma' < \gamma\}$. The second fact is that if $\gamma < \alpha_{n-1}$, then $\alpha_n$ is $(S)_{<\gamma}$-stable; i.e., $(L_{\alpha_n}, (S)_{<\gamma} \cap (\alpha_n \times \alpha_n))$ is a $\Sigma_1$-elementary substructure of $(L_{\alpha_n}, (S)_{<\gamma})$.

To demonstrate the first fact note that $S \cap (\alpha_n \times \alpha_n)$ is $\alpha_n$-RE as $\alpha_n$ is $\alpha$-stable and $S$ has a parameter-free $\Sigma_1(L_\alpha)$ definition. Note that $\{\beta \mid \beta$ is an $\alpha_n$-cardinal$\}$ is finite and hence $\alpha_n$-finite. So $S \cap (\alpha_n \times \alpha_n)$ is in fact $\alpha_n$-recursive since if $\gamma, \delta$ are limit ordinals $\gamma < \alpha_n$, $\gamma, \delta \notin S$ iff $\exists \alpha_n$-cardinals $\kappa, \lambda$ s.t. $(\kappa, \gamma), (\lambda, \delta) \in S$ and $\kappa \neq \lambda$.

As for the second fact note that it suffices to establish the $T$-stability of $\alpha_n$ where $T = S(\omega) \cup S(\mathbb{N}_1^I) \cup \cdots \cup S(\mathbb{N}_n^I)$. But this follows from the remark made between Theorems 1 and 2 of Friedman [1].

Our proof follows the same outline as the proof of Shoenfield's Thickness Lemma given in Soare [7]. The facts above are used to bound the lim inf of the restraint imposed by a proper initial segment of negative requirements. We use Soare's notation. Let $\Phi_{e,\sigma}(X; y)$ be the result, if any, of performing the $\sigma$th partial $\alpha$-recursive reduction with oracle $X$ to argument $y$ through stage $\sigma < \alpha$. Also let $\Phi_e(X) = \bigcup_\sigma (\lambda y, \Phi_{e,\sigma}(X; y))$. We have in mind the $\alpha$-recursive enumeration of our desired $\alpha$-RE set $A$, written as $\{A^\sigma \mid \sigma < \alpha\}$. $A^{<\sigma} = \bigcup \{A^{\sigma'} \mid \sigma' < \sigma\}$. The definition of this enumeration is guided by some auxiliary
functions:
\[
    u(e, x, \sigma) = \begin{cases} 
        \min\{z \mid \Phi_{e,\sigma}(A^\sigma[z]; x) \text{ is defined}\} & \text{if } z \text{ exists}, \\
        0 & \text{otherwise},
    \end{cases}
\]
\[
a_\sigma = \begin{cases} 
        \text{least } x[x \in A^\sigma - A^{<\sigma}] & \text{if } A^\sigma - A^{<\sigma} \neq \emptyset, \\
        \sup(A^\sigma \cup \{\sigma\}) & \text{otherwise}.
    \end{cases}
\]
\[
    \Phi_{e,\sigma}(A^\sigma; x) = \begin{cases} 
        \Phi_{e,\sigma}(A^\sigma; x) & \text{if defined and } u(e, x, \sigma) < a_\sigma, \\
        \text{undefined} & \text{otherwise},
    \end{cases}
\]
\[
    \hat{u}(e, x, \sigma) = \begin{cases} 
        u(e, x, \sigma) & \text{if } \Phi_{e,\sigma}(A^\sigma; x) \text{ is defined}, \\
        0 & \text{otherwise}.
    \end{cases}
\]

Thus we use the modified computation function \(\hat{\Phi}\). It has the property that at a true stage \(\sigma\) any apparent computation \(\Phi_{e,\sigma}(A^\sigma; x)\) is a true computation \(\Phi_e(A; x)\). The set of true stages \(T\) is defined by:
\[
    T = \{\sigma \mid A^\sigma[a_\sigma] = A[a_\sigma]\}.
\]

We also use:
\[
    \hat{l}(e, \sigma) = \sup\{x \mid \forall y < x \ (K^\sigma(y) = \Phi_{e,\sigma}(A^\sigma; y))\},
\]
\[
    \hat{r}(e, \sigma) = \sup\{\hat{u}(e, x, \sigma) \mid x \leq \hat{l}(e, \sigma)\},
\]
\[
    \hat{R}(e, \sigma) = \sup\{\hat{r}(e', \sigma) \mid e' \leq e\}.
\]

These are the length of agreement function, restraint function and full restraint function, respectively. The set \(K\) is the complete \(\alpha\)-RE set defined by \(K = \{(e, x) \mid \Phi_{e,\sigma}(\emptyset, x) \text{ is defined for some } \sigma\}\). Thus \(\hat{l}\) measures the length of agreement between \(K\) and \(\Phi_e(A)\) at stage \(\sigma\). \(\hat{r}\) indicates how large an initial segment of \(A\) must remain unchanged for the sake of preserving \(\Phi_e(A)\) through this length of agreement.

The element \(x\) injures \(e\) at stage \(\sigma\) if \(x \in A^\sigma - A^{<\sigma}\) and \(x \leq \hat{r}(e, \sigma)\). So we are thinking of \(e\) synonymously with the requirement \(N_e : K \neq \Phi_e(A)\) (where \(K\) is identified with its characteristic function). The strategy for achieving \(N_e\) is to preserve agreements between \(K\) and \(\Phi_e(A)\). If \(x\) injures \(e\), then \(x\) is interfering with this strategy. We define the injury sets
\[
    \hat{l}_{e,\sigma'} = \{x \mid \exists \sigma' \leq \sigma [x \leq \hat{r}(e, \sigma') \text{ and } x \in A^{\sigma'} - A^{<\sigma'}]\},
\]
\[
    \hat{l}_{e, < \sigma} = \bigcup \{\hat{l}_{e,\sigma'} \mid \sigma' < \sigma\},
\]
\[
    \hat{l}_e = \bigcup \{\hat{l}_{e,\sigma} \mid \sigma < \alpha\}.
\]

In terms of the above definitions our construction proceeds as follows: First choose an \(\alpha\)-recursive enumeration \(\{S^\sigma \mid \sigma < \alpha\}\) of the \(\alpha\)-RE set \(S\). We enumerate \(x\) into \((A^\sigma)_\gamma\) if \(x \in (S^\sigma)_\gamma\) and \(x > \hat{r}(i, \sigma)\) for all \(i \leq \gamma\). Let \(A = \bigcup\{A^\sigma \mid \sigma < \alpha\}\).

We must show that \((A)_\gamma\) contains a final segment of \((S)_\gamma\) and \(K \neq \Phi_{e}(A)\), for
each \( \gamma < \alpha \). This is done by establishing the following claims by induction on \( \gamma < \alpha \):

(i) For \( \gamma < \alpha_n \), \( \lim \{ \bar{R}(\gamma, \sigma) \mid \sigma \in T_{\gamma} \} < \alpha_{n+1} \) where

\[
T_{\gamma} = \{ \sigma \mid (A)_{\gamma}[a_{\sigma}] = (A^\sigma)_{\gamma}[a_{\sigma}] \}
\]

= true stages for enumeration of \( (A)_{\gamma} \).

(ii) For \( \gamma < \alpha_n \), \( (S)_{\gamma} - (A)_{\gamma} \) is a bounded subset of \( \alpha_{n+1} \).

Suppose (i), (ii) hold for all \( \gamma' < \gamma \) and we seek to establish (i), (ii) for \( \gamma \geq 0 \). (Thus the base case of the induction is included here.) Note that for \( \gamma' < \gamma \), \( T_{\gamma'} \subseteq T_{\gamma} \). So \( \lim \{ \sup_{\gamma' < \gamma} \bar{R}(\gamma', \sigma) \mid \sigma \in T_{\gamma'} \} < \alpha_{n+1} \) by the regularity of \( \alpha_{n+1} \). But then \( (S)_{\gamma} - (A)_{\gamma} \) is a bounded subset of \( \alpha_{n+1} \) since at a stage \( \sigma \in T_{\gamma} \) nothing prevents \( x \in (S^\sigma)_{\gamma} \) from entering \( (A^\sigma)_{\gamma} \) if \( x \) is greater than the above limit. It follows that \( \hat{I}_{\gamma} \) is \( \alpha \)-recursive in \( (S)_{\gamma} \) since \( (A)_{\gamma} \subseteq (S)_{\gamma} \) and:

\[
x \in \hat{I}_{\gamma} \text{ if and only if } x \in (A)_{\gamma} \text{ and } x \in \hat{I}_{n, \sigma} \text{ where } \sigma \text{ is the stage such that } x \in (A^\sigma)_{\gamma} - (A^\sigma)_{\gamma}.
\]

Note that the reduction \( \hat{I}_{\gamma} \leq \alpha (S)_{\gamma} \) only uses parameters \( < \alpha_{n+1} \) and we have that \( \hat{I}_{\gamma, < \alpha_{n+1}} \leq \alpha_{n+1} (S)_{\gamma} \cap (\alpha_{n+1} \times \alpha_{n+1}) \).

We claim now that \( K \cap \alpha_{n+1} \neq \Phi_\gamma (A) \cap \alpha_{n+1} \). First note that \( K \cap \alpha_{n+1} \neq \Phi_{y, < \alpha_{n+1}} (A^{< \alpha_{n+1}}) \). For otherwise, as \( K \cap \alpha_{n+1} = K^{< \alpha_{n+1}} \) by the \( \alpha \)-stability of \( \alpha_{n+1} \), we would have \( \lim \{ \bar{I}(\gamma, \sigma) \mid \sigma < \alpha_{n+1} \} = \alpha_{n+1} \). But now we can compute \( K \cap \alpha_{n+1} \) in an \( \alpha_{n+1} \)-recursive way from \( \hat{I}_{y, < \alpha_{n+1}} \): To compute \( K(\rho) \) for \( \rho < \alpha_{n+1} \) find some stage \( \sigma < \alpha_{n+1} \) such that \( \hat{I}(\gamma, \sigma) > \rho \) and

\[
\forall x \leq \rho \forall z \left[ z \leq u(\gamma, x, \sigma) \rightarrow (z \notin \hat{I}_{y, < \alpha_{n+1}} \text{ or } z \in (A^\sigma)) \right].
\]

Then \( K(\rho) = \Phi_{y, \sigma} (A^\sigma; \rho) \): To see this it suffices to argue that for \( \tau > \sigma \), \( \tau < \alpha_{n+1} \) we have \( \hat{I}(\gamma, \tau) > \rho \) and \( \hat{I}(\gamma, \tau) \supseteq \sup \{ u(\gamma, x, \sigma) \mid x \leq \rho \} \). If \( \tau \) were the least counterexample to this claim, then we must have \( K^\tau(x) \neq K^{< \tau}(x) \) for some \( x \leq \rho \). But then \( K(x) = K^\tau(x) \neq \Phi_{y, < \alpha_{n+1}} (A^{< \alpha_{n+1}})(x) \) as the information about \( A^\sigma \) used in establishing \( \Phi_{y, \sigma} (A^\sigma; x) \) cannot change before stage \( \alpha_{n+1} \). We have therefore shown \( K \cap \alpha_{n+1} \leq \hat{I}_{y, < \alpha_{n+1}} \). But then \( K \cap \alpha_{n+1} \leq (S)_{\gamma} \cap (\alpha_{n+1} \times \alpha_{n+1}) \) which is impossible since \( K \cap \alpha_{n+1} = \{(e, x) \mid \Phi_{\sigma} (\emptyset; x) \} \) is defined for some \( \sigma < \alpha_{n+1} \) is complete \( \alpha_{n+1} \)-RE and \( (S)_{\gamma} \cap (\alpha_{n+1} \times \alpha_{n+1}) \) is \( \alpha_{n+1} \)-recursive.

Second, we argue that \( \Phi_{y, < \alpha_{n+1}} (A^{< \alpha_{n+1}}) = \Phi_{y} (A) \cap \alpha_{n+1} \). For the \( \alpha \)-stability of \( \alpha_{n+1} \) implies that \( A^{< \alpha_{n+1}} = A \cap \alpha_{n+1} \) so clearly \( \Phi_{y, < \alpha_{n+1}} (A^{< \alpha_{n+1}}) \subseteq \Phi_{y} (A) \). But conversely, if \( \Phi_{y} (A)(x) = y \) where \( x, y < \alpha_{n+1} \), then we have:

\[
\exists \sigma < \alpha [ \Phi_{y, \sigma} (A^\sigma; x) = y \text{ and } \forall z \left[ z \leq u(\gamma, x, \sigma) \rightarrow (z \notin \hat{I}_{y} \text{ or } z \in A^\sigma) \right]].
\]

So since \( \alpha_{n+1} \) is \( (S)_{\gamma} \)-stable and \( \hat{I}_{y} \leq \alpha (S)_{\gamma} \) (with parameter \( < \alpha_{n+1} \) for the reduction):

\[
\exists \sigma < \alpha_{n+1} [ \Phi_{y, \sigma} (A^\sigma; x) = y \text{ and } \forall z \left[ z \leq u(\gamma, x, \sigma) \rightarrow (z \notin \hat{I}_{y} \text{ or } z \in A^\sigma) \right]].
\]

But then \( \Phi_{y, < \alpha_{n+1}} (A^{< \alpha_{n+1}}; x) = y \).
Now we can establish (i), (ii), for $\gamma$. Define $\rho = \text{least } x \text{ s.t. } K(x) \neq \Phi_{\gamma}(A)(x)$. Then $\rho < \alpha_{n+1}$. By the preceding paragraph we can choose $\sigma < \alpha_{n+1}$ such that for all $\tau \geq \sigma$: $\forall x \leq \rho (K(x) = K^*(x))$ and $\forall x < \rho (\Phi_{\gamma,\sigma}(A^*; x) = \Phi_{\gamma}(A)(x))$. If for all $\tau \in T_\gamma$, $\tau \geq \sigma$ $\Phi_{\gamma,\tau}(A^*; \rho)$ is undefined, then for any $\tau \geq \sigma$, $\tau \in T_\gamma \rightarrow \hat{I}(\gamma, \tau) = \rho$ and $\hat{I}(\gamma, \tau) = \sup \{ \hat{u}(\gamma, x, \sigma) \mid x < \rho \}$. So $\lim \{ \hat{r}(\gamma, \tau) \mid \tau \in T_\gamma \}$ exists and is less than $\alpha_{n+1}$. If $\Phi_{\gamma,\tau}(A^*; \rho)$ is defined for some $\tau \geq \sigma$, $\tau \in T_\gamma$, then for any $\sigma' \geq \tau$ $K(\rho) \neq \Phi_{\gamma,\sigma}(A'^*; \rho)$ since the computation $\hat{I}(\gamma, \alpha'; \rho) = y$ cannot be injured at any stage $\sigma' \geq \tau$ (as $\tau$ is a true stage for the enumeration of $(A)_{<\gamma}$). Thus for all $\sigma' \geq \tau$ $\hat{I}(\gamma, \sigma') = \rho$ and $\hat{r}(\gamma, \sigma') = \hat{r}(\gamma, \tau)$. So once again $\lim \{ \hat{r}(\gamma, \sigma') \mid \sigma' \in T_\gamma \}$ exists. But note that $\tau$ can be chosen to be less than $\alpha_{n+1}$ by the $(S)_{<\gamma}$-stability of $\alpha_{n+1}$, since $T_\gamma \leq (A)_{<\gamma} \leq (S)_{<\gamma}$ and all of these reductions only involve parameters less than $\alpha_{n+1}$. We have therefore established (i) for $\gamma$.

We can now easily conclude (ii) for $\gamma$ since at a stage $\sigma \in T_\gamma$ nothing can prevent $x \in (S^\sigma)_{<\gamma}$ from entering $(A^\sigma)_{<\gamma}$ provided $x > \lim \{ R(y, \sigma') \mid \sigma' \in T_\gamma \}$ and this last ordinal is less than $\alpha_{n+1}$ by (i) for $\gamma$.

Finally note that in the course of establishing (i), (ii) we demonstrated that $K \neq \Phi_{\gamma}(A)$ for each $\gamma < \alpha$. Also (ii) immediately implies that $A$ is a thick subset of $S$. This completes the proof of Theorem 1.1.

2. $S(S_{\omega}^L)$, $S(S_{\omega}^L)$ do not form a minimal pair

To establish this result we follow a strategy communicated to us by Carl Jockusch. If $A, B$ are RE sets of integers, choose recursive onto functions $f: \omega \rightarrow A$, $g: \omega \rightarrow B$. An RE set recursive simultaneously in $A$ and $B$ is $C = \{(x, y) \mid$ for some $n, x = f(n), y = g(n)\}$. This can be a useful way of showing that $A, B$ do not form a minimal pair.

Thus let $A = S(S_{\omega}^{L_\omega}) = \{ \beta < S_{\omega}^{L_\omega} \mid L - \text{cof}(\beta) = S_{\omega}^{L_\omega} \}$, $B = S(S_{\omega}^{L_\omega}) = \{ \beta < S_{\omega}^{L_\omega} \mid L - \text{cof}(\beta) = S_{\omega}^{L_\omega} \}$. In order to produce $C$ as above we first make a definition.

Definition. If $\beta \in S(S_{\omega}^{L_\omega})$ let $\hat{\beta}$ be the least ordinal $\gamma$ such that there is a cofinal $f : S_{\omega}^{L_\omega} \rightarrow \beta$ which is definable over $L_\gamma$.

Then set $C = \{ (\beta_1, \beta_2) \mid \beta_1 \in S(S_{\omega}^{L_\omega}), \beta_2 \in S(S_{\omega}^{L_\omega}) \text{ and } \hat{\beta}_1 = \hat{\beta}_2 \}$. It is easy to see that $C$ is $S_{\omega}^{L_\omega}$-recursive in each of $S(S_{\omega}^{L_\omega}), S(S_{\omega}^{L_\omega})$. For, given $(\beta_1, \beta_2)$ first check if $\beta_1 \in S(S_{\omega}^{L_\omega})$; if not, then $(\beta_1, \beta_2) \notin C$. If so, then compute $\hat{\beta}_1$ effectively and check if $L_{\hat{\beta}_1 + 1}$ contains a cofinal increasing $f : S_{\omega}^{L_\omega} \rightarrow \beta_2$. If it does but $L_{\hat{\beta}_2}$ does not, then $(\beta_1, \beta_2) \notin C$. Otherwise $(\beta_1, \beta_2) \notin C$. This shows that $C$ is $S_{\omega}^{L_\omega}$-recursive in $S(S_{\omega}^{L_\omega})$. A similar argument works for $S(S_{\omega}^{L_\omega})$.

It remains to show that $C$ is not $S_{\omega}^{L_\omega}$-recursive. Suppose it is and let $\exists y \phi(y, \beta_1, \beta_2)$ define the complement of $C$ over $(L_{\omega}, \in)$ where $\phi$ is $\Delta_0$ with parameter $p$. Choose $k$ so large that $m, n < k$ and $p \in L_{\omega}^{\omega}$. Note that $(\omega_{k+1}^{L_\omega}, \omega_{k+2}^{L_\omega}) \notin C$ so we can choose $y'' \in L^{\omega}_k$ such that $\phi(y'', \omega_{k+1}^{L_\omega}, \omega_{k+2}^{L_\omega})$. We get our
desired contradiction by producing $\beta_1, \beta_2, y \in L_{\alpha^+}$, such that $(\beta_1, \beta_2) \in \mathcal{C}$ but $\phi(y, \beta_1, \beta_2)$.

The desired $\beta_1, \beta_2, y$ are obtained by applying a Gödel collapse argument to $\mathcal{N}_{k+1}^L, \mathcal{N}_{k+2}^L, y''$. The construction is very similar to the proof of Theorem 1 in Friedman [1].

Choose a $\Sigma_2$-admissible $\alpha'' < \mathcal{N}_\omega^L$ large enough so that $y'' \in L_{\alpha''}, \mathcal{N}_{k+2}^L < \alpha''$. Define an $\mathcal{N}_m^L$-sequence by:

$$H_0 = \Sigma_1 \text{ Skolem hull of } \mathcal{N}_{k+1}^L \cup \{y'', \mathcal{N}_{k+2}^L\} \text{ in } L_{\alpha''},$$

$$\gamma_0 = H_0 \cap \mathcal{N}_{k+2}^L,$$

$$H_{\delta+1} = \Sigma_1 \text{ Skolem hull of } \mathcal{N}_{k+1}^L \cup \{y'', \mathcal{N}_{k+2}^L, \gamma_\delta\} \text{ in } L_{\alpha''},$$

$$\gamma_{\delta+1} = H_{\delta+1} \cap \mathcal{N}_{k+2}^L,$$

$$H_\lambda = \bigcup \{H_\delta \mid \delta < \lambda\} \text{ for limit } \lambda,$$

$$\gamma_\lambda = \sup\{\gamma_\delta \mid \delta < \lambda\}.$$

Finally set $H = \bigcup \{H_\delta \mid \delta < \mathcal{N}_m^L\}, \beta_2 = \sup\{\gamma_\delta \mid \delta < \mathcal{N}_m^L\}$. Let $\pi: H = L_{\alpha''}$ and define $y' = \pi(y'')$. Note that $\pi(\mathcal{N}_{k+2}^L) = \beta_2$ and $L_{\alpha''} \models \phi(y', \mathcal{N}_{k+1}^L, \beta_2)$ since $\pi$ is the identity on $L_{\alpha''}$. And most importantly $L_{\alpha''} \models \beta_2$ is regular but there is a $\Sigma_2(L_{\alpha''})$ cofinal increasing $f: \mathcal{N}_m^L \to \beta_2$, given by $f(\delta) = \gamma_\delta$. So $\beta_2 = \alpha'$.

Now we repeat the above with an $\mathcal{N}_m^L$-iteration of $\Sigma_2$ Skolem hulls inside $L_{\alpha''}$. Thus:

$$K_0 = \Sigma_2 \text{ Skolem hull of } \mathcal{N}_k^L \cup \{y', \beta_2, \mathcal{N}_{k+1}^L\} \text{ in } L_{\alpha''},$$

$$\mu_0 = K_0 \cap \mathcal{N}_{k+1}^L,$$

$$K_{\delta+1} = \Sigma_2 \text{ Skolem hull of } \mathcal{N}_k^L \cup \{y', \beta_2, \mu_\delta, \mathcal{N}_{k+1}^L\} \text{ in } L_{\alpha''},$$

$$\mu_{\delta+1} = K_{\delta+1} \cap \mathcal{N}_{k+1}^L,$$

$$K_\lambda = \bigcup \{K_\delta \mid \delta < \lambda\}, \lambda \text{ limit},$$

$$\mu_\lambda = K_\lambda \cap \mathcal{N}_{k+1}^L, \lambda \text{ limit}.$$

Finally set $K = \bigcup \{K_\delta \mid \delta < \mathcal{N}_m^L\}, \beta_1 = \sup\{\mu_\delta \mid \delta < \mathcal{N}_m^L\}$. Let $\sigma: K = L_{\alpha''}, y = \sigma(y'), \beta_2 = \sigma(\beta_2')$. Note that $\sigma(\mathcal{N}_{k+1}^L) = \beta_1$ and $L_{\alpha''} \models \phi(y, \beta_1, \beta_2)$. As $\mathcal{N}_{k+1}^L, \beta_2'$ are regular in $L_{\alpha''}$, we have that $\beta_1, \beta_2$ are regular in $L_{\alpha''}$. But there is a $\Sigma_2(L_{\alpha''})$ cofinal increasing $g: \mathcal{N}_m^L \to \beta_1$, namely $g(\delta) = \mu_\delta$. And if $f: \mathcal{N}_m^L \to \beta_2'$ is the cofinal, increasing $\Sigma_2(L_{\alpha''})$ function mentioned earlier, we have that $\sigma \circ f \cdot \mathcal{N}_m^L \to \beta_2$ is cofinal, increasing and $\Sigma_2(L_{\alpha''})$ since $K$ is a $\Sigma_2$ elementary submodel of $L_{\alpha''}$ containing the defining parameters for $f$ (namely $y', \beta_2', \mathcal{N}_m^L$). So $\beta_1 = \beta_2 = \alpha$. As $\beta_1 \in S(\mathcal{N}_m^L), \beta_2 \in S(\mathcal{N}_m^L)$ we have proved:

**Theorem 2.1.** $S(\mathcal{N}_m^L), S(\mathcal{N}_n^L)$ do not form a minimal pair for any $m, n < \omega$.

The same proof shows:

**Corollary 2.2 (to proof).** $S(\mathcal{N}_n^L), \ldots, S(\mathcal{N}_n^L)$ have a common nonzero lower bound for any $n_1, \ldots, n_k < \omega$. 
A refinement of the proof shows: The set
\[ \{ (\beta_1, \ldots, \beta_k) \mid \beta_i \in S(\mathbf{N}_n^L) \text{ all } i, \hat{\beta}_1 = \hat{\beta}_2 = \cdots = \hat{\beta}_k \} \]
is a non $\mathbf{N}_{\omega}^L$-recursive lower bound for $S(\mathbf{N}_n)$ if and only if $n = n_i$ for some $i$. This can be used to produce an order-reversing embedding of the tree $\omega^{<\omega}$ into the $\mathbf{N}_{\omega}^L$-RE degrees (and without use of the priority method).

3. Characters of $\mathbf{N}_{\omega}^L$-RE sets

We continue to fix $\alpha = \mathbf{N}_{\omega}^L$. The set $S(\omega)$, as was shown in Friedman [1], has the property that
\[ (k^+, S(\omega) \cap \mathbf{N}_n^L) <_1 (\omega, S(\omega)) \]
for each $n > 1$ ($<_1$ denotes `$\Sigma_1$-elementary substructure`). Not every $\alpha$-r.e. set has this property. In particular, let $C$ be of complete degree. Then $C$ is not hyperregular and so there is a function $f: \omega \to \alpha$ such that $f \leq_{\text{w}} C$ and $\sup\{ f(n) \mid n < \omega \} = \alpha$. By introducing parameters if necessary, one may assume that $|f(n)| < |f(n + 1)|$ for all $n$. This implies that for all sufficiently large $n$, $\langle L_{\mathbf{N}_n^L}, C \cap \mathbf{N}_n^L \rangle$ is not a $\Sigma_1$-elementary substructure of $\langle L, C \rangle$. We are thus naturally led to the following definition.

**Definition 3.1.** Let $A$ be a subset of $\alpha$. The *character* of A, denoted $\text{char}(A)$, is the set of all $n < \omega$ such that
\[ \langle L_{\mathbf{N}_n^L}, A \cap \mathbf{N}_n^L \rangle <_1 \langle L, A \rangle. \]

**Proposition 3.2.** Let $A$ and $B$ in $L$ be such that $A \leq_{\omega} B$. Then $\text{char}(A) \supseteq \text{char}(B)$, where $\supseteq$ means containment module finite sets.

**Proof.** Let $\exists y \varphi$ and $\exists y \bar{\varphi}$ be $\Sigma_1(B)$ sentences such that for all $\alpha$-finite $K$:
\[ K \subseteq A \iff \langle L, B \rangle \models \exists y \varphi(y, K) \]
and
\[ K \subseteq \bar{A} \iff \langle L, B \rangle \models \exists y \bar{\varphi}(y, K). \]

Let $n_0$ be chosen so that $\mathbf{N}_{n_0}$ is greater than the parameters occurring in $\varphi$ and $\bar{\varphi}$. Let $n > n_0$ and $n \in \text{char}(B)$. Let $\exists x \psi(x)$ be $\Sigma_1(A)$ with parameters less than $\mathbf{N}_n^L$. Suppose that
\[ \langle L, A \rangle \models \exists x \psi(x). \]
Then
\[ \langle L, B \rangle \models \exists K_1 K_2 \exists y_1 y_2 \exists y (\varphi(y_1, K_1) \& \bar{\varphi}(y_2, K_2) \& \langle L, K_1, K_2 \rangle \models \exists x \psi(x)) \]
where $\psi'$ is obtained from $\psi$ by replacing `$z \in A$' by $z \in K_1$, `$z \notin A$' by $z \in K_2$. By the $\Sigma_1(B)$-stability of $\mathbf{N}_n^L$, the statement $(*)$ is true in the structure $\langle L_{\mathbf{N}_n^L}, B \cap \mathbf{N}_n^L \rangle$. 


In other words,

\[
\langle L_{\mathbb{N}_{\omega}}, B \cap \mathbb{N}_{\omega}^1 \rangle \models \exists K_1 K_2 \exists y_1 y_2 \exists \gamma (\phi(y_1, K_1) \land \bar{\phi}(y_2, K_2) \\
\land \langle L_{\gamma}, K_1, K_2 \rangle \models \exists x \psi(x))
\]

Hence \( \langle L_{\mathbb{N}_{\omega}}, A \cap \mathbb{N}_{\omega}^1 \rangle \models \exists x \psi(x) \), so that \( n \in \text{Char}(A) \).

**Remark.** It follows from Proposition 3.2 that if \( A \equiv_\alpha B \), then \( \text{char}(A) = \text{char}(B) \) modulo finite sets (written \( \text{char}(A) = * \text{char}(B) \)). The converse is not true. Indeed since characters are degree-theoretically invariant (modulo finite sets), and there are only \( \aleph_1 \) possibilities for characters while there are \( \alpha = \aleph_1 \) many \( \alpha \)-r.e. degrees, there certainly exist many different \( \alpha \)-r.e. degrees whose characters are only finitely different.

In the Sacks–Simpson [3] construction the incomparable \( \alpha \)-r.e. degrees have characters \( * \omega \), as with the \( S(n) \), for \( n = \aleph_0, \aleph_1, \ldots, \aleph_n, \ldots \).

We now prove a theorem which is a weak form of the density theorem. The theorem says that for each constructible \( K \subset \text{char}(A) \), there is a \( B \) such that \( A \equiv_\alpha B \) and \( \text{char}(B) = * K \). This result has the following consequences: (i) (Representability) Every constructible \( K \subset \omega \) is the character of an \( \alpha \)-r.e. set \( A_K \); (ii) (Friedberg–Muchnik–Sacks–Simpson theorem) There exist \( \aleph_1 \)-many pairwise incomparable \( \alpha \)-r.e. degrees; (iii) (Upward Density Theorem) If \( \text{char}(A) \neq * \emptyset \), then there exist \( \aleph_1 \)-many \( \alpha \)-r.e. degrees \( d \) such that \( \deg(A) <_\alpha d <_\alpha 0' \). While (ii) and (iii) are known results (in particular, Shore [6] has proved the full Density theorem), our method of proof is different and does not use a priority argument. Furthermore, (i) provides a classification of hyperregular \( \alpha \)-r.e. degrees according to the measure of stability (in terms of characters) of the sets sitting in them. This may provide a useful tool in the study of the fine structure of hyperregular \( \alpha \)-r.e. degrees.

**Theorem 3.3.** Let \( A \) be \( \alpha \)-r.e. and let \( K \subset \text{char}(A) \) be constructible. There exists an \( \alpha \)-r.e. set \( B \geq_\alpha A \) such that \( \text{char}(B) = * K \).

**Proof.** If \( \text{char}(A) = * \emptyset \), then we let \( B = A \). Hence we may assume that \( \text{char}(A) \neq * \emptyset \) and let \( K \subset \text{char}(A) \) be constructible.

The set \( B \) will consist of pairs \( (x, y) \) such that

\[
\forall x ((x, 0) \in B \iff x \in A).
\]

This ensures that \( A \leq_\alpha B \). We let \( B_0 = \{ x \mid (x, 0) \in B \} \).

Let \( B_1 = \{ y \mid (0, y) \in B \} \). We construct \( B \) so that for each \( n \not\in K \), \( B_1 \equiv (\mathbb{N}^1_{n-1}, \mathbb{N}^1_n) \).

Without loss of generality we may assume \( n > 1 \). Furthermore, it is safe to assume that \( \omega - K \neq * \emptyset \) since otherwise we may again take \( B = A \) to prove the theorem.

The objective is to construct a \( B \) such that \( B \geq_\alpha A \) and for all but finitely many \( n \in \omega \),

\[
\langle L_{\mathbb{N}_{\omega}}, B \cap \mathbb{N}_{\omega}^1 \rangle <_1 \langle L_{\alpha}, B \rangle
\]

if and only if \( n \in K \).
To kill the $B$-stability of $\mathbb{N}_L^L$, $n \notin K$, note that if $B_1 \models (\mathbb{N}_{n-1}^L, \mathbb{N}_n^L)$, then
\[
\langle L_\alpha, B \rangle \models \exists x (x \notin B_1 \land x > \mathbb{N}_{n-1}^L)
\]
but $\langle L_{\omega_1}, B \cap \mathbb{N}_n^L \rangle$ fails to satisfy the same sentence.

At stage $\sigma$, let $A^{<\sigma}$ be the set of ordinals which have been enumerated in $A$ before $\sigma$, and define $B^{<\sigma}, B_i^{<\sigma}$ ($i = 0, 1$) similarly. Let $\sigma'$ be the least $\tau > \sigma$ such that:

(i) $L_{\sigma'} \models \exists n < \omega (|\sigma| = \mathbb{N}_n^L)$,

(ii) let $\tau_n$ denote the $n$th infinite cardinal of $L_{\sigma'}$. Suppose that $|\sigma|^{\tau_n} = \tau_m$. Then for each $n \leq m$ such that $n \in K$,

\[
\langle L_{\tau_n}, A^{<\tau} \cap \tau_n \rangle < \langle L_{\sigma'} A^{<\tau} \rangle.
\]

Note that $i \in \text{char}(A)$, $\sigma \notin \mathbb{N}_L^L$ implies that $\tau = \mathbb{N}_L^L$ obeys (i), (ii). So $\sigma'$ exists.

Let $n(\sigma)$ be the integer $m$ such that $|\sigma| = \sigma_m$ in $L_{\sigma'}$. If $n(\sigma) + 1 \in K$, let

\[
B^{\sigma} = B^{<\sigma} \cup \{ (x, 0) \mid x \in A^{\sigma} \}.
\]

If $n(\sigma) + 1 \notin K$, then let

\[
B^{\sigma} = B^{<\sigma} \cup \{ (x, 0) \mid x \in A^{\sigma} \} \cup \{ (0, y) \mid y \in (\sigma_n(\sigma), \sigma) \}.
\]

Finally let

\[
B = \bigcup_{\sigma \prec \alpha} B^{\sigma}.
\]

Lemma 3.4. If $n \notin K$, then $B_1 \models (\mathbb{N}_{n-1}^L, \mathbb{N}_n^L)$.

Proof. This follows easily from our construction. Let $\sigma$ be chosen so that $\sigma'_{n(\sigma)} = \mathbb{N}_{n-1}^L$. Then $(\mathbb{N}_{n-1}^L, \sigma) \subseteq B_1^{\sigma}$. As such $\sigma$'s occur unboundedly often in $\mathbb{N}_L^L$, we have the lemma.

Lemma 3.5. Suppose $n \in K$ and $L_{\omega_1}^{\mathbb{N}_n^L}$ contains the parameters which define $A$ and $B$. Then

\[
\langle L_{\mathbb{N}_n^L}, B \cap \mathbb{N}_n^L \rangle < \langle L_\omega, B \rangle.
\]

Proof. Let $n \in K$ be as given and let $\exists x \phi(x, a_1, \ldots, a_r, B)$ be a $\Sigma_1(B)$-sentence with parameters $a_1, \ldots, a_r$ in $L_{\mathbb{N}_n^L}$. Suppose that this sentence is true in $\langle L_\omega, B \rangle$. Then

\[
\langle L_\alpha, A \rangle \models \exists \theta \exists x [B^{<\sigma} = A \cap \sigma = A^{<\sigma} \land \phi(x, a_1, \ldots, a_r, B^{<\sigma})].
\]

As $K \subseteq \text{char}(A)$, we have $n \in \text{char}(A)$ and so $\langle L_{\mathbb{N}_n^L}, A \cap \mathbb{N}_n^L \rangle$ satisfies this sentence which we denote by $\psi$.

Choose $\hat{\sigma}$ to be the least $\sigma \gg \mathbb{N}^L_{n-1}$ for which $\psi$ is true in $\langle L_{\hat{\sigma}}, A \cap \sigma \rangle$. Our goal is to show that $B^{<\hat{\sigma}} = B \cap \hat{\sigma}$ and therefore that $\exists x \phi$ is true in $\langle L_{\hat{\sigma}}, B \cap \hat{\sigma} \rangle$.

Claim. $n(\hat{\sigma}) + 1 = n$.  

Clearly $n(\hat{\sigma}) + 1 \geq n$.

Suppose $n(\hat{\sigma}) > n - 1$. In the notation of our construction, let $\hat{\sigma}'_{n(\hat{\sigma})}$ be the cardinality of $\hat{\sigma}$ in $L_{\hat{\sigma}_i}$. Then by (ii) in the definition of $\hat{\sigma}'$, we have $\hat{\sigma}'_{n(\hat{\sigma})} \leq \hat{\sigma}'_{n(\hat{\sigma})}$ and

$$\langle L_{\hat{\sigma}_i}, A_{\hat{\sigma}'_i} \setminus \hat{\sigma}'_{n(\hat{\sigma})} \rangle \prec \langle L_{\hat{\sigma}_i}, A_{\hat{\sigma}'} \rangle.$$ 

In particular $\langle L_{\hat{\sigma}_i}, A_{\hat{\sigma}'_i} \setminus \hat{\sigma}'_{n(\hat{\sigma})} \rangle = \langle L_{\hat{\sigma}_i}, A_{\hat{\sigma}'_i} \setminus \hat{\sigma}'_{n(\hat{\sigma})} \rangle$ satisfies $\psi$. Hence there is a $\nu < \hat{\sigma}'_{n(\hat{\sigma})}$ for which $\psi$ is true in $\langle L_{\nu}, A \setminus \nu \rangle = \langle L_{\nu}, A_{\hat{\sigma}'_i} \setminus \hat{\sigma}'_{n(\hat{\sigma})} \rangle$. But this contradicts the choice of $\hat{\sigma}$, and the claim is proved.

It follows from our construction that $B_{<\hat{\sigma}} = B \cap \hat{\sigma}$, since $A_{<\hat{\sigma}} = A \cap \hat{\sigma}$ and no $L_\tau$ for $\tau > \hat{\sigma}'$ will tell us that $\hat{\sigma}$ has smaller cardinality than it has in $L_{\hat{\sigma}_i}$. Thus $\langle I_{\hat{\sigma}_i}, B \cap \hat{\sigma} \rangle \not\models \exists x \phi$ and so $\langle I_{\hat{\sigma}_i}, B \cap N_{\hat{\sigma}_i} \rangle \prec \langle I_{\omega}, B \rangle$. This proves Theorem 3.3, as it was observed earlier that for $n \notin K$, $(N_{n-1}, N_n) \subseteq B$.

**Corollary 3.6.** Let $K \subseteq \omega$ be constructible. Then there is an $\alpha$-r.e. set $A$ such that $\text{char}(A) = K$.

**Proof.** Let $A$ be $\alpha$-recursive. Then $\text{char}(A) = K$. As $K \subseteq \omega$, by Theorem 3.3, we have an $\alpha$-r.e. set $A_K$ such that $\text{char}(A_K) = K$.

**Corollary 3.7.** There exist $\aleph_1$-many pairwise incomparable $\alpha$-r.e. sets.

**Proof.** There exist $\aleph_1$ many almost disjoint constructible reals $K$. For each such $K$, let $A_K$ be $\alpha$-r.e. such that $\text{char}(A_K) = K$. By Proposition 3.2, these sets are pairwise $\alpha$-incomparable.

**Corollary 3.8.** Let $\text{char}(A) \neq \emptyset$. Then there exist $\aleph_1$-many $\alpha$-r.e. degrees which lie strictly between $\text{deg}(A)$ and $0'$.

**Proof.** There exist $\aleph_1$ many almost disjoint constructible reals $K \subseteq \text{char}(A)$ such that $(\text{char}(A) - K) \neq \emptyset$ and $K \neq \emptyset$. Then by Proposition 3.2 each $\text{deg}(A_K)$ where $\text{char}(A_K) = K$ and $K$ is as above lies strictly between $\text{deg}(A)$ and $0'$.

7. Some open problems

(a) Prove the following Character Density theorem.

If $\alpha = \aleph_\omega$ and $A \leq \alpha B$ are $\alpha$-RE, $\text{char}(A) \supseteq K \supseteq \text{char}(B)$, $K \subseteq L$, then there is an $\alpha$-RE $C$ such that $A \leq \alpha C \leq \alpha B$ and $\text{char}(C) = K$.

(b) Let $C = \{(\beta_1, \beta_2) \mid \beta_1 \in S(\aleph_\omega), \beta_2 \in S(\aleph_\omega) \text{ and } \beta_1 = \beta_2\}$, as in Section 2. Is $C$ a greatest lower bound for $S(\aleph_\omega)$, $S(\aleph_\omega)$ in the sense of $\aleph_\omega$-degree?
(c) Find a non $\mathbf{N}_\omega$-recursive $A$ which is $\mathbf{N}_\omega$-recursive in each of the sets $S(\mathbf{N}_n)$, $n \in \omega$.

References