Using almost disjoint coding we prove the consistency of the existence of a $\Pi^1_2$ definable $\omega$-mad family of infinite subsets of $\omega$ (resp. functions from $\omega$ to $\omega$) together with $b = 2^\omega = \omega_2$.

1. Introduction

A classical result of Mathias [7] states that there exists no $\Sigma^1_1$ definable mad family of infinite subsets of $\omega$. One of the two main results of [4] states that there is no $\Sigma^1_1$ definable $\omega$-mad family of functions from $\omega$ to $\omega$. It is the purpose of this paper to analyse how low in the projective hierarchy one can consistently find a mad subfamily of $[\omega]^\omega$ or $\omega^\omega$.

Recall that $a, b \in [\omega]^\omega$ are called almost disjoint, if $a \cap b$ is finite. An infinite set $A$ is said to be an almost disjoint family of infinite subsets of $\omega$ (or an almost disjoint subfamily of $[\omega]^\omega$) if $A \subset [\omega]^\omega$ and any two elements of $A$ are almost disjoint. $A$ is called a mad family of infinite subsets of $\omega$ (abbreviated from “maximal almost disjoint”), if it is maximal with respect to inclusion among almost disjoint families of infinite subsets of $\omega$. Given an almost disjoint family $A \subset [\omega]^\omega$, we denote by $L(A)$ the set $\{b \in [\omega]^\omega : b$ is not covered by finitely many $a \in A\}$. Following [6] we define a mad subfamily $A$ of $[\omega]^\omega$ to be $\omega$-mad, if for every $B \in [L(A)]^\omega$ there exists $a \in A$ such that $|a \cap b| = \omega$ for all $b \in B$.

Two functions $a, b \in \omega^\omega$ are called almost disjoint, if they are almost disjoint as subsets of $\omega \times \omega$, i.e. $a(k) \neq b(k)$ for all but finitely many $k \in \omega$. A set $A$ is said to be an almost disjoint family of functions (or an almost disjoint subfamily of $\omega^\omega$) if $A \subset \omega^\omega$ and any two elements of $A$ are almost disjoint. $A$ is called a mad family of functions, if it is maximal with respect to inclusion among almost disjoint families of functions. Given an almost disjoint family $A \subset \omega^\omega$, we denote by $\mathcal{L}(A)$ the set $\{b \in \omega^\omega : b$ is not covered by finitely many $a \in A\}$. A mad subfamily $A$ of $\omega^\omega$ is $\omega$-mad\(^1\), if for every $B \in [\mathcal{L}(A)]^\omega$ there exists $a \in A$ such that $|a \cap b| = \omega$ for all $b \in B$.

The following theorems are the main results of this paper.

**Theorem 1.** It is consistent that $2^\omega = b = \omega_2$ and there exists a $\Pi^1_2$ definable $\omega$-mad family of infinite subsets of $\omega$.

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\(^1\)Such families of functions are called strongly maximal in [4, 9]. We call them $\omega$-mad just to keep the analogy with the case of subsets of $[\omega]^\omega$. 

Theorem 2. It is consistent that $2^\omega = b = \omega_2$ and there exists a $\Pi^1_2$ definable $\omega$-mad family of functions.

By [8, Theorem 8.23], in $L$ there exists a mad subfamily of $[\omega]^\omega$ which is $\Pi^1_2$ definable. Moreover, $V = L$ implies the existence of a $\Pi^1_2$ definable $\omega$-mad subfamily $A$ of $\omega^\omega$, see [4, §3]. It is easy to check that $A \cup \{ \{ n \} \times \omega : n \in \omega \}$ is actually an $\omega$-mad family of subsets of $\omega \times \omega$ for every $\omega$-mad subfamily $A$ of $\omega^\omega$, and hence $\Pi^1_2$ definable $\omega$-mad subfamilies of $[\omega]^\omega$ exist under $V = L$ as well.

Regarding the models of $\neg$CH, it is known that $\omega$-mad subfamilies of $[\omega]^\omega$ remain so after adding any number of Cohen subsets, see [5] and references therein. Combining Corollary 53 and Theorem 65 from [9], we conclude that the ground model $\omega$-mad families of functions remain so in forcing extensions by countable support iterations of a wide family of posets including Sacks and Miller forcings. If $A \in V$ is a $\Pi^1_2$ definable almost disjoint family whose $\Pi^1_2$ definition is provided by formula $\varphi(x)$, then $\varphi(x)$ defines an almost disjoint family in any extension $V'$ of $V$ (this is a straightforward consequence of the Shoenfield’s Absoluteness Theorem). Thus if a ground model $\Pi^1_2$ definable mad family remains mad in a forcing extension, it remains $\Pi^1_2$ definable by means of the same formula. From the above it follows that the $\Pi^1_2$ definable $\omega$-mad family in $L$ of functions constructed in [4, §3] remains $\Pi^1_2$ definable and $\omega$-mad in $L[G]$, where $G$ is a generic over $L$ for the countable support iteration of Miller forcing of length $\omega_2$. Thus the essence of Theorems 1 and 2 is the existence of projective $\omega$-mad families combined with the inequality $b > \omega_1$, which rules out all mad families of size $\omega_1$.

It is not known whether in ZFC one can prove the existence of $\Sigma^1_1$ mad families of functions or of $\omega$-mad families of functions; see [9].

2. Preliminaries

In this section we introduce some notions and notation needed for the proofs of Theorems 1 and 2, and collect some basic facts about $T$-proper posets, see [2] for more details.

Proposition 3. (1) There exists an almost disjoint family $R = \{ r_{(\zeta,\xi)} : \zeta \in \omega \cdot 2, \xi \in \omega_1^\\}$ in $L$ of infinite subsets of $\omega$ such that $R \cap M = \{ r_{(\zeta,\xi)} : \zeta \in \omega \cdot 2, \xi \in (\omega_1^\\)^M \}$ for every transitive model $M$ of $\text{ZF}^-$. 

(2) There exists an almost disjoint family $F = \{ f_{(\zeta,\xi)} : \zeta \in \omega \cdot 2, \xi \in \omega_1^\\} \in L$ of functions such that $F \cap M = \{ f_{(\zeta,\xi)} : \zeta \in \omega \cdot 2, \xi \in (\omega_1^\\)^M \}$ for every transitive model $M$ of $\text{ZF}^-$. 

Proof sketch. Let $r^*_{\zeta,\xi}$ be the $L$-least real coding the ordinal $(\omega^2 \cdot \xi) + \zeta$ and let $r_{\zeta,\xi}^*$ be the set of numbers coding a finite initial segment of $r^*_{\zeta,\xi}$. Similarly for functions. □

One of the main building blocks of the required $\omega$-mad family will be suitable sequences of stationary in $L$ subsets of $\omega_1$ given by the following proposition which may be proved in the same way as [1, Lemma 14].

Say that a transitive $\text{ZF}^-$ model $M$ is suitable iff $M \models \omega_2$ exists and $\omega_2 = \omega_2^L$.
Proposition 4. There exists a $\Sigma_1$ definable over $L_{\omega_2}$ tuple $\langle T_0, T_1, T_2 \rangle$ of mutually disjoint $L$-stationary subsets of $\omega_1$ and $\Sigma_1$ definable over $L_{\omega_2}$ sequences $\bar{S} = \langle S_\alpha : \alpha < \omega_2 \rangle$, $\bar{S}' = \langle S'_\alpha : \alpha < \omega_2 \rangle$ of pairwise almost disjoint $L$-stationary subsets of $\omega_1$ such that

- $S_\alpha \subset T_2$ and $S'_\alpha \subset T_1$ for all $\alpha \in \omega_2$;
- Whenever $M, N$ are suitable models of $ZF^-$ such that $\omega_1^M = \omega_1^N$, $\bar{S}^M$ agrees with $\bar{S}^N$ on $\omega_2^M \cap \omega_2^N$. Similarly for $\bar{S}'$.

The following standard fact gives an absolute way to code an ordinal $\alpha < \omega_2$ by a subset of $\omega_2$.

Fact 5. There exists a formula $\phi(x, y)$ and for every $\alpha < \omega_2^f$ a set $X_\alpha \in (\omega_1^{\omega_1^\omega})^L$ such that

1. For every suitable model $M$ containing $X_\alpha \cap \omega_1^M$, $\phi(x, X_\alpha \cap \omega_1^M)$ has a unique solution in $M$, and this solution equals $\alpha$ provided $\omega_1^M = \omega_1^f$;
2. For arbitrary suitable models $M, N$ with $\omega_1^M = \omega_1^N$ and $X_\alpha \cap \omega_1^M \in M \cap N$, the solutions of $\phi(x, X_\alpha \cap \omega_1^M)$ in $M$ and $N$ coincide.

Let $\gamma$ be a limit ordinal and $r : \gamma \to 2$. We denote by $\text{Even}(r)$ the set \{ $\alpha < \gamma : r(2\alpha) = 1$ \}. For ordinals $\alpha < \beta$ we shall denote by $\beta - \alpha$ the ordinal $\gamma$ such that $\alpha + \gamma = \beta$. If $B$ is a set of ordinals above $\alpha$, then $B - \alpha$ stands for \{ $\beta - \alpha : \beta \in B$ \}. Observe that if $\zeta$ is an indecomposable ordinal (e.g., $\omega_1^M$ for some countable suitable model of $ZF^-$), then $((\alpha + B) \cap \zeta) - \alpha = B \cap \zeta$ for all $B$ and $\alpha < \zeta$. This will be often used for $B = X_\alpha$.

For $x, y \in \omega^\omega$ we say that $y$ dominates $x$ and write $x \leq^* y$ if $x(n) \leq y(n)$ for all but finitely many $n \in \omega$. The minimal size of a subset $B$ of $\omega^\omega$ such that there is no $y \in \omega^\omega$ dominating all elements of $B$ is denoted by $b$. It is easy to see that $\omega < b \leq 2^\omega$. We say that a forcing notion $\mathbb{P}$ adds a dominating real if there exists $y \in \omega^\omega \cap V^\mathbb{P}$ dominating all elements of $\omega^\omega \cap V$.

Definition 6. Let $T \subset \omega_1$ be a stationary set. A poset $\mathbb{P}$ is $T$-proper, if for every countable elementary submodel $M$ of $H_\theta$, where $\theta$ is a sufficiently large cardinal, such that $M \cap \omega_1 \in T$, every condition $p \in \mathbb{P} \cap M$ has an $(M, \mathbb{P})$-generic extension $q$.

The following theorem includes some basic properties of $T$-proper posets.

Theorem 7. Let $T$ be a stationary subset of $\omega_1$.

1. Every $T$-proper poset $\mathbb{P}$ preserves $\omega_1$. Moreover, $\mathbb{P}$ preserves the stationarity of every stationary set $S \subset T$.
2. Let $\langle \mathbb{P}_\xi, \mathbb{Q}_\zeta : \xi \leq \delta, \zeta < \delta \rangle$ be a countable support iteration of $T$-proper posets. Then $\mathbb{P}_\delta$ is $T$-proper. If, in addition, CH holds in $V$, $\delta \leq \omega_2$, and the $\mathbb{Q}_\zeta$’s are forced to have size at most $\omega_1$, then $\mathbb{P}_\delta$ is $\omega_2$-c.c. If, moreover, $\delta < \omega_2$, then CH holds in $V^{\mathbb{P}_\delta}$.

\[\text{In what follows the phrase "X codes an ordinal } \beta \text{ in a suitable } ZF^- \text{ model } M\text{" means that there exists } \alpha < \omega_2^f \text{ such that } X = \omega_1^M \cap X_\alpha \in M \text{ and } \phi(\beta, X) \text{ holds in } M.\]
3. Proof of Theorem 1

We start with the ground model \( V = L \). Recursively, we shall define a countable support iteration \( \langle P_\alpha, \check{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle \). The desired family \( A \) is constructed along the iteration: for cofinally many \( \alpha \)'s the poset \( Q_\alpha \) takes care of some countable family \( B \) of infinite subsets of \( \omega \) which might appear in \( L(A) \) in the final model, and adds to \( A \) some \( a_\alpha \in [\omega]^{>\omega} \) almost disjoint from all elements of \( A_\alpha \) such that \( [a \cap b] = \omega \) for all \( b \in B \) (here \( A_\alpha \) stands for the set of all elements of \( A \) constructed up to stage \( \alpha \)). Our forcing construction will have some freedom allowing for further applications.

We proceed with the definition of \( P_{\omega_2} \). For successor \( \alpha \) let \( \check{Q}_\alpha \) be a \( P_\alpha \)-name for some proper forcing of size \( \omega_1 \) adding a dominating real. For a subset \( s \) of \( \omega \) and \( l \in |s| \) (= \( \text{card}(s) \leq \omega \)) we denote by \( s(l) \) the \( l \)'th element of \( s \). In what follows we shall denote by \( E(s) \) and \( O(s) \) the sets \( \{ s(2i) : 2i \in |s| \} \) and \( \{ s(2i + 1) : 2i + 1 \in |s| \} \), respectively. Let us consider some limit \( \alpha \) and a \( P_\alpha \)-generic filter \( G_\alpha \). Suppose also that

\[
\forall B \in [A_\alpha]^{<\omega} \forall r \in R \left( |E(r) \setminus B| = |O(r) \setminus B| = \omega \right)
\]

Observe that equation \((*)\) yields \( |E(r) \setminus B| = |O(r) \setminus B| = \omega \) for every \( B \in [R \cup A_\alpha]^{<\omega} \) and \( r \in R \setminus B \). Let us fix some function \( F : \text{Lim} \cap \omega_2 \to L_{\omega_2} \) such that \( F^{-1}(x) \) is unbounded in \( \omega_2 \) for every \( x \in L_{\omega_2} \). Unless the following holds, \( Q_\omega \) is a \( P_\alpha \)-name for the trivial poset. Suppose that \( F(\alpha) \) is a sequence \( \langle b_i : i \in \omega \rangle \) of \( P_\alpha \)-names such that \( b_i = b_i^{G_\alpha} \in [\omega]^{>\omega} \) and none of the \( b_i \)'s is covered by a finite subfamily of \( A_\alpha \). In this case \( Q_\omega := Q_\check{G}_\alpha \) is the two-step iteration \( \mathbb{K}_0 \ast \mathbb{K}_1 \) defined as follows.

In \( V[G_\alpha] \), \( \mathbb{K}_0 \) is some \( T_0 \cup T_2 \)-proper poset of size \( \omega_1 \). Our proof will not really depend on \( \mathbb{K}_0 \). \( \mathbb{K}_0 \) is reserved for some future applications, see section 5.

Let us fix some \( \mathbb{K}_0 \)-generic filter \( h_\alpha \) over \( V[G_\alpha] \) and find a limit ordinal \( \eta_\alpha \in \omega_1 \) such that there are no finite subsets \( J, E \) of \( (\omega \cdot 2) \times (\omega_1 \setminus \eta_\alpha) \), \( A_\alpha \), respectively, and \( \in \omega \), such that \( b_i \cup \bigcup_{(i, \xi) \in J} r_{(\xi, \xi)} \cup \bigcup E \). (The almost disjointness of the \( r_{(\xi, \xi)} \)'s imply that if \( b_i \cup \bigcup \cap A' \) for some \( R' \in [R]^{<\omega} \) and \( A' \in [A_\alpha]^{<\omega} \), then \( b_i \setminus A' \) has finite intersection with all elements of \( R \setminus R' \). Together with equation \((*)\) this easily yields the existence of such an \( \eta_\alpha \).) Let \( z_\alpha \) be an infinite subset of \( \omega \) coding a surjection from \( \omega \) onto \( \eta_\alpha \). For a subset \( s \) of \( \omega \) we denote by \( \bar{s} \) the set \( \{ 2k + 1 : k \in s \} \cup \{ 2k : k \in (\sup s \setminus s) \} \). In \( V[G_\alpha \ast h_\alpha] \), \( \mathbb{K}_1 \) consists of sequences \( \langle (s, s^*), (c_k, y_k : k \in \omega) \rangle \) satisfying the following conditions:

\begin{enumerate}
  \item \( c_k \) is a closed, bounded subset of \( \omega_1 \setminus \eta_\alpha \) such that \( S_{\alpha+k} \cap c_k = \emptyset \) for all \( k \in \omega \);
  \item \( y_k : |y_k| = 2, |y_k| > \eta_\alpha, y_k \upharpoonright \eta_\alpha = 0, \text{ and Even}(y_k) = (\{ \eta_\alpha \} \cup \{ \eta_\alpha + X_\alpha \}) \upharpoonright |y_k| ;
  \item \( s \in [\omega]^{<\omega}, s^* \in \{ r_{(m, \xi)} : m \in s, \xi \in c_m \} \cup \{ r_{(\omega+m, \xi)} : m \in s, y_m(\xi) = \emptyset \} \cup A_\alpha \}^{<\omega} \). In addition, for every \( 2n \in |s \cap r_{(0,0)}|, n \in z_\alpha \) if and only if there exists \( m \in \omega \) such that \( (s \cap r_{(0,0)})2n = r_{(0,0)}(2m) \); and
\end{enumerate}

\footnote{The tuples \( (s, s^*) \) and \( (c_k, y_k : k \in \omega) \) will be referred to as the \textit{finite part} and the \textit{infinite part} of the condition \( \langle (s, s^*), (c_k, y_k : k \in \omega) \rangle \), respectively.}
(iv) For all $k \in s \cup (\omega \setminus (\max s))$, limit ordinals $\xi \in \omega_1$ such that $\eta_\alpha < \xi \leq |y_k|$, and suitable $\mathsf{ZF}^-$ models $M$ containing $y_k \downarrow \xi$ and $c_k \cap \xi$ with $\omega^M_1 = \xi$, $\xi$ is a limit point of $c_k$, and the following holds in $M$: (Even($y_k$) − min Even($y_k$)) $\cap \xi$ codes a limit ordinal $\bar{\alpha}$ such that $S^M_{\bar{\alpha} + k}$ is non-stationary.

For conditions $\bar{p} = \langle \langle s, s^* \rangle, \langle c_k, y_k : k \in \omega \rangle \rangle$ and $\bar{q} = \langle \langle t, t^* \rangle, \langle d_k, z_k : k \in \omega \rangle \rangle$ in $K^1_{\omega_1}$, we let $\bar{q} \leq \bar{p}$ (by this we mean that $\bar{q}$ is stronger than $\bar{p}$) if and only if

(v) $(t, t^*)$ extends $(s, s^*)$ in the almost disjoint coding, i.e. $t$ is an end-extension of $s$ and $t \setminus s$ has empty intersection with all elements of $s^*$;

(vi) If $m \in \bar{t} \cup (\omega \setminus (\max \bar{t}))$, then $d_m$ is an end-extension of $c_m$ and $y_m \subseteq z_m$.

This finishes our definition of $\mathbb{P}_{\omega_2}$. Before proving that the statement of our theorem holds in $V^\mathbb{R}_{\omega_2}$ we shall establish some basic properties of $K^1_{\omega_1}$.

Claim 8. (Fischer, Friedman [1, Lemma 1].) For every condition $\bar{p} = \langle \langle s, s^* \rangle, \langle c_k, y_k : k \in \omega \rangle \rangle \in K^1_{\omega_1}$ and every $\gamma \in \omega_1$ there exists a sequence $\langle d_k, z_k : k \in \omega \rangle$ such that $\langle \langle s, s^* \rangle, \langle d_k, z_k : k \in \omega \rangle \rangle \in K^1_{\omega_1}$, $\langle \langle s, s^* \rangle, \langle d_k, z_k : k \in \omega \rangle \rangle \leq \bar{p}$, and $|z_k|, \max d_k \geq \gamma$ for all $k \in \omega$.

Claim 9. For every $\bar{p} \in K^1_{\omega_1}$ and open dense $D \subset K^1_{\omega_1}$ there exists $\bar{q} \leq \bar{p}$ with the same finite part as $\bar{p}$ such that whenever $\bar{p}_1$ is an extension of $\bar{q}$ meeting $D$ with finite part $\langle r_1, r^*_1 \rangle$, then already some condition $\bar{p}_2$ with the same infinite part as $\bar{q}$ and finite part $\langle r_1, r^*_2 \rangle$ for some $r^*_2$ meets $D$.

Proof. Let $\bar{p} = \langle \langle t_0, t^*_0 \rangle, \langle d^0_k, z^0_k : k \in \omega \rangle \rangle$ and let $\mathcal{M}$ be a countable elementary submodel of $H_\theta$ containing $K^1_{\omega_1}$, $\bar{p}$, $X_\alpha$, and $D$, and such that $j := \mathcal{M} \cap \omega_1 = \bigcup_{\epsilon \in X_\alpha \cup \omega (\max \epsilon)} S_{\alpha + k}$.

Let $\{\langle \bar{r}_n, s_n \rangle : n \in \omega \}$ be a sequence in which every pair $\langle \bar{r}, s \rangle \in (K^1_{\omega_1} \cap \mathcal{M}) \times [\omega]^<\omega$ with $\bar{p} \geq \bar{r}$ appears infinitely often. Let $\langle j_n : n \in \omega \rangle$ be increasing and cofinal in $j$. Using Claim 8, by induction on $n$ construct sequences $\langle d^n_k, z^n_k : k \in \omega \rangle$ in $\mathcal{M}$ as follows:

If there exists $\bar{r}_{n, n} \in D \cap \mathcal{M}$ below both $\bar{r}_n$ and $\langle \langle t_0, t^*_0 \rangle, \langle d^n_k, z^n_k : k \in \omega \rangle \rangle$ and with finite part of the form $\langle s_n, s^*_n \rangle$ for some $s^*_n$, then let $\langle d^{n + 1}_k, z^{n + 1}_k : k \in \omega \rangle$ be the infinite part of $\bar{r}_{1, n}$, extended further in such a way that $\langle \langle t_0, t^*_0 \rangle, \langle d^{n + 1}_k, z^{n + 1}_k : k \in \omega \rangle \rangle \in K^1_{\omega_1}$ and $|z^{n + 1}_k|, \max d^{n + 1}_k \geq j_n$ for all $n \in \omega$ and $k \in t_0 \cup \langle \omega \setminus (\max t_0) \rangle$. If there is no such $\bar{r}_{1, n}$, then let $d^{n + 1}_k$ be an arbitrary end-extension of $d^n_k$ and $z^{n + 1}_k$ be an extension of $z^n_k$ such that $|z^{n + 1}_k|, \max d^{n + 1}_k \geq j_n$ for all $n \in \omega$ and $k \in t_0 \cup \langle \omega \setminus (\max t_0) \rangle$, and $\langle \langle t_0, t^*_0 \rangle, \langle d^{n + 1}_k, z^{n + 1}_k : k \in \omega \rangle \rangle \in K^1_{\omega_1}$.

Set $d_k = \bigcup_{n \in \omega} d^n_k \cup \{j\}$ and $z_k = \bigcup_{n \in \omega} z^n_k$ for all $k \in \omega \setminus F$, $d_k = z_k = \emptyset$ for $k \in F$, and $\bar{q} = \langle \langle t_0, t^*_0 \rangle, \langle d_k, z_k : k \in \omega \rangle \rangle$. We claim that $\bar{q}$ is as required.

Let us show first that $\bar{q} \in K^1_{\omega_1}$. Only item (iv) of the definition of $K^1_{\omega_1}$ for $k \in t_0 \cup \langle \omega \setminus (\max t_0) \rangle$ and $\xi = j$ must be verified. Fix such a $k$ and suitable $\mathsf{ZF}^-$ model $M$ containing $z_k$ and $d_k$ with $\omega^M_1 = j$. Let $\bar{M}$ be the Mostowski collapse of $\mathcal{M}$ and $\pi : \mathcal{M} \rightarrow \bar{M}$ be the corresponding
Let us note that $j = \omega_1^M = \omega_1^\alpha$. Since $X_\alpha \in M$, and $M$ is elementary submodel of $H_\theta$, $\alpha$ is the unique solution of $\phi(x, X_\alpha)$ in $M$, and hence $\alpha := \pi(\alpha)$ is the unique solution of $\phi(x, X_\alpha \cap j = \pi(X_\alpha))$ in $M$. In addition, $S^M_{\alpha+k} = \pi(S_{\alpha+k}) = S_{\alpha+k} \cap j$ for all $k \in \omega$. Applying Fact 5(2) and Proposition 4, we conclude that $\phi(\alpha, X_\alpha \cap j)^M$ holds and $S^M_{\alpha+k} = S^M_{\alpha+k} = S_{\alpha+k} \cap j$. Since $d_k \in M$, $d_k \cap S_{\alpha+k} = \emptyset$, and $d_k \setminus \{j\}$ is unbounded in $j = \omega_1^M$ by the construction of $d_k$, we conclude that $S^M_{\alpha+k}$ is not stationary in $M$. This proves that $\vec{q} \in \mathbb{K}^1_\alpha$.

Now suppose that $\vec{p}_1 = \langle \langle t_1, c_1, y_1 : k \in \omega \rangle \rangle \leq \vec{q}$ and $\vec{p}_2 \in D$. Since $r_1, r_1^* \in \omega$ are finite, there exists $m \in \omega$ such that $\vec{r} := \langle \langle r_1, r_1^* \cap \mathcal{M}, \langle d^n_k, z^n_k : k \in \omega \rangle \rangle \cap \mathcal{M}$. Let $n \geq m$ be such that $\vec{s}_n, s_n = r_1 \in \mathcal{M}$. Since $\vec{p}_1$ is obviously a lower bound of $\vec{r}_n$ and $\langle \langle t_0, t_0^* \rangle, \langle d^n_k, z^n_k : k \in \omega \rangle \rangle$ with finite part $\langle s_n, r_1^* \rangle$, there exists $\vec{p}_2 \in \mathcal{M} \cap D$ below both $\vec{r}_n$ and $\langle \langle t_0, t_0^* \rangle, \langle d^n_k, z^n_k : k \in \omega \rangle \rangle$ with finite part $\langle s_n, r_2^* \rangle$ for some suitable $r_2^* \in \mathcal{M}$. Thus the first (nontrivial) alternative of the construction of $d_{k+1}^n, z_{k+1}^n$’s took place. Without loss of generality, $r_1, n = r_2$. A direct verification shows that $\vec{p}_2 = \langle \langle s_n = r_1, r_2^* \rangle, \langle d_k, z_k : k \in \omega \rangle \rangle$ is as required.

Claim 10. Let $\mathcal{M}$ be a countable elementary submodel of $H_\theta$ for sufficiently large $\theta$ containing all relevant objects with $i = \mathcal{M} \cap \omega_1$ and $\vec{p} \in \mathcal{M} \cap \mathbb{K}^1_\alpha$. If $\vec{q} \not\in \bigcup_{\vec{i} \in \mathbb{K}^1_\alpha} S_{\alpha+n}$ then there exists an $(\mathcal{M}, \mathbb{K}^1_\alpha)$-generic condition $\vec{q} \leq \vec{p}$ of the same finite part as $\vec{p}$.

Proof. Let $\vec{p} = \langle \langle s, s^* \rangle, \langle c_k, y_k : k \in \omega \rangle \rangle \langle D_n : n \in \omega \rangle$ and $\langle D_n : n \in \omega \rangle$ be the collection of all open dense subsets of $\mathbb{K}^1_\alpha$ which are elements of $\mathcal{M}$, and $\langle i_n : n \in \omega \rangle$ be an increasing sequence of ordinals converging to $i$. Using Claims 8 and 9, inductively construct a sequence $\langle \vec{q}_n : n \in \omega \rangle \subset \mathcal{M} \cap \mathbb{K}^1_\alpha$, where $\vec{q}_n = \langle \langle s, s^* \rangle, \langle d^n_k, z^n_k : k \in \omega \rangle \rangle$ and $\vec{q}_0 = \vec{p}$, such that

(i) $d^n_{k+1}$ is an end-extension of $d^n_k$ and $z^{n+1}_k$ is an extension of $z^n_k$ for all $n \in \omega$ and $k \in s \cup (\omega \setminus (\text{max} s))$;

(ii) $|z^n_k|, \max d^n_k \geq i_n$ for all $n \geq 1$ and $k \in s \cup \omega \setminus (\text{max} s)$ and

(iii) For every $n \geq 1$ and $\vec{r} = \langle \langle r, r^* \rangle, \langle d^n_k, z^n_k : k \in \omega \rangle \rangle \leq \vec{q}_n, \vec{r} \in \mathcal{D}_n$, there exists $r^*_2$ such that $\vec{r}_2 := \langle \langle r_1, r_2^* \rangle, \langle d^n_k, z^n_k : k \in \omega \rangle \rangle \in \mathcal{D}_n$ and $\vec{r}_2 \leq \vec{q}_n$.

Set $d_k = \bigcup_{n \in \omega} d^n_k \cup \{i\}$ and $z_k = \bigcup_{n \in \omega} z^n_k$ for all $k \in s \cup (\omega \setminus (\text{max} s))$, $d_k = z_k = \emptyset$ for all other $k \in \omega$, and $\vec{q} = \langle \langle t_0, t_0^* \rangle, \langle d_k, z_k : k \in \omega \rangle \rangle$. We claim that $\vec{q}$ is as required, i.e., $\vec{q} \in \mathbb{K}^1_\alpha$ and $\mathcal{D}_n \cap \mathcal{M}$ is pre-dense below $\vec{q}$ for every $n \in \omega$. The fact that $\vec{q} \in \mathbb{K}^1_\alpha$ can be shown in the same way as in the proof of Claim 9.

Let us fix $n \in \omega$ and $\vec{r}_1 = \langle \langle t_1, t_1^* \rangle, \langle d^n_k, z^n_k : k \in \omega \rangle \rangle \leq \vec{q}$.

Without loss of generality, $\vec{r}_1 \in \mathcal{D}_n$. Since $\vec{r}_1 \leq \vec{q}_n$, $\vec{r}_1 \in \mathcal{D}_n$. It is clear that $\vec{r}_2 \in \mathcal{M}$. We claim that $\vec{r}_2$ and $\vec{r}_1$ are compatible. Indeed, set $\vec{r}_3 = \langle \langle t_1, t_2^* \cup t_1^* \rangle, \langle d^n_k, z^n_k : k \in \omega \rangle \rangle$ and note that $\vec{r}_3 \leq \vec{r}_1, \vec{r}_2$.

Let $H_\alpha$ be a $\mathbb{K}^1_\alpha$-generic filter over $L[G_\alpha * h_\alpha]$. Set $Y_{k}^\alpha = \bigcup_{\vec{p} \in H_\alpha} y_k$, $C_k^\alpha = \bigcup_{\vec{p} \in H_\alpha} c_k$, $a_\alpha = \bigcup_{\vec{p} \in H_\alpha} s$, $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$, and $S^\alpha = \bigcup_{\vec{p} \in H_\alpha} s^\alpha$, where
\(\bar{p} = \langle (s, s^*), (c_k, y_k : k \in \omega) \rangle\). The following statement is a consequence of the definition of \(K^1_\alpha\) and the genericity of \(H_\alpha\).

**Claim 11.**

1. \(S^* = \{r_{(m, \xi)} : m \in \overline{\alpha}, \xi \in C_\alpha \} \cup \{r_{(\omega + m, \xi)} : m \in \overline{\alpha}, Y_m(\xi) = 1\} \cup A_\alpha;\)
2. \(a_\alpha \in [\omega]^{\omega};\)
3. If \(m \in \overline{\alpha}\), then \(\text{dom}(Y_m) = \omega_1\) and \(C_\alpha\) is a club in \(\omega_1\) disjoint from \(S_{\alpha + m};\)
4. \(a_\alpha\) is almost disjoint from all elements of \(A_\alpha;\)
5. If \(m \in \overline{\alpha}\), then \(|a_\alpha \cap r_{(m, \xi)}| < \omega\) if and only if \(\xi \in C_\alpha;\)
6. If \(m \in \overline{\alpha}\), then \(|a_\alpha \cap r_{(\omega + m, \xi)}| < \omega\) if and only if \(Y_m(\xi) = 1;\)
7. \(|a_\alpha \cap b_i| = \omega\) for all \(i \in \omega;\)
8. For every \(n \in \omega, n \in z_\alpha\) if and only if there exists \(m \in \omega\) such that \(\langle a_\alpha \cap r_{(0, 0)}(2n) = r_{(0, 0)}(2m)\rangle;\) and
9. Equation (*) holds for \(\alpha + 1, i.e. for every \(r \in R\) and a finite subfamily \(B\) of \(A_{\alpha + 1}\), \(B\) covers neither a cofinite part of \(E(r)\) nor of \(O(r)\).

**Proof.** Items (1), (2), (4), and (9) are straightforward. Items (2), (5), (6), and (8) follow from the inductive assumption (*). Item (3) is a consequence of Claim 8.

We are left with the task to prove (7). Let us fix \(l, i \in \omega\) and denote by \(D_{l, i}\) the set of conditions \(\langle (s, s^*), (c_k, y_k : k \in \omega) \rangle \in K^1_\alpha\) such that \((s \setminus l) \cap b_i \neq \emptyset\). It suffices to show that \(D_{l, i}\) is dense in \(K^1_\alpha\). Fix a condition \(\bar{p} = \langle (s, s^*), (c_k, y_k : k \in \omega) \rangle \in K^1_\alpha\) and set \(x = b_i \cup s^*\). Note that \(x \in [\omega]^{\omega}\) by our choice of \(\eta_\alpha\) and items (i), (ii) of the definition of \(K^1_\alpha\). Two cases are possible.

1. \(|x \setminus r_{(0, 0)}| = \omega\). Then \(\bar{q} := \langle (s \cup \{\min(x \setminus (r_{(0, 0)} \cup l \cup \max s)\}), s^*), (c_k, y_k : k \in \omega)\rangle\) is an element of \(D_{l, i}\) and is stronger than \(\bar{p}\).
2. \(x \subset^* r_{(0, 0)}\). Without loss of generality, \(x \setminus r_{(0, 0)} \subset l\). Suppose that \(|s \cap r_{(0, 0)}| = 2j - 1\) for some \(j \in \omega\) (the case of even \(|s \cap r_{(0, 0)}|\) is analogous and simpler). Let \(y = r_{(0, 0)} \setminus \max s^*\) and note that \(x \subset^* y\). By (*), \(|y \cap E(r_{(0, 0)})| = |y \cap O(r_{(0, 0)})| = \omega\). Denote by \(m_e\) and \(m_o\) the minima of the sets \((y \cap E(r_{(0, 0)})) \setminus (l \cup (\max s + 1))\) and \((y \cap O(r_{(0, 0)})) \setminus (l \cup (\max s + 1))\), respectively. Set \(\bar{r} := \langle (s \cup \{m_e\} \cup \{\min(x \setminus (m_e + 1))\}, s^*), (c_k, y_k : k \in \omega)\rangle\) if \(j \in z_\alpha\) and \(\bar{r} := \langle (s \cup \{m_o\} \cup \{\min(x \setminus (m_o + 1))\}, s^*), (c_k, y_k : k \in \omega)\rangle\) otherwise. A direct verification shows that \(\bar{r} \in D_{l, i}\) and \(\bar{r} \leq \bar{p}\). \(\square\)

**Corollary 12.** \(\hat{Q}_\alpha\) is \(T_0\)-proper. Consequently, \(\mathbb{P}_{\omega_2}\) is \(T_0\)-proper and hence preserves cardinals.

More precisely, for every condition \(\bar{p} = \langle (s, s^*), (c_k, y_k : k \in \omega) \rangle \in K^1_\alpha\) the poset \(\{\bar{r} \in K^1_\alpha : \bar{r} \leq \bar{p}\} \) is \(\omega_1 \setminus \bigcup_{n \in \mathbb{R}(\omega \setminus (\max s))} S_{\alpha + n}\)-proper. Consequently, \(S_{\alpha + n}\) remains stationary in \(V^n_{\omega_2}\) for all \(n \in \omega \setminus \overline{\alpha}\).
For every countable suitable model $A$ is a $\Pi^1_2$ definable subset of $[\omega]^\omega$ in $L[G]$ and thus finishes the proof of Theorem 1.

**Lemma 13.** In $L[G]$ the following conditions are equivalent:

1. $a \in A$;
2. For every countable suitable model $M$ of $\text{ZF}^-$ containing $a$ as an element there exists $\bar{a} < \omega_2^M$ such that $S_{\bar{a} + k}^M$ is nonstationary in $M$ for all $k \in \bar{a}$.

**Proof.** (1) $\rightarrow$ (2). Fix $a \in A$ and find $\alpha < \omega_2$ such that $a = a_\alpha$. Fix also a countable suitable model $M$ of $\text{ZF}^-$ containing $a_\alpha$ as an element. By Claim 11(5, 6, 8), $z_\alpha \in M$ and $C^\alpha_k \cap \omega_1^M, Y^\alpha_k \restriction \omega_1^M \in M$ for all $k \in \bar{\alpha}$. Therefore $\eta_{\alpha} < \omega_1^M$. Since $\langle \langle 0, 0 \rangle, (C^\alpha_k \cap (\omega_1^M + 1), Y^\alpha_k \restriction \omega_1^M : k \in \omega) \rangle$ is a condition in $\mathbb{K}^1_\alpha$, item (iv) of the definition of $\mathbb{K}^1_\alpha$ ensures that for every $k \in \bar{\alpha}$, $\text{Even}(Y^\alpha_k \restriction \omega_1^M) - \text{min Even}(Y^\alpha_k \restriction \omega_1^M)$ codes a limit ordinal $\bar{\alpha}_k \in \omega_2^M$ such that $S_{\bar{\alpha}_k + k}^M$ is nonstationary in $M$. By item (ii) of the definition of $\mathbb{K}^1_\alpha$,

$$\text{Even}(Y^\alpha_k \restriction \omega_1^M) - \text{min Even}(Y^\alpha_k \restriction \omega_1^M) = X_\alpha \cap \omega_1^M$$

for every $k \in I$, and hence $\bar{\alpha}_k$’s do not depend on $k$.

(2) $\rightarrow$ (1). Let us fix $a$ fulfilling (2) and observe that by Löwenheim-Skolem, (2) holds for arbitrary (not necessarily countable) suitable model of $\text{ZF}^-$ containing $a$. In particular, it holds in $M = L_{\omega_2}[G]$. Observe that $\omega_2^M = \omega_2^L[G] = \omega_2^\beta$, $S^M = \tilde{S}$, and the notions of stationarity of subsets of $\omega_1$ coincide in $M$ and $L[G]$. Thus there exists $\alpha < \omega_2$ such that $S_{\bar{\alpha} + k}^M$ is nonstationary for all $k \in \bar{\alpha}$. Since the stationarity of some $S_{\bar{\alpha} + k}$’s has been destroyed, Corollary 12 together with the $T_2$-properness of $\mathbb{K}^0_\xi$’s implies that $\check{Q}_\alpha$ is not trivial. Now, the last assertion of Corollary 12 easily implies that $a = a_\alpha$. \hfill $\square$

4. PROOF OF THEOREM 2

The proof is completely analogous to that of Theorem 1. Therefore we just define the corresponding poset $\mathbb{P}_{\omega_2}$, the use of the poset $\mathbb{M}^1_\alpha$ defined below instead of $\mathbb{K}^1_\alpha$ at the $\alpha$’s stage of iteration being the only significant change. We leave it to the reader to verify that the proof of Theorem 1 can be carried over.

For successor $\alpha$ let $\check{Q}_\alpha$ be a $\mathbb{P}_\alpha$-name for some proper forcing of size $\omega_1$ adding a dominating real. Let us consider some limit $\alpha$ and a $\mathbb{P}_\alpha$-generic filter $G_\alpha$. Suppose also that we have already constructed an almost disjoint family $A_\alpha \subset \omega^\omega$ such that

$$\forall E \in [A_\alpha]^{<\omega} \forall f \in F \ (\vert f \restriction (2\omega) \setminus E \vert = \vert f \restriction (2\omega + 1) \setminus E \vert = \omega) \quad (**)$$

Equation (**) yields

$$\forall E \in [F \cup A_\alpha]^{<\omega} \forall f \in F \setminus E \ (\vert f \restriction (2\omega) \setminus E \vert = \vert f \restriction (2\omega + 1) \setminus E \vert = \omega).$$

Let $F : \text{Lim} \cap \omega_2 \rightarrow L_{\omega_2}$ be the same as in the proof of Theorem 1. Unless the following holds, $\check{Q}_\alpha$ is a $\mathbb{P}_\alpha$-name for the trivial poset. Suppose that $F(\alpha)$ is a sequence $\langle b_i : i \in \omega \rangle$ of $\mathbb{P}_\alpha$-names such that $b_i = b_i^{\check{Q}_\alpha} \in \omega^\omega$.
and none of the \( b_i \)'s is covered by a finite subfamily of \( A_\alpha \). In this case \( \mathcal{Q}_\alpha := \mathcal{Q}_\alpha^G \) is the two-step iteration \( \mathbb{K}_\alpha^0 \ast \mathbb{M}_\alpha \) defined as follows.

In \( V^{\mathcal{P}_\alpha \ast \mathbb{K}_\alpha^0} \), \( \mathbb{K}_\alpha^0 \) is some \( T_0 \cup T_2 \)-proper poset of size \( \omega_1 \).

Let us fix a recursive bijection \( \psi : \omega \times \omega \rightarrow \omega \) and \( s \in \omega^{<\omega} \). Set \( \text{sq}(s) = \text{dom}(s) \times (\text{dom}(s) + \text{ran}(s)) \) and

\[
\bar{s} = \{2k + 1 : k \in \psi(s)\} \cup \{2k : k \in \psi(\text{sq}(s) \setminus s)\}.
\]

In \( V^{\mathcal{P}_\alpha \ast \mathbb{K}_\alpha^0} \) find an ordinal \( \eta_\alpha \in \omega_1 \) such that there are no finite subsets \( J, E \) of \( (\omega \cdot 2) \times (\omega_1 \setminus \eta_\alpha) \), \( A_\alpha \), respectively, and \( i \in \omega \), such that \( b_i \subset \bigcup_{(\xi,\ell) \in J} f(\xi,\ell) \cup E \). \( M_\alpha^1 \) consists of sequences \( \langle \langle s, s^* \rangle, \langle c_k, y_k : k \in \omega \rangle \rangle \) satisfying the following conditions:

- \( (i) \_f \) Conditions \((i)-(ii)\) from the definition of \( \mathbb{K}_\alpha^1 \) in the proof of Theorem 1 hold;
- \( (ii) \_f \) \( s \in \omega^{<\omega} \), \( s^* \in \left\{ \{f(m,\xi) : m \in \bar{s}, \xi \in c_m\} \cup \{f(\omega + m,\xi) : m \in \bar{s}, y_m(\xi) = 1\} \cup A_\alpha \right\}^{<\omega} \). In addition, for every \( 2n \in [s \cap f(0,0), \omega_1 \setminus \eta_\alpha] \), \( n \in z_\alpha \) if and only if there exists \( m \in \omega \) such that \( s(j) = f(0,0)(2m) \), where \( j \) is the \( 2n \)th element of the domain of \( s \cap f(0,0) \); and
- \( (iii) \_f \) For all \( m \in \bar{s} \cup \{2k, 2k + 1 : k \in \psi(\omega \setminus \text{dom}(s)) \times \omega) \} \), limit ordinals \( \xi \in \omega_1 \) such that \( \eta_\alpha < \xi \leq |y_m| \), and suitable \( \text{ZF}^- \) models \( M \) containing \( y_m \upharpoonright \xi \) and \( c_m \cap \xi \) with \( \omega_1^M = \xi \), \( \xi \) is a limit point of \( c_m \), and the following holds in \( M \): \((\text{Even}(y_m) - \text{min Even}(y_m)) \cap \xi \) codes a limit ordinal \( \bar{\alpha} \) such that \( S_{\alpha + m}^M \) is non-stationary.

For conditions \( \bar{p} = \langle \langle s, s^* \rangle, \langle c_k, y_k : k \in \omega \rangle \rangle \) and \( \bar{q} = \langle \langle t, t^* \rangle, \langle d_k, z_k : k \in \omega \rangle \rangle \) in \( M_\alpha^1 \), \( \bar{q} \leq \bar{p} \) if and only if

- \( (iv) \_f \) \( s \subset t \), \( s^* \subset t^* \), and \( t \setminus s \) has empty intersection with all elements of \( s^* \);
- \( (v) \_f \) If \( m \in \bar{s} \cup \{2k, 2k + 1 : k \in \psi(\omega \setminus \text{dom}(s)) \times \omega) \} \), then \( d_m \) is an end-extension of \( c_m \) and \( y_m \subset z_m \).

5. Final remarks

The fact that \( S_\alpha^1 \cap S_\beta = \emptyset \) for all \( \alpha, \beta < \omega_2 \) together with the freedom to choose \( \mathbb{K}_\alpha^0 \) to be an arbitrary \( T_0 \cup T_2 \)-proper forcing of size \( \omega_1 \) allow for combining the proofs of Theorems 1, 2 and [1, Theorem 1]. In addition, we could take \( \mathbb{K}_\alpha^0 \) to be a name for a two-step iteration with second component equal to the poset used in the proof of [1, Theorem 1] at stage \( \alpha \), and first component equal to a name of a c.c.c poset of size \( \omega_1 \) (Theorem 7(2) allows us to arrange a suitable bookkeeping of such names). This gives us the following statements.

**Theorem 14.** It is consistent with Martin’s Axiom that there exists a \( \Delta_3^1 \) definable wellorder of the reals and a \( \Pi_3^1 \) definable \( \omega \)-mad family of infinite subsets of \( \omega \).

**Theorem 15.** It is consistent with Martin’s Axiom that there exists a \( \Delta_3^1 \) definable wellorder of the reals and a \( \Pi_3^1 \) definable \( \omega \)-mad family of functions.
The following questions remain open. In all questions we are interested in families of infinite subsets of $\omega$ as well as in families of functions from $\omega$ to $\omega$.

**Question 16.** Is it consistent to have $b > \omega_1$ with a $\Sigma^1_2$ definable ($\omega$-)mad family?

**Question 17.** Is it consistent to have $\omega_1 < b < 2^\omega$ with a $\Pi^1_2$ definable ($\omega$-)mad family?

In the proofs of Theorems 1 and 2 we ruled out all mad families of size $\omega_1$ by making $b$ big. Alternatively, one could use the methods developed in [1] and prove the consistency of $\omega_1 = b < a = \omega_2$ together with a $\Delta^1_3$ definable $\omega$-mad family. This suggests the following

**Question 18.** Is it consistent to have $b < a$ and a $\Pi^1_2$ definable ($\omega$-)mad family?

**Question 19.** Is a projective ($\omega$-)mad family consistent with $b \geq \omega_3$?

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