PATTERNS OF STATIONARY REFLECTION

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Abstract. We consider the behavior of the stationary reflection property $\text{RP}(\kappa \cap \text{cof}(\aleph_n))$ across the class of all cardinals and prove that it has only trivial $\text{ZFC}$ constraints.

In this paper we will examine the global behavior of stationary reflection. A stationary subset $S \subseteq \delta$ reflects if there is some ordinal $\alpha < \delta$ of uncountable cofinality such that $S \cap \alpha$ is a stationary subset of $\alpha$. Stationary reflection is a basic notion of compactness that is studied widely in the set-theoretic literature because it serves to distinguish inner models like Gödel’s Constructible Universe $L$ from models containing very large cardinals. The question of whether a given cardinal $\delta$ has a stationary subset is known to be independent of $\text{ZFC}$. For example, if $\kappa$ is weakly compact, then every stationary subset of $\kappa$ reflects, but if $\kappa$ is a successor cardinal in $L$, then every stationary subset of $\kappa$ has a non-reflecting stationary subset.

Our project here is to prove an Easton-style result for stationary reflection across the class of all cardinals: we show that, given a fixed cofinality $\lambda$, the existence of a non-reflecting stationary subset of $\kappa \cap \text{cof}(\lambda)$ does not depend on the existence of non-reflecting stationary subsets of $\mu \cap \text{cof}(\lambda)$ for $\mu < \kappa$ as long as $\kappa$ is regular. In other words, there is no Silver’s Theorem for stationary reflection at a fixed cofinality. For the sake of exposition we prove our result for the fixed cofinalities $\aleph_n$ for $n < \omega$ because a result of Shelah allows us to handle the approachability ideal in a convenient manner. However, we believe that our methods will generalize to any fixed cofinality.

The naive approach of simply forcing non-reflecting stationary sets wherever desired does not work, because we risk adding unintended non-reflecting sets. Hence, our work here will be of interest in part because of the methods we use in order to surmount this obstacle. First, the class forcing that we employ is a hybrid between iterated forcing and product forcing in the sense that the forcing at a particular stage depends on the prior stages, while at the same time the forcing can be factored as a product of successors of regular cardinals. Second, we make use of $\text{PCF}$-theoretic ideas by introducing Easton-supported scales on wide products and exploiting the good points of these scales. (The hybrid notion resembles a technique of Cummings and Shelah, but our presentation is quite different because of the scales that we use [5].) The good points allow us to restore stationary reflection when necessary. The techniques that we introduce with these scales should be applicable to further results.

Set theorists have long been interested in the global phenomena. The most notable example is Easton’s result that he proved that the continuum function $\kappa \mapsto 2^\kappa$ is constrained only by monotonicity and König’s Theorem (which states

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that $2^\kappa > \kappa$) when restricted to regular cardinals \[6\]. At first it was expected that this result would be extended to include singular cardinals, but singular cardinals turn out to be much more complicated: Silver proved that GCH cannot fail for the first time at a singular of uncountable cofinality \[13\]. Generally, singular cardinals present a challenge to global results.

Here we consider $\text{RP}(S)$ where $S$ is stationary in some $\kappa$, which states every stationary subset $T \subseteq S$ reflects. More precisely, we examine $\text{RP}(\kappa \cap \text{cof}(\aleph_n))$. This property winds up being compelling precisely because we can precisely determine the ZFC constraints on its behavior. The constraints are:

- $\text{RP}(\kappa \cap \text{cof}(\aleph_n))$ holds vacuously if $\kappa \leq \aleph_n$;
- $\aleph_{n+1} \cap \text{cof}(\aleph_n)$ has a non-reflecting stationary subset;
- If $\lambda$ is a singular cardinal, then $\lambda$ has a non-reflecting stationary subset if and only if $\text{cf} \lambda$ has a non-reflecting stationary subset.

One might object that the third bullet point makes singulars uninteresting, but in our case it is the successors of singulars that are difficult to handle—so the difficult case nonetheless pertains to singular cardinals.

Our main theorem, which takes up the whole paper, is the following:

**Theorem 1.** Suppose $\chi$ is a supercompact cardinal in $V$ such that GCH holds above $\chi$. Let $F$ be a definable 2-valued function on the class of regular cardinals $\geq \chi$. Then there is an extension $W \supseteq V$ in which $\chi = \aleph_{n+2}$, cofinalities $\geq \chi$ are preserved, GCH is preserved above $\chi$, and for all regular $\kappa \in W$ such that $\kappa \geq \aleph_{n+2}$, there is a non-reflecting stationary subset of $\kappa \cap \text{cof}(\aleph_n)$ if and only if $F(\kappa) = 1$.

It follows that there is a lot of flexibility around when $\text{RP}(\kappa \cap \text{cof}(\aleph_n))$ holds for different $\kappa$:

**Corollary 2.** Relative to the consistency of a supercompact cardinal, it is consistent that there is a model in which a given regular $\kappa \geq \aleph_{n+2}$ has a non-reflecting stationary subset of $\kappa \cap \text{cof}(\aleph_n)$ precisely when:

- $\kappa$ is a successor of a regular cardinal;
- $\kappa$ is the successor of a singular cardinal;
- $\kappa$ is inaccessible;
- $\kappa$ is not inaccessible.

We assume familiarity with basics of forcing and large cardinals \[8\].

1. **Methods**

1.1. **Large Cardinal Notions.** We need large cardinals to obtain stationary reflection. In fact, stationary reflection at a successor of a singular cardinal $\lambda^+$ implies the failure of $\square_\lambda$, which has strong inner model-theoretic consequences \[11\]. Hence, some large cardinal hypothesis is necessary for our result.

**Definition 1.** A cardinal $\kappa$ is $\lambda$-supercompact for $\lambda \geq \kappa$ if there is an elementary embedding $j : V \to M \subseteq V$ from the class of all sets to a proper subclass such that $j \upharpoonright V_\kappa = \text{id} \upharpoonright V_\kappa$, $j(\kappa) > \lambda$, and $M^\lambda \subseteq M$. A cardinal $\kappa$ is supercompact if $\kappa$ is $\lambda$-supercompact for all $\lambda \geq \kappa$.

We plan to force over a model with a supercompact cardinal and then lift the supercompact embedding to recover reflection. Our ability to lift this embedding without destroying stationarity subsets is the technical core of this paper.
We will need weakly compact cardinals for approachability, but their definition can be a black box [8]. And if we have a supercompact cardinal, then we have many weakly compact cardinals.

**Fact 1.** If $\kappa$ is supercompact, then there are $\kappa$-many weakly compact cardinals below $\kappa$.

This is because a supercompact $\kappa$ is measurable, and under the measurable embedding $j: V \to M$, $M \models "\kappa$ is weakly compact".

### 1.2. Forcing Techniques.

We will consider partial orders that are not quite closed, so we will define a weakening of closure that appears frequently in the literature.

**Definition 2.** Fix a poset $P$ and consider the following game played by two players. The play of the game is a decreasing sequence of conditions $\langle p_\xi : \xi < \tau \rangle$ in which Player II chooses conditions $p_\xi$ at limits $\xi$ as well as even successor ordinals (successors of the form $\xi + n$ where $\xi$ is a limit and $n$ is even), and Player I chooses conditions at odd successor ordinals. Player II wins a play $\langle p_\xi : \xi \leq \eta \rangle$ of length $\eta$ if it is possible for the players to make a move at every step of the play, i.e. there is a lower bound of the sequence $\langle p_\xi : \xi < \xi' \rangle$ for all $\xi' \leq \eta$.

We say that $P$ is $\eta$-weakly strategically closed, or simply that $P$ is $\eta$-strategically closed, if Player II has a winning strategy for all games of length $\eta$. In other words, $P$ is $\eta$-strategically closed if the there is a function $\sigma: \leq^\eta P \to P$ such that for all $\xi \leq \eta$, if $p_\xi = \sigma(\langle p_\zeta : \zeta < \xi \rangle)$ for all $\xi < \eta$ such that $\xi$ is either a limit or an even successor, then $\langle p_\zeta : \zeta < \xi \rangle$ has a lower bound.

Now consider a similar game on $P$ between Player I and Player II where instead Player II only plays at even successors while Player I plays at all other ordinals. If Player II still has a winning strategy for this game, we say that $P$ is $\kappa$-strongly strategically closed.

We need strategic closure because we will use it to get preservation of cardinals in our forcing extension.

**Fact 2.** If $\kappa$ is regular and $P$ is $\eta$-strategically closed for all $\eta < \kappa$, then $P$ is $\kappa$-distributive, meaning that it does not add new sequences of ordinals of length $< \kappa$.

Now we introduce a poset that for adding a non-reflecting stationary subset of $\kappa \cap \text{cof}(\lambda)$. The poset that we will use in our construction will be akin to a product of instances of this poset—with some additional details.

**Definition 3.** If $\kappa$ and $\lambda$ are regular uncountable cardinals with $\lambda < \kappa$, then $S(\kappa, \lambda)$ is the poset consisting of conditions $p$ such that:

- $p$ is a function from $\alpha + 1$ to $\{0, 1\}$ for some $\alpha < \kappa$;
- if $p(\beta) = 1$ for some $\beta \leq \alpha$, then $\text{cf}(\beta) = \lambda$;
- for all $\beta \leq \text{max dom } p$ such that $\text{cf} \beta > \omega$, there is a closed unbounded subset $c \subseteq \beta$ such that $c \cap \{ \gamma \in \text{dom } p : p(\gamma) = 1 \} = \emptyset$.

If $p, q \in S(\kappa, \lambda)$ then $p \leq q$ precisely when $q \subseteq p$.

The notation we are using in terms of the symbols $S$, the parentheses, and so forth, is not strictly standard. The non-reflecting stationary set added by $S(\kappa, \lambda)$ is the function defined by the generic.
Fact 3. If $G$ is $S(\kappa, \lambda)$-generic, then $\{ \alpha < \kappa : \exists p \in G, p(\alpha) = 1 \}$ is a non-reflecting stationary subset if $\kappa \cap \text{cof}(\lambda)$.

The poset $S(\kappa, \lambda)$ has several reasonable properties.

Facts 4.

1. $S(\kappa, \lambda)$ is $\lambda$-closed.
2. $S(\kappa, \lambda)$ is $\eta$-strategically closed for all $\eta < \kappa$, and hence $\kappa$-distributive.
3. If GCH holds, then $|S(\kappa)| = \kappa$ and therefore $S(\kappa)$ satisfies the $\kappa^+$-chain condition.

Hence, under GCH, the poset $S(\kappa, \lambda)$ preserves cardinals and cofinalities. However, it will take more work to show that one can add non-reflecting sets at class-many cardinals without affecting cardinals and cofinalities.

Because $S(\kappa, \lambda)$ is not $\kappa$-closed, we need to define a poset that will represent the quotient of $S(\kappa, \lambda)$ with a $\kappa$-closed poset. This will help us lift the supercompact embedding at the end of our construction.

Definition 4. If $G$ is $S(\kappa, \lambda)$-generic and $S = \{ \alpha < \kappa : \exists p \in G, p(\alpha) = 1 \}$, then in the extension by $G$ we can define $T(\kappa, \lambda)$ to be the set of closed bounded sets $t \subseteq \kappa$ such that $t \cap S = \emptyset$. The conditions are ordered by end-extension, i.e. $t \leq s$ if and only if $t \cap (\max s + 1) = s$.

Fact 5. $S(\kappa, \lambda) \ast T(\kappa, \lambda)$ has a $\kappa$-directed closed dense subset, and therefore $T(\kappa, \lambda)$ is $\kappa$-distributive.

Again, it will take more work to show that we can utilize this stationary-killing forcing for class-many cardinals. However, we will make use of the limited closure properties of this poset.

Fact 6. $T(\kappa, \lambda)$ is $\lambda$-closed (and $T(\kappa, \kappa_\alpha)$ is not closed).

The last forcing concept that we will crucially use pertains to the support of the (ersatz) product that we employ for our construction.

Definition 5. Suppose $I$ is a set of cardinals, and consider the product of posets $P = \prod_{\kappa \in I} P_\kappa$. Given $p \in P$, the support of $p$, denoted $\text{sprt}(p)$, is the set of $\kappa \in I$ such that $p(\kappa) \neq 1_{P_\kappa}$. We say that a subset $Q$ of the full-support product $P$ is an Easton product—or has Easton support—if $Q$ consists of those $p \in P$ such that for all regular $\mu$, $|\text{sprt}(p) \cap \mu| < \mu$. This definition also applies when $I$ is a proper class.

1.3. Approachability and the $H_\Theta$ Technique. In this paper we will make ample use of a technique developed by Shelah. The general idea begins by listing the parameters $a_1, \ldots, a_k$ relevant to an argument and to choose a regular cardinal $\Theta$ which is “large enough” in the sense that $\Theta > |\text{tc}(a_1)|, \ldots, |\text{tc}(a_k)|$, and moreover $\Theta$ is large enough that the set $H_\Theta$ of all sets $x$ with $|\text{tc}(x)| < \Theta$ witnesses the result of a statement that needs to be proved. Then we choose a well-order $<_\Theta$ on $H_\Theta$, and we consider the model $\mathcal{H} := H_\Theta(\in, <_\Theta, a_1, \ldots, a_k)$, and run an argument in the model $\mathcal{H}$. The point of this technique is usually that we want to use a club $\langle M \cap \mu : M < \mathcal{H} \rangle$ for some cardinal $\mu$, but writing down the actual definition of such a club in terms of its closure properties is infeasible. There are several good sources for further reading [7] [4].

Occasionally the $H_\Theta$ technique requires the notion of approachability. When we consider models $M \not\prec H_\Theta$, the models are not necessarily closed, so approachability is a way for these models to be in some sense closed enough.
Definition 6. Suppose $\mu$ is regular with $\mu^{<\mu} = \mu$ and $\bar{a} = (a_\alpha : \alpha < \mu)$ is an enumeration of $[\mu]^{<\mu}$. Then a point $\alpha < \mu$ is approachable with respect to $\bar{a}$ if and only if there is an unbounded set $A \subseteq \alpha$ with $\text{ot} \ A = \text{cf} \alpha$ such that for all $\beta < \alpha$ there is some $\gamma < \alpha$ such that $A \cap \beta = a_\gamma$. We write $S \in I[\mu]$ if there is a club $C \subseteq \mu$ such that every point in $S \cap C$ is approachable with respect to $\bar{a}$.

When we prove our theorem, we will not have full approachability—this is the case if $\mu \in I[\mu]$. However, it is important to point out that the extent of approachability that we will have is well-defined:

Fact 7. If $\mu^{<\mu} = \mu$, then $I[\mu]$ is well-defined modulo a club regardless of a specific enumeration of $[\mu]^{<\mu}$.

1.4. PCF Theory on Wide Products. The purpose of this section is to define Easton-supported scales and to give conditions by which these scales have good points. These good points will in turn be used for the interleaving argument that will be the crux of our strategic closure and stationary preservation arguments.

Definitions 7.

1. If $\lambda$ is a singular cardinal and $L \subseteq \lambda$ is a set of regular cardinals unbounded in $\lambda$, let $f \in \prod L$ mean that $\text{dom} \ f$ is an Easton subset of $L$, i.e. $\text{dom} \ f \cap \kappa < \kappa$ for every regular $\kappa < \lambda$, and that for all $\kappa \in \text{dom} \ f$, $f(\kappa) < \kappa$. We say that $\prod L$ is a product of width $\lambda$.

2. If $\prod L$ is a product of width $\lambda$ and $D, E \subseteq L$ are Easton sets (for all regular $\kappa < \lambda$, $|D \cap \kappa|, |E \cap \kappa| < \kappa$), then we write $D \subseteq^* E$ if there is some $\tau < \lambda$ such that $D \subseteq (\tau, \lambda) \subseteq E$, and we say that this relation is witnessed by $\tau$. We write that $f <^* g$ if there is some $\tau < \lambda$ witnessing that $\text{dom} \ f \subseteq^* \text{dom} \ g$ and such that for all $\kappa \in \text{dom} \ f \cap (\tau, \lambda)$, $f(\kappa) < g(\kappa)$. The corresponding notions for equality and for non-strict inequality are denoted $f =^* g$ and $f \leq^* g$ respectively.

3. If $\prod L$ is a product of length $\lambda$, then we say that two $<^*$-increasing sequences $\bar{f} = (f_\beta : \beta < \alpha)$ and $\bar{g} = (g_\beta : \gamma < \delta)$ cofinally interleave each other if for all $\beta_0 < \alpha$, there is some $\beta_0 < \gamma$ such that $f_{\beta_0} <^* g_{\beta_0}$, and for all $\delta_0 < \gamma$, there is some $\beta_0 < \alpha$ such that $g_{\beta_0} <^* f_{\beta_0}$.

4. Given a product $L$, a wide scale is a sequence $(f_\alpha : \alpha < \lambda^+)$ of functions such that:

(a) $\forall \alpha < \lambda^+, \ f_\alpha \in \prod L$;
(b) $\forall \alpha < \beta < \lambda^+, \ f_\alpha <^* f_\beta$;
(c) $\forall g \in \prod L$, there is some $\alpha < \lambda^+$ such that $g <^* f_\alpha$.

Note that these resemble the definitions of products and scales as used in standard PCF theory, but we are generally considering more than $\text{cf} \lambda$-many points below each singular $\lambda$.

Proposition 3. If $2^\lambda = \lambda^+$, and $L$ is a product of width $\lambda$, then there is a wide scale on $L$.

Proof. Using the fact that $|\prod L| = 2^\lambda = \lambda^+$, let $\langle g_\alpha : \alpha < \lambda^+ \rangle$ be an enumeration of $\prod L$. Build a sequence $(f_\alpha : \alpha < \lambda^+) \subseteq \prod L$ by induction. If $\alpha = \beta + 1$, let $\text{dom} \ f_\alpha = \text{dom} \ f_\beta \cup \text{dom} \ g_\beta$ and let $f_\alpha(\kappa) = \max \{f_\beta(\kappa), g_\beta(\kappa)\} + 1$ for all $\kappa \in \text{dom} \ f_\alpha$. If $\alpha$ is a limit, pick $A \subseteq \alpha$ unbounded of order-type $\text{cf} \alpha$. Then let $\text{dom} \ f_\alpha = (\text{cf} \alpha, \lambda) \cap \bigcup_{\beta \in A} \text{dom} f_\beta$, noting in particular that if $\kappa$ is regular and
\( \kappa \in (\text{cf} \, \alpha, \lambda) \), then \( \bigcup_{\beta \in A} \text{dom} \, f_\beta \cap \kappa \) has cardinality less than \( \kappa \). And for \( \kappa \in \text{dom} \, f_\alpha \), let \( f_\alpha(\kappa) = \sup_{\beta \in A} f_\beta(\kappa) \), so \( f_\alpha(\kappa) < \kappa \) for the same reason. Then \( f_\alpha \) is a good point for all \( \gamma < \alpha \) because \( \langle \gamma, \alpha \rangle \) is an ordering. If \( f_\beta \) witnesses that \( \text{dom} \, f_\beta \subseteq^* \text{dom} \, f_\alpha \) and \( \alpha \in \text{dom} \, f_\beta \cap (\gamma, \alpha) \), then by induction \( f_\gamma <^* f_\beta <^* f_\alpha \).

So that we do not repeat ourselves, we fix a product \( \prod L \) of width \( \lambda \) and a wide scale \( \vec{f} \) on this product for the remainder of the section.

**Definition 8.** A point \( \alpha < \lambda^+ \) such that \( \text{cf} \, \alpha 
eq \lambda \) is **good** if there is some unbounded \( A \subset \alpha \) of order-type \( \text{cf} \, \alpha \) and some \( \tau < \lambda \) such that for all \( \beta, \gamma \in A \) with \( \beta < \gamma \), \( \tau \) witnesses that \( \text{dom} \, f_\beta \subseteq^* \text{dom} \, f_\gamma \) and for all \( \kappa \in \text{dom} \, f_\beta \cap (\tau, \lambda) \), \( f_\beta(\kappa) < f_\gamma(\kappa) \).

Observe that every point \( \alpha \) such that \( \text{cf} \, \alpha < \text{cf} \, \lambda \) is good. Again, this resembles the definition of goodness from standard PCF theory. However, one of the most standard definitions of goodness—the one regarding exact upper bounds—breaks down when we consider wide products. Hence, we have two useful equivalent definitions of goodness.

**Proposition 4.** The following are equivalent for \( \alpha \) such that \( \text{cf} \, \alpha > \text{cf} \, \lambda \):

1. \( \alpha \) is a good point for \( \vec{f} \).
2. There exists a sequence \( \langle h_\xi : \xi < \text{cf} \, \alpha \rangle \subset \prod L \) such that:
   1. for some \( \tau < \lambda \) and every \( \xi, \eta < \text{cf} \, \alpha \) with \( \xi < \eta \), \( \tau \) witnesses that \( \text{dom} \, h_\xi \subseteq^* \text{dom} \, h_\eta \) and for all \( \kappa \in \text{dom} \, h_\xi \cap (\tau, \lambda) \), \( h_\xi(\kappa) < h_\eta(\kappa) \);
   2. \( \langle h_\xi : \xi < \text{cf} \, \alpha \rangle \) and \( \langle f_\beta : \beta < \alpha \rangle \) cofinally interleave each other.

**Proof.** First suppose \( \alpha \) is a good point for \( \vec{f} \). If \( A := \langle \beta_\xi : \xi < \text{cf} \, \alpha \rangle \) and \( \tau \) witness goodness at \( \alpha \), then let \( h_\xi := f_{\beta_\xi} \).

Now we prove the converse. Let \( \langle \lambda_i : i < \text{cf} \, \lambda \rangle \subset L \) converge to \( \lambda \). We use the so-called Sandwich Argument. Pick \( A := \langle \beta_\xi : \xi < \text{cf} \, \alpha \rangle \) such that for all \( \xi < \text{cf} \, \alpha \), \( h_\xi <^* f_{\beta_\xi} \subseteq^* h_{\xi+1} \) (by thinning out the enumeration of \( h_\xi \)’s if necessary). For each \( \xi \in A \), let \( i(\xi) \) be such that \( \lambda_{i(\xi)} \geq \tau \) witnesses both \( h_\xi <^* f_{\beta_\xi} \) and \( f_{\beta_\xi} \subseteq^* h_{\xi+1} \). By the Pigeonhole Principle, there is some unbounded \( X \subset \text{cf} \, \alpha \) and some \( j < \text{cf} \, \lambda \) such that \( i(\xi) = j \) for all \( \xi \in X \). Let \( A' := \langle \beta_\xi : \xi \in X \rangle \). Then if \( \xi, \eta \in A' \) and \( \xi < \eta \), then \( \lambda_i \) witnesses that

\[
\text{dom} \, f_{\beta_\xi} \subseteq^* \text{dom} \, h_{\xi+1} \subseteq^* \text{dom} \, h_\eta \subseteq^* \text{dom} \, f_{\beta_\eta},
\]

and moreover if \( \kappa \in \text{dom} \, f_{\beta_\xi} \cap (\lambda_j, \lambda) \), then it follows that,

\[
f_{\beta_\xi}(\kappa) \leq h_{\xi+1}(\kappa) \leq h_\eta(\kappa) < f_{\beta_\eta}(\kappa).
\]

This shows that \( A' \) and \( \lambda_j \) witness goodness of \( \alpha \). \( \square \)

Using Proposition 4, we can show that approachable sets give us good points.

**Lemma 5.** If \( S \in I[\lambda^+] \), then there is some club \( C \subset \lambda^+ \) such that every \( \alpha \in S \cap C \) of cofinality greater than \( \text{cf} \, \lambda \) is a good point for \( \vec{f} \).

**Proof.** Work consider \( H_\Theta \), the set of hereditarily \( \Theta \)-sized sets where \( \Theta \) is large enough that \( H_\Theta \) correctly witnesses statements in the following proof. Let \( D \) be the club such that all points in \( D \cap S \) are approachable with respect to some enumeration \( \vec{a} = \langle \alpha : \alpha < \lambda^+ \rangle \). We work with \( \mathcal{H} := H_\Theta(\epsilon, <_{\lambda^+}, D, \vec{f}, \prod L, \vec{a}) \).

Let \( \langle M_\xi : \xi < \lambda^+ \rangle \) be a continuous and strictly-increasing sequence of elementary submodels of \( \mathcal{H} \) of cardinality \( \lambda \). Then if \( \delta_\xi := M_\xi \cap \lambda^+ \) for \( \xi < \lambda^+ \), it follows that
\( \vec{\delta} := \langle \delta_\xi : \xi < \lambda^+ \rangle \) is a club in \( \lambda^+ \). In particular, \( \vec{\delta} \subseteq D \). We will argue that \( \vec{\delta} \) consists of good points for \( \vec{f} \).

Let \( \delta = \delta_\xi \) and \( M = M_\xi \) for some \( \xi < \lambda^+ \). We will use the second definition of good points from Proposition 4. By approachability, there is some unbounded \( A \subseteq \delta \) with \( \text{ot} A = \text{cf} \delta \) such that for all \( \beta < \delta, A \cap \beta = a_\gamma \) for some \( \gamma < \delta \). In particular, this means that all initial segments of \( A \) are in the model \( M \) by elementarity because \( M \cap \lambda^+ = \delta \).

Now we can work in \( M \). Let \( \langle \beta_i : i < \text{cf} \delta \rangle \) enumerate \( A \). We will define a sequence \( \langle h_i : i < \text{cf} \delta \rangle \) of functions in \( \prod L \) and we will ensure that for all \( \kappa > \text{cf} \delta \), \( \langle h_i(\kappa) : i \in X \rangle \) is strictly increasing. Namely, within \( M \), let \( \text{dom} h_i = \bigcup_{j < i} \text{dom} h_j \), and for \( \kappa \in \text{dom} h_i \cap (\text{cf} \delta, \lambda) \), let \( h_i(\kappa) := \max \{ f_{\beta_j}(\kappa), \sup_{j < i} h_j(\kappa) \} + 1 \). We know that \( h_i \) is definable in \( M \) for limits \( i \) because \( \langle \beta_j : j < i \rangle \in M \). Furthermore, by elementarity and the fact that \( M \cap \lambda^+ = \delta \), for every \( h_i \) there is some \( \beta_k < \delta \) such that \( h_i <^* f_{\beta_k} \). \( \square \)

Then we use good points to show that certain interleaving arguments define unique (up to \( ^* \)) functions in \( \prod L \).

**Lemma 6.** Suppose \( \alpha \) is a good point and \( \text{cf} \alpha \neq \text{cf} \lambda \). Suppose also that \( A \subseteq \alpha \) is unbounded in \( \alpha \) and \( \text{ot} A = \text{cf} \alpha \), and that \( \langle g_\xi : \xi < \text{cf} \alpha \rangle \) cofinally interleaves \( \langle f_\beta : \beta < \alpha \rangle \). Define \( \text{dom} f = (\text{cf} \alpha, \lambda) \cap \bigcup_{\beta \in A} \text{dom} f_\beta, \) and for \( \kappa \in \text{dom} f \), define \( f(\kappa) = \sup_{\beta \in A} f_\beta(\kappa) \). Similarly, define \( \text{dom} \bar{g} = (\text{cf} \alpha, \lambda) \cap \bigcup_{\xi < \text{cf} \alpha} \text{dom} g_\xi \) and for \( \kappa \in \text{dom} \bar{g} \) define \( g(\kappa) = \sup_{\xi < \text{cf} \alpha} g_\xi(\kappa) \).

Then \( f =^* \bar{g} \). In particular, this works if \( g_\xi := f_{\tau_\xi} \) where \( \langle \gamma_\xi : \xi < \text{cf} \alpha \rangle \) enumerates an arbitrary cofinal sequence in \( \alpha \).

**Proof.** Enumerate \( A \) as \( \langle \beta_\xi : \xi < \text{cf} \alpha \rangle \).

Suppose first that \( \text{cf} \alpha < \text{cf} \lambda \). For each \( \{ \xi, \eta \} \in [\text{cf} \alpha]^2 \), let \( \tau_{\xi,\eta} \) witness that \( f_\xi <^* f_\eta \) or vice versa, depending on whether \( \xi \) or \( \eta \) is bigger. Let \( \tau := \sup \{ \tau_{\xi,\eta} : \{ \xi, \eta \} \in [\text{cf} \alpha]^2 \} \) and for all \( \xi < \text{cf} \alpha \) we have \( f(\kappa) = g(\kappa) \).

Now suppose that \( \text{cf} \alpha > \text{cf} \lambda \). This uses the same idea as the Sandwich Argument. By picking subsequences, we can assume without loss of generality that for all \( \xi < \text{cf} \alpha, g_\xi <^* f_{\beta_{\xi+1}} \) and \( f_{\beta_{\xi}} <^* g_{\xi+1} \). Furthermore, let \( \tau \) be large enough to witness goodness of \( \alpha \) with respect to \( A \). Pick a sequence \( \langle \lambda_i : i < \text{cf} \lambda \rangle \) converging to \( \lambda \). Let \( \lambda_{\xi(\eta)} \geq \tau \) witness both \( g_\xi <^* f_{\beta_{\xi+1}} \) and \( f_{\beta_{\xi}} <^* g_{\xi+1} \). Then there is some unbounded \( X \subseteq \alpha \) and some \( j < \text{cf} \lambda \) such that for all \( \xi \in X, i(\xi) = j \). It follows that for all \( \xi, \eta \in X \) with \( \xi < \eta \), \( \lambda_j \) witnesses that

\[
\text{dom} g_\xi \subseteq^* \text{dom} f_{\beta_{\xi+1}} \subseteq^* \text{dom} f_{\beta_\eta} \subseteq^* \text{dom} g_{\eta+1},
\]

so it follows that \( \lambda_i \) witnesses \( \text{dom} f =^* \text{dom} \bar{g} \). Furthermore, for any \( \xi, \eta \in X \) with \( \xi < \eta \) and any \( \kappa \in \text{dom} \bar{g} \cap (\lambda_i, \lambda) \),

\[
g_\xi(\kappa) < f_{\beta_{\xi+1}}(\kappa) \leq f_{\beta_\eta}(\kappa) < g_{\eta+1}(\kappa).
\]

It follows that for such \( \kappa, f(\kappa) = g(\kappa) \). Hence \( f =^* \bar{g} \). \( \square \)

**Definition 9.** If \( \alpha \) is a good point for \( f \), we say that \( f \) is **continuous** at \( \alpha \) if there is some unbounded \( A \subseteq \alpha \) of order-type \( \text{cf} \alpha \) and some \( \tau < \lambda \) such that \( \text{dom} f_\alpha \cap (\tau, \lambda) = \bigcup_{\beta \in A} \text{dom} f_\beta \), and such that for all \( \kappa \in (\tau, \lambda), f_\alpha(\kappa) = \sup_{\beta \in A} f_\beta(\kappa) \).
Corollary 7. If \( S \in I[\lambda^+] \), then for any wide product \( \prod L \) on \( \lambda \), there is a club \( C \subseteq \lambda^+ \) and a scale \( f = \langle f_\alpha : \alpha < \lambda^+ \rangle \) that is continuous at all points in \( S \cap C \).

As a restatement of Lemma 6, we have the proposition that will later on (in subsection 2.3, subsection 2.5, and subsection 2.7) vindicate our discussion of good points:

Proposition 8. Suppose \( \alpha \) is a good point and \( \vec{f} \) is continuous at \( \alpha \). Suppose also that \( \langle g_\xi : \xi < \text{cf} \alpha \rangle \) cofinally interleave \( \langle f_\beta : \beta < \alpha \rangle \), and that the function \( g \) is defined such that \( \text{dom} \ g = (\text{cf} \alpha, \lambda) \cap \bigcup_{\xi < \text{cf} \alpha} \text{dom} \ g_\xi \), and such that for all \( \kappa \in \text{dom} \ g \), \( g(\kappa) = \sup_{\xi < \text{cf} \alpha} g_\xi(\kappa) \). Then \( g =^* f_\alpha \).

2. Constructing the Model

Now we will commence with the proof of Theorem 1.

2.1. Preparation of the Ground Model. Let \( \chi \) be a supercompact cardinal. We can obtain a model in which GCH holds above \( \chi \)—just take a Laver-ndestructible supercompact \( \chi \) and force GCH above it using a product of Cohen forcings—and then call this model \( V \). Our present goal is to define a suitable forcing extension \( V[G] \) in which \( \chi = \aleph_{n+2} \). We will let \( W := V[G] \) and work in \( W \) in subsection 2.2 through subsection 2.7. Then in subsection 2.8 we will refer back to \( V \).

There are two cases that we consider. If we are trying to prove the result about stationary subsets of \( \kappa \cap \text{cof}(\aleph_0) \), then we do not actually need to deal with approachability, so we may simply take the Lévy Collapse \( \mathbb{C} = \text{Col}(\aleph_1, < \chi) \) and let \( G \) be \( \mathbb{C} \)-generic, so that \( V[G] \models \chi = \aleph_2 \).

If we are trying to prove the result for \( \aleph_n, n > 0 \), then we need to make arrangements so that we will have enough approachability when we need it. We require the following result of Shelah (which appears in a stronger form as Fact 2.10 in the paper about forcing approachability with set-sized forcing [12]):

Fact 8. If \( \lambda \) is a singular strong limit and \( \nu = \lambda^+ \) in \( V \), and \( W \) is a forcing extension of \( V \) in which \( \lambda \) is still a singular strong limit and \( \nu \) is still its successor, then:

\[
W \models \{ \alpha < \nu : V \models "\text{cf} \alpha \text{ is weakly compact}" \} \in I[\nu].
\]

Because \( \chi \) is supercompact, there are \( \chi \)-many weakly compact cardinals below it. We pick any weakly compact \( \psi < \chi \). Let \( C_1 = \text{Col}(\aleph_{n-1}, < \psi) \), let \( C_2 = \text{Col}(\psi^+, < \chi) \), and let \( C = C_1 \times C_2 \). Then \( V[G] \) will be our \( W \), where \( W \models \chi = \aleph_{n+2} \). The point is that enough approachability will persist in mild forcing extensions of \( W \).

Proposition 9. If \( \lambda \) is a singular strong limit and \( \nu \) is the successor of \( \lambda \) in \( V[G] \) and \( n > 0 \), then \( \nu \cap \text{cof}(\aleph_n) \in I[\nu] \) in any \( \aleph_n \)-distributive forcing extension of \( V[G] \).

Proof. If \( (\text{cf} \alpha)^{V[G]} = \aleph_n \), then the cofinality of \( \alpha \) is preserved in any \( \aleph_n \)-distributive forcing extension of \( V[G] \). Moreover, any such extension will preserve the conditions of Fact 8. \( \square \)

Observe that we cannot use the Lévy Collapse to get \( \psi \) equal to \( \aleph_{n+1} \) (or any successor of a singular). This is precisely why the results of this paper cover \( \text{RP}(\kappa \cap \text{cof}(\aleph_n)) \) for fixed \( n \).
2.2. Defining the Forcing. Now we work in \( W \). For ease of use, we classify regular cardinals as follows:

(A) If \( F(\kappa) = 1 \), then \( \kappa \in \mathcal{C}_A \).

(B) If \( F(\kappa) = 0 \) and \( \kappa \) is inaccessible, a successor of a regular cardinal, a successor of a singular cardinal of cofinality equal to \( \aleph_n \), or a successor of a singular cardinal \( \lambda \) of cofinality not equal to \( \aleph_n \) such that \( \{ \kappa < \lambda : F(\kappa) = 1 \} \) is bounded in \( \lambda \), then \( \kappa \in \mathcal{C}_B \).

(C) If \( F(\lambda^+) = 0 \) and \( \lambda \) is a singular of cofinality not equal to \( \aleph_n \) such that \( \mathcal{C}_B \) is unbounded in \( \lambda \), then \( \lambda^+ \in \mathcal{C}_C \).

So \( \mathcal{C}_A \) is the class of cardinals \( \kappa \) that will have non-reflecting subsets of \( \kappa \cap \text{cof}(\aleph_n) \), \( \mathcal{C}_B \) is the class of cardinals \( \kappa \) where \( \text{RP}(\kappa \cap \text{cof}(\aleph_n)) \) will hold but where nothing special needs to be done with the forcing, and \( \mathcal{C}_C \) will be those cardinals where in order to make \( \text{RP}(\kappa \cap \text{cof}(\aleph_n)) \) hold, we will need to force an extra club through \( \kappa \).

We define a class partial order \( \mathcal{S} \) for adding nonreflecting stationary subsets to cardinals \( \kappa \) where \( F(\kappa) = 1 \). Since we have committed to \( \aleph_n \), let \( \mathcal{S}(\kappa) = \mathcal{S}(\kappa, \aleph_n) \) to simplify the notation.

**Definition 10.** For every \( \lambda^+ \in \mathcal{C}_C \), we fix a wide scale \( f^\lambda_A := \langle f^\alpha_A : \alpha < \lambda^+ \rangle \) on \( \prod (\mathcal{C}_A \cap \lambda) \) that is continuous at every \( \alpha \in \text{lim} D^\lambda_A \cap \text{cof}(\aleph_n) \) for some club \( D^\lambda_A \subset \lambda^+ \).

We can do this because of [Corollary 7]

We define \( \mathcal{S} \) to consist of conditions \( p \) such that:

1. \( p \) has Easton support: \( \text{sprt} p \) is a set of regular cardinals \( \geq \chi \) such that if \( \kappa \) is inaccessible, then \( |\text{sprt}(p) \cap \kappa| < \kappa \).
2. If \( \kappa \in \mathcal{C}_A \), then \( p(\kappa) \) is a condition in \( \mathcal{S}(\kappa) \).
3. If \( \kappa \in \mathcal{C}_B \), then \( p(\kappa) \) is the trivial forcing.
4. If \( \lambda^+ \in \mathcal{C}_C \), then \( p(\lambda^+) \) is a closed bounded subset \( c \subset \lambda^+ \) such that if \( \alpha \in \text{lim} c \cap \text{cof}(\aleph_n) \cap D^\lambda_A \), then the following condition holds:

   There is some \( \tau < \lambda \) such that \( \text{dom} f^\lambda_A \cap (\tau, \lambda] \subset \text{sprt} p \) and such that for all \( \kappa \in \text{dom} f^\lambda_A \cap (\tau, \lambda], f^\lambda_A(\kappa) \in \text{dom} p(\kappa) \) and \( p(\kappa)(f^\lambda_A(\kappa)) = 0 \).

   If this condition holds for \( \alpha \), we say that \( \alpha \) has the *Annulment Property*.

If \( p, q \in \mathcal{S} \), then \( p \preceq q \) if:

(a) \( \text{sprt} q \subseteq \text{sprt} p \);  
(b) for all \( \kappa \in \text{sprt} q \cap \mathcal{C}_A, p(\kappa) \restriction (\text{max} \text{dom } q(\kappa) + 1) = q(\kappa) \);  
(c) for all \( \lambda^+ \in \text{sprt} q \cap \mathcal{C}_C, p(\lambda^+) \cap (\text{max} q(\lambda^+) + 1) = q(\lambda^+) \).

We have considerable freedom to extend conditions.

**Proposition 10.** Suppose:

- \( p \in \mathcal{S} \), \( X \) is a set of regular cardinals \( \geq \chi \) and is such that \( |\kappa \cap X| < \kappa \) for all inaccessible \( \kappa \) and \( \text{sprt} p \cap X = \emptyset \);
- \( \gamma_\kappa \in (\text{max} \text{dom } p(\kappa), \kappa) \) for all \( \kappa \in \text{sprt} p \cap \mathcal{C}_A \), and \( \delta_{\lambda^+} \in (\text{max} p(\lambda^+), \lambda^+) \) for all \( \lambda^+ \in \text{sprt} p \cap \mathcal{C}_C \);
- For \( \gamma_\kappa \in \kappa \) for all \( \kappa \in X \cap \mathcal{C}_A \) and \( \delta_{\lambda^+} < \lambda^+ \) for all \( \lambda^+ \in X \cap \mathcal{C}_C \).

Consider the function \( q \) with support \( \text{sprt} p \cup X \) defined such that:

- \( q(\kappa) \) is any extension of \( p(\kappa) \) such that \( \text{max} \text{dom } q(\kappa) = \gamma_\kappa \) for \( \kappa \in \text{sprt} p \cap \mathcal{C}_A \);
- \( q(\lambda^+) = p(\lambda^+) \cup \{ \delta_{\lambda^+} \} \) for \( \lambda^+ \in \text{sprt} p \cap \mathcal{C}_C \);
• $q(\kappa)$ is any condition in $S(\kappa)$ such that $\text{max dom } q(\kappa) = \gamma_\kappa$ for $\kappa \in X \cap \mathcal{C}_A$;

• $q(\lambda^+) = \{\delta_{\lambda^+}\}$ for $\lambda^+ \in X \cap \mathcal{C}_C$.

Then $q \in S$.

Proof. We need to show that we have not violated that Annulment Property for points in $p(\lambda^+)$, $\lambda^+ \in \text{sprt } p \cap \mathcal{C}_C$. To this end, there are two observations to be made. The $\delta_{\lambda^+}$’s that we chose are not limit points of $q(\lambda^+)$, so the Annulment Property is dealt with vacuously. As for the old limit points $\alpha \in D_\kappa^*$ of cofinality $\aleph_{\kappa}$ from $p(\lambda^+)$, the Annulment Property is already verified from some $\tau < \lambda$ and cardinals from $\text{dom } f_\alpha^\lambda \cap (\tau, \lambda) \subseteq \text{sprt } p$, so expanding the domain of $p$ does not violate the Annulment Property for these points. \qed

As we pointed out in the introduction, the question of whether $\text{RP}(\kappa \cap \text{cof}(\aleph_n))$ is trivially settled for $\kappa \leq \aleph_{n+1}$, so we only concern ourselves with cardinals $\kappa \geq (\aleph_{n+1})^W$. Hence $\chi$ (which is supercompact in $V$ but is equal to $\aleph_{n+2}$ in $W$) is the smallest cardinal where $S$ can possibly be nontrivial.

We adopt the convention whereby $S[\mu, \nu]$ and $S[\mu, \nu]$ refer to $S$ restricted to intervals.

**Proposition 11.** For all regular $\mu$, $S \cong S[\chi, \mu] \times S[\mu^+, \text{ON}]$, and more generally, $S[\kappa, \nu] \cong S[\kappa, \mu] \times S[\mu^+, \nu]$ for regular $\mu$.

Proof. The map $p \mapsto (p \mid [\chi, \mu], p \mid [\mu^+, \text{ON}])$ maps $S$ into the product $S[\chi, \mu] \times S[\mu^+, \text{ON}]$. Moreover, if $\mu$ is regular. The Annulment Property is the only nontrivial point, and initial segments of cardinals do not affect whether it holds. \qed

It is important to note that the above proposition fails if $\mu^+ \subseteq \mathcal{C}_C$. For this reason, when we write $S[\mu^+, \lambda)$, we will assume that $\mu$ is regular.

2.3. **Preservation Properties of the Forcing.** Because of the use of Easton support, a simple counting argument yields an upper bound on the cardinality of $S[\chi, \mu]$ for regular $\mu$:

**Proposition 12.** For all regular $\mu$, $S[\chi, \mu]$ has size $\leq \mu$. Hence $S[\chi, \mu]$ has the $\mu^+$-chain condition.

The effective analysis of $p \in S$ requires us to take careful consideration of functions on $\text{sprt } p \cap \lambda$ for singular cardinals $\lambda$ such that $\lambda^+ \in \mathcal{C}_C$. This requires us to commit to some notation.

**Definition 11.** Given $p \in S$ and $\lambda^+ \subseteq \mathcal{C}_C \cap \text{dom } p$,

- let $f_\alpha^\lambda$ be $f_\alpha^\lambda$ where $\alpha = \text{max } p(\lambda^+)$;

- and let $g_\alpha^\lambda$ be the function on $\text{sprt } p \cap \mathcal{C}_A \cap \lambda$ such that $g_\alpha^\lambda(\kappa) = \text{max dom } p(\kappa)$ if $\kappa \in \text{dom } g_\alpha^\lambda$.

**Lemma 13.** Given $q \in S$, there is some $r \leq q$ such that for all $\lambda^+ \in \text{sprt } q \cap \mathcal{C}_C$, $f_q^\lambda \leq^* g_q^\lambda$. This also applies to the restrictions $S[\mu^+, \nu)$, $S[\mu^+, \text{ON}]$.

Notice the crucial use of Easton support in the proof.

Proof. We will argue for $S$ because the argument for its restrictions is the same. Let $q \in S$ and assume without loss of generality that $\text{sprt } q \cap \mathcal{C}_C$ has maximal element $\Lambda^+$. We will define a sequence of functions $h_\lambda \in \prod (\mathcal{C}_A \cap \lambda)$ by induction on $\lambda^+ \in \text{sprt } q \cap \mathcal{C}_C$ such that $h_\mu \leq^* h_\lambda \mid \mu$ for all $\mu < \lambda$ and such that $f_q^\lambda \leq^* h_\lambda$ for
all $\lambda \leq \Lambda$ such that $\lambda^+ \in \text{sprt } q \cap \mathcal{C}_C$. The base case $\lambda^+ = \min\{\text{sprt } q \cap \mathcal{C}_C\}$ works easily by letting $h_\lambda = f^\lambda_q$. For the successor case, there is a greatest element $\mu$ of $\text{sprt } q \cap \mathcal{C}_C$ below $\lambda$, so we let $h_\lambda(\kappa) = h_\mu(\kappa)$ for $\kappa \in (\text{sprt } q) \cap \mu$ and $h_\lambda(\kappa) = f^\lambda_q(\kappa)$ for $\kappa \in \text{sprt } q \cap (\mu, \lambda)$.

For the limit case, suppose we have defined $h_\mu$ for all elements of $\text{sprt } q \cap \mathcal{C}_C$ below $\lambda$. Let $\delta := \sup(\text{sprt } q \cap \mathcal{C}_C \cap \lambda)$, noting that $\delta \leq \lambda$, possibly strictly. We know that $\delta$ is singular because otherwise it would be inaccessible (we are working in a model of GCH) and we would have $\text{sprt } q$ unbounded in an inaccessible cardinal, but we chose to define $\mathbb{S}$ with Easton support. Hence, we let $\langle \lambda_i : i < cf \delta \rangle$ be a sequence in $\text{sprt } q \cap \mathcal{C}_C$ converging to $\delta$ such that $cf \delta < \lambda_0$. Let $h_\lambda$ be a function with the domain $h_{\lambda_0} \cup (\{\lambda_0, \delta\} \cap \bigcup_{i < cf \delta} \text{dom } h_{\lambda_i}) \cup (\{\delta, \lambda\} \cap \text{dom } f^\lambda_q)$ such that:

$$h_\lambda(\kappa) = \begin{cases} h_{\lambda_0}(\kappa) & \kappa < \lambda_0 \\
 \max(\sup_{i < cf \delta} h_{\lambda_i}(\kappa), f^\lambda_q(\kappa)) & \kappa \in [\lambda_0, \delta] \\
 f^\lambda_q(\kappa) & \kappa \in [\delta, \lambda] \end{cases}$$

It is immediately apparent that $f^\lambda_q \leq^+ h_\lambda$. If $\mu < \lambda$ and $\mu^+ \in \text{sprt } q \cap \mathcal{C}_C$, then there is some $i$ such that $\mu < \lambda_i$, and so $f^\mu_q \leq^+ h_{\lambda_i} \upharpoonright \mu \leq^+ h_\lambda \upharpoonright \mu$.

At the end of this process, we obtain $h_\Lambda$. Let $r \leq q$ be a condition such that $\text{dom } h_\Lambda \subseteq \text{sprt } r$ and such that $\max \text{dom } r(\kappa) \geq \max\{\max \text{dom } q(\kappa), h_\Lambda(\kappa)\}$ for all $\kappa \in \text{sprt } r \cap \mathcal{C}_A$.

Now we are in a position to get distributivity through strategic closure.

**Lemma 14.** For all regular $\mu$, $\mathbb{S}[\mu^+, \text{ON})$ is $\eta$-strategically closed for all $\eta < \mu^+$. The same holds for $\mathbb{S}[\mu^+, \nu)$ for any $\nu > \mu^+$.

**Proof.** We will do the proof for $\mathbb{S}[\mu^+, \text{ON})$ because the argument for $\mathbb{S}[\mu^+, \nu)$ is the same. Suppose Players I and II are constructing a descending sequence $\langle p_\xi : \xi < \mu^+ \rangle$ in $\mathbb{S}[\mu^+, \text{ON})$. We will demonstrate a strategy for Player II such that the play can continue at any $\xi < \mu^+$.

For even successors $\xi = \eta + 1$, let $p_\xi$ be a condition such that for all $\kappa \in \text{dom } p_\eta \cap \mathcal{C}_A$, $\gamma^\xi_\eta := \max \text{dom } p_\kappa(\eta) > \max \text{dom } p_\eta(\kappa)$ and $p_\kappa(\gamma^\xi_\eta) = 0$, and furthermore, that for all $\lambda^+ \in \text{dom } p_\eta \cap \mathcal{C}_C$, $\max \text{dom } p_\kappa(\lambda^+) > \max \text{dom } p_\eta(\lambda^+)$. We also want to employ an interleaving argument, we so consider two sub-cases. If $\xi$ is of the form $\xi' + 4k$ where $\xi'$ is a limit and $k < \omega$, then Player II will in addition make sure that $\max p_\kappa(\lambda^+)$ is large enough so that $g^\lambda_{p_\kappa} \leq^* f^\lambda_{p_\kappa}$ using Proposition 10. If $\xi$ is of the form $\xi' + 4k + 2$ where $\xi'$ is a limit and $k < \omega$, Player II will apply Lemma 13 to find $p_\xi$ such that for all $\lambda^+ \in \text{sprt } p_\eta \cap \mathcal{C}_C$, $f^\lambda_{p_\kappa} \leq^* g^\lambda_{p_\kappa}$.

At limits $\xi$, Player II will choose $p_\xi$ as follows: First, let $\text{sprt } p_\xi = \bigcup_{\eta < \xi} \text{sprt } p_\eta$. If $\lambda^+ \in \text{dom } p_\xi \cap \mathcal{C}_C$, then $p_\kappa(\lambda^+) = \bigcup_{\eta' \leq \eta < \xi} p_\eta(\lambda^+) \cup \{\sup_{\eta' \leq \eta < \xi} \max \text{dom } p_\eta(\lambda^+)\}$ for big enough $\eta'$. If $\kappa \in \text{dom } p_\eta \cap \mathcal{C}_A$, then let $p_\kappa(\eta)$ have a domain with maximum $\gamma^\kappa_\xi := \sup_{\eta < \xi} \max \text{dom } p_\eta(\kappa)$ such that $p_\kappa(\xi) \upharpoonright \text{dom } p_\eta(\kappa) = p_\eta(\kappa)$ for $\eta < \xi$ and $p_\kappa(\xi)(\gamma^\kappa_\xi) = 0$.

This produces a valid condition: If $\kappa \in \text{sprt } p_\xi \cap \mathcal{C}_A$, then $\langle \gamma^\kappa_\xi : \eta < \xi \rangle$ is a club avoiding $\{\alpha < \kappa : p_\xi(\alpha) = 1\}$. If $\lambda^+ \in \text{sprt } p_\xi \cap \mathcal{C}_C$, then consider $\delta := \max p_\kappa(\lambda^+)$. If $\xi$ has cofinality not equal to $\aleph_n$ then the same is true of $\delta$, and so the Annulment Property is satisfied vacuously. The Annulment Property is also vacuously satisfied if $\max p_\kappa(\lambda^+) \notin D^\lambda_n$, the club such that points of cofinality $\aleph_n$ are continuity
points of $\tilde{f}_\lambda$. The serious case is if $\xi$, and hence $\delta$, has cofinality equal to $\aleph_n$ and $\delta \in D^*_\lambda$. Then for all $\lambda^+ \in \text{sprt} \ p_\xi \cap \mathcal{C}$, $(f^\lambda_\gamma : \gamma < \delta)$ and $(g^\lambda_\eta : \eta < \xi)$ cofinally interleave each other (because of Player II’s choices at successor steps), so it follows from Proposition 8 that $f^\lambda_{\tilde{\delta}} = * g^\lambda_{\tilde{\delta}}$. Because we have guaranteed that $p(\kappa)(\gamma^\xi) = 0$ for all $\kappa \in \text{sprt} \ p_\xi \cap \mathcal{C}_A$, the Annulment Property is satisfied. □

We can use a weak version of Easton’s Lemma:

**Fact 9.** If $\mathbb{P}$ and $\mathbb{Q}$ are forcing posets where $|\mathbb{P}| < \mu$ and $\mathbb{Q}$ is $\mu$-distributive, then $\Vdash \mathbb{Q}$ is $\mu$-distributive.

Fact 9 works because Easton’s Lemma is proved in two basic steps: First, we show that $\Vdash \mathbb{P}$ has the $\mu$-chain condition”, which follows immediately from the hypotheses of Fact 9. The second step, in which we prove that $\Vdash \mathbb{Q}$ is $\mu$-distributive, proceeds the same way as the original version of Easton’s Lemma.

In turn, Fact 9 is used to prove that $\mathbb{S}$ preserves cardinals and cofinalities in the same manner as the original Easton construction.

**Proposition 15.** $\mathbb{S}$ preserves cardinals, cofinalities, and GCH. In particular, it preserves inaccessible cardinals.

The acute reader may observe that we have not guaranteed that the wide scales $\tilde{f}_\lambda$ are still wide scales in $W^{\mathbb{S}}$. In other words, we do not know that in $W^{\mathbb{S}}$, that $\tilde{f}_\lambda$ is still cofinal in $\mathcal{C}_A \cap \lambda$. This is one preservation property that we lack. However, it will turn out in the key moment that this does not matter.

2.4. Adding Non-Reflecting Stationary Sets. Most of this construction will focus on showing that $\text{RP}(\kappa \cap \text{cof}(\aleph_n))$ holds for $\kappa$ such that $F(\kappa) = 0$, but let us show that we add the non-reflecting sets that we intended to add. An important point to keep in mind is that $\mathbb{S}$ is ($\aleph_n + 1$)-strategically closed.

**Lemma 16.** For all $\kappa \in \mathcal{C}_A$, $\Vdash \kappa \cap \text{cof}(\aleph_n)$ has a non-reflecting stationary set”.

**Proof.** We claim that for all $\kappa$ such that $F(\kappa) = 1$, if $H$ is $\mathbb{S}$-generic over $W$, then the set $S_\kappa = \bigcup_{p \in H} p(\kappa)$ is a non-reflecting stationary set. This set is non-reflecting at any given $\gamma < \kappa$ as witnessed by any $p \in H$ with $\text{max dom} p(\kappa) \geq \gamma$, so we must give a reason why it is stationary. Work in $W$ and suppose $\Vdash \mathbb{S}$ is club in $\kappa$.

Use strategic closure to build a decreasing sequence $(p_\xi : \xi < \aleph_n)$ of conditions in $\mathbb{S}$ (we will suppress the distinction of even successors for the sake of readability) and a continuous and increasing sequence $(\alpha_\xi : \xi < \aleph_n)$ in $\kappa$ as follows: Given $p_\xi, \alpha_\xi$, let $\alpha_{\xi+1} \in (\alpha_\xi, \kappa)$ be such that there is some $q \leq p_\xi$ with $q \Vdash \alpha_{\xi+1} \in \check{C}$.

Then let $p_{\xi+1} \leq q$ be such that $\text{max dom} p_{\xi+1}(\kappa) > \alpha_{\xi+1}$. If $\xi$ is a limit, let $p_\eta$ be a lower bound of $(p_\xi : \xi < \eta)$ and let $\alpha_\xi = \sup_{\eta < \xi} \alpha_\eta$, noting that $p_\eta \Vdash \alpha_\xi \in \check{C}$.

Then $\sup_{\xi < \aleph_n} \alpha_\xi = \sup_{\xi < \aleph_n} \text{max dom} p_\xi(\kappa)$, and we can denote this ordinal by $\gamma$. Let $\check{p}$ be a condition below each of the $p_\xi$’s such that $\check{p}(\kappa)(\gamma) = 1$. Then $\check{p} \Vdash \gamma \in \check{C} \cap S_\kappa$. □

2.5. Defining the Quotient Poset. The purpose of this section is to define a poset that will be used to lift supercompact embeddings, and to prove that it has some reasonable properties.

Recall that $T(\kappa, \lambda)$ is the poset that shoots a club of order-type $\kappa$ through the stationary set added by $\mathbb{S}(\kappa, \lambda)$. Hence we let $T(\kappa) = T(\kappa, \aleph_n)$. 
Definition 12. Let \( p \in S[\mu^+, \nu] \).

- We say that \( T \in E(p) \) if \( T \) is function with domain \( \text{sprt} \ p \cap C_A \) such that \( p(\kappa) \forces_T (\kappa) \in T(\kappa) \).
- If \( T \in E(p) \), \( p^{-T} \) is a condition such that \( \text{sprt}(p^{-T}) = \text{sprt} \ p \), \( (p(\kappa), T(\kappa)) = (p(\kappa), T(\kappa)) \) \( \kappa \in C_A \), and \( (p^{-T})(\kappa) = p(\kappa) \) if \( \kappa \in C_C \).
- \( \mathcal{D}[\mu^+, \nu] \) is the poset of conditions of the form \( p^{-T} \) for \( p \in S[\mu^+, \nu] \), \( T \in E(p) \). If \( p, q \in \mathcal{D}[\mu^+, \nu] \), then \( p \preceq q \) if \( \text{sprt} \ q \subseteq \text{sprt} \ p \) and \( p(\kappa) \) is coordinate-wise stronger than \( q(\kappa) \) for all \( \kappa \in \text{sprt} \ q \).

Lemma 17. \( \mathcal{D}[\mu^+, \nu] \) is \( \eta \)-strongly strategically closed for every \( \eta < \mu^+ \).

Proof. We describe a decreasing sequence of conditions \( \langle r_\xi : \xi < \mu^+ \rangle \) in \( Q(\mu^+, \nu) \) and describe a strategy for Player II that allows play to continue at any \( \xi < \mu^+ \). To do this, we will describe conditions \( \langle p_\xi : \xi < \mu^+ \rangle \) in \( \mathcal{D}(\mu^+, \nu) \) and extensions \( \langle T_\xi : \xi < \mu^+ \rangle \) such that \( r_\xi = p_\xi \forces T_\xi \). We will also use a sequence \( \langle d_\xi^p : \eta_\kappa \leq \xi < \mu^+, \xi \text{ an even successor} \rangle \) where \( \eta_\kappa \) is such that \( \xi \geq \eta_\kappa \) implies \( \kappa \in \text{sprt} \ r_\xi \) and \( d_\xi^p \) is a closed bounded subset of \( \kappa \) that is an element of \( W \).

Player II only plays at even successors \( \xi = \eta + 1 \). Then let \( s \preceq p_\eta \) be a condition such that for all \( \kappa \in \text{sprt} \ p_\eta \cap C_A \), there is some closed bounded \( c^\kappa \subseteq \kappa \) such that \( p(\kappa) \forces_T (\kappa) = c^\kappa \). Choose \( s' \leq s \) such that \( \text{max \ dom} \ s'(\kappa) > c^\kappa \) for all \( \kappa \in \text{sprt} \ p_\eta \cap C_A \) and \( \text{max} \ s'(\lambda^+) > \text{max} \ p_\eta(\lambda^+) \) for all \( \lambda^+ \in \text{sprt} \ p_\eta \cap C_C \). Then let \( p_\xi = s' \), and let \( d_\xi^p \) be defined for all \( \kappa \in \text{sprt} \ p_\xi \) such that \( d_\xi^p = c^\kappa \cup \{ \text{max} \text{ dom} \ q_\kappa(\kappa) \} \). As in the (weak) strategic closure of \( S(\mu^+, \Omega) \), we consider two sub-cases for the purpose of an interleaving argument: If \( \xi \) is of the form \( \xi' + 4k \) where \( \xi' \) is a limit and \( k < \omega \), then Player II will additionally guarantee that \( \text{max} \ p_\xi(\lambda^+) \) is large enough so that \( g_\eta^{\lambda^+} \leq^* f_\xi^{\lambda^+} \), and if \( \xi \) is of the form \( \xi' + 4k + 2 \) where \( \xi' \) is a limit and \( k < \omega \), then Player II will use Lemma 13 to guarantee that \( f_\xi^{\lambda^+} \leq^* g_\eta^{\lambda^+} \).

It remains to argue that play can continue at any limit stage \( \xi \). Specifically, we claim that if \( \langle r_\eta : \eta < \xi \rangle \) have already been defined—and hence \( p_\xi : \xi < \mu^+ \) and \( T_\xi : \xi < \mu^+ \) have already been defined—then we can find a lower bound regardless of whether Player I chooses to play that specific lower bound. Let \( p_\xi \) be defined so that \( \text{sprt} \ p_\xi = \bigcup_{\eta < \xi} \text{sprt} \ p_\eta \), such that \( p_\xi(\kappa) := \bigcup_{\eta < \xi} p_\eta(\kappa) \cup \{ \sup_{\eta' < \eta \leq \xi} \text{max} \text{ dom} \ p_\eta(\kappa), 0 \} \) for large enough \( \eta' \) and \( \kappa \in \text{sprt} \ p_\xi \cap C_A \), and such that \( p_\xi(\lambda^+) := \bigcup_{\eta' < \eta \leq \xi} p_\eta(\lambda^+) \cup \{ \sup_{\eta' < \eta \leq \xi} \text{max} \ p_\eta(\lambda^+) \} \) for \( \lambda^+ \in \text{sprt} \ p_\xi(\lambda^+) \).

Then \( p_\xi \) is a lower bound for \( (p_\eta : \eta < \xi) \). First, \( \bigcup_{\eta < \xi} d_\eta^p \) is a club avoiding \( p_\kappa(\kappa) \) for all \( \kappa \in \text{sprt} \ p_\xi \cap C_A \). Furthermore, we can argue that all points of cofinality \( \aleph_\alpha \) from \( p_\xi(\lambda^+) \) for \( \lambda^+ \in \text{sprt} \ p_\xi \cap C_C \) satisfy the Annullation Property, in which case the discussion proceeds exactly as in the proof of strategic closure of \( S(\mu^+, \Omega) \), the key point being Proposition 8. Finally, for \( \kappa \in \text{sprt} \ r_\xi \cap C_A \), let \( T(\kappa) := \bigcup_{\eta' < \eta \leq \xi} \text{ ev. succ.} \ d_\eta^p \cup \{ \sup_{\eta' < \eta \leq \xi} \text{ ev. succ. max} \ d_\eta^p \} \). Then we can see that \( r_\xi := p_\xi \forces T \) is a lower bound for \( (r_\eta : \eta < \xi) \).

Proposition 18. There is a complete embedding \( \iota \) from \( S(\mu^+, \nu) \) to \( \mathcal{D}[\mu^+, \nu] \).

Proof. The map \( \iota \) sends \( p \) to \( p^{-T} \) where \( T(\kappa) = 1_{\iota(\kappa)} \) for each \( \kappa \in \text{sprt} \ p \cap C_B \). This map evidently preserves \( \leq \) and \( \perp \), and it is a complete embedding because given \( r \in \mathcal{D}[\mu^+, \nu] \), if \( p \in S(\mu^+, \nu) \) is the version of \( r \) without the “T-part,” then \( p \) is the reduction of \( r \) modulo the embedding. \( \square \)
Definition 13. Let $\mathbb{Q}[\chi, \nu]$ be $\mathbb{D}[\chi, \nu] / \iota(H)$ where $H$ is $\mathbb{S}[\chi, \nu]$-generic. More generally, $\mathbb{Q}[\mu^+, \nu]$ is defined as $\mathbb{D}[\mu^+, \nu] / \iota(H')$ where $H'$ is $\mathbb{S}[\mu^+, \nu]$-generic.

In particular, $\mathbb{S}[\chi, \nu] \ast \mathbb{Q}[\chi, \nu]$ is forcing-equivalent to $\mathbb{D}[\chi, \nu]$.  

**Proposition 19.** The poset $\mathbb{Q}[\chi, \nu]$ is a subset of $W$.

**Proof.** Working in $W$, suppose that $p \in \mathbb{S}[\chi, \nu]$ and $p \models \"\dot{q} \in \mathbb{Q}[\chi, \nu]\"$. This means that if $p \in H$, then $\iota(p)\|q$. Extend $p$ to $p'$ such that $\text{sprt} \dot{q} \subseteq \text{sprt} p'$. Then extend $p'$ to $p''$ such that for each $\kappa \in \mathbb{E}_A \cap [\chi, \nu]$ and some $d_\kappa \in W$, $p''(\kappa) \models \dot{q}(\kappa) = d_\kappa$. It follows that $p'' \Vdash \dot{q} \in W$. □

**Proposition 20.** For regular $\mu$:
- $\forces_\mathbb{S} \"\mathbb{Q}[\chi, \mu] \text{ has size } \leq \mu\"$.
- $\forces_\mathbb{S} \"\mathbb{Q}[\mu^+, \nu] \text{ is } \mu^+\text{-distributive}\"$.

**Proof.** The first point follows from a counting argument. The second follows from the fact that $\mathbb{Q}[\mu^+, \nu]$ is a factor of the $< \mu^+$-strongly strategically closed, and hence $\mu^+$-distributive, poset $\mathbb{D}[\mu^+, \nu]$. □

**Proposition 21.** If $\mu$ is a regular cardinal, then $\mathbb{Q}[\chi, \mu] = \mathbb{Q}[\chi, \mu^+]$ preserves stationary subsets of $\mu^+$.

Despite these ostensibly nice properties, $\mathbb{Q}[\chi, \mu]$ is not $\aleph_{n+1}$-closed, which means that we need to be innovative to prove stationary preservation.

2.6. **Freezing Arguments for Stationary Preservation.** Our immediate goal is to prove that $\mathbb{Q}[\chi, \nu]$ preserves stationary subsets of $\nu \cap \text{cof}(\aleph_n)$ for $\nu \in \mathbb{E}_B$. First we establish a general lemma:

**Lemma 22.** (Freezing Lemma) Let $\mu$ be a regular cardinal, and $\mathbb{P}_1$ and $\mathbb{P}_2$ be posets such that $\forces_{\mathbb{P}_2} \"\mu \text{ is regular}\"$ and $\forces_{\mathbb{P}_2} \"\mathbb{P}_1 \text{ is } \mu\text{-c.c.}\"$. (In particular, we can suppose that $\mathbb{P}_1$ is a poset of size $< \mu$ and $\mathbb{P}_2$ is $\mu$-distributive.) If $(p, q) \in \mathbb{P}_1 \times \mathbb{P}_2$ and $(p, q) \models \"C \subseteq \mu \text{ is a club}\"$, then for all $\beta < \mu$, there is some $q' \leq q$ and some $\alpha \in (\beta, \mu)$ such that $(p, q') \models \\"\alpha \in \dot{C}\\"$.

**Proof.** Let $G_2$ be $\mathbb{P}_2$-generic over $V$. In $V[G_2]$ there is a $\mathbb{P}_1$-name $\dot{C}$ such that for any $\mathbb{P}_1$-generic $G_1$ over $V[G_2]$, $\dot{C}_{G_1 \times G_2} = C_{G_1}$. This is because, without loss of generality, $\dot{C}$ is a nice name. Hence we let $(\dot{\alpha}, r) \in \dot{C}$ if and only if $(\dot{\alpha}, (r, s)) \in \dot{C}$ for some $s \in G_2$ (bearing in mind that there is an abuse of notation when referring to $\dot{\alpha}$ because the definition of $\dot{\alpha}$ depends on the poset being used). So if $(p, q) \models \\"C \text{ is a club in } \mu\\"$, then $p \Vdash \\"\dot{C} \text{ is a club in } \mu\\"$.

Working in $V[G_2]$, the $\mu$-c.c. of $\mathbb{P}_1$ implies that $(\alpha < \mu : p \Vdash \alpha \in \dot{C})$ is a club in $\mu$. Hence there is some $\alpha \in (\beta, \mu)$ such that $p \Vdash \\"\alpha \in \dot{C}\\"$. Let $q' \leq q$, $q' \in G_2$ witness this. Then $(p, q') \Vdash \\"\alpha \in \dot{C}\\"$. □

There are two notable sub-cases for cardinals in $\mathbb{E}_B$. First we consider certain inaccessible cardinals.

**Lemma 23.** Suppose $H$ is $\mathbb{S}[\chi, \mu]$-generic over $W$. If $\mu$ is an inaccessible cardinal and $\mathbb{S} \subseteq \mu \cap \text{cof}(\aleph_n)$ is stationary in $W[H]$, then the stationarity of $\mathbb{S}$ is preserved by $\mathbb{Q}[\chi, \mu]$. 
Proof. Work in $W[H]$ and fix a stationary set $S \subseteq \mu \cap \text{cof}(\aleph_n)$. Let $q_0 \in Q[\chi, \mu]$, $q_0 \models \lnot \forall C \subseteq \mu^+$ is a club. Let $\Theta$ be large enough for the following discussion, and consider the structure $\mathcal{H} := H_\Theta(\in, \in, Q[\chi, \mu], q_0, C, \mu)$. Using the stationarity of $S$, we can find an elementary submodel $M \prec H$ such that $\delta := M \cap \mu = \text{sup}(M \cap \mu) \in S$. Moreover, since $\mu$ is inaccessible (and we are working with $\text{GCH}$), we can find such an $M$ that is closed under sequences of length less than $\aleph_n$. Fix a sequence $\langle \delta_\xi : \xi < \aleph_n \rangle$ converging to $\delta$.

We will define a decreasing sequence of conditions $\langle q_\xi : \xi < \aleph_n \rangle \subseteq M$ in $Q[\chi, \mu]$, an increasing sequence of ordinals $\langle \alpha_\xi : \xi < \aleph_n \rangle$, and an increasing sequence of cardinals $\langle \kappa_\xi : \xi < \aleph_n \rangle$ such that $\kappa_0 = \chi$, if $\xi$ is a successor then $\kappa_\xi$ is regular, and $\text{sprt} q_\xi \subseteq \kappa_\xi$. For successor stages, we are given $q_\xi, \alpha_\xi$, and we view $Q[\chi, \mu]$ as $Q[\chi, \kappa_\xi] \times Q[\kappa_\xi^+, \mu]$ and use the Freezing Lemma to find $\alpha_{\xi+1} \in (\text{max}\{\alpha_\xi, \delta_\xi\}, \delta)$ and $q_{\xi+1} \models q_\xi$ such that $q_{\xi+1} \models [\chi, \kappa_\xi] = q_\xi$ and $q_{\xi+1} \models \alpha_{\xi+1} \in \dot{C}$. And of course, $\kappa_{\xi+1}$ is a regular cardinal large enough that $\text{sprt} q_{\xi+1} \subseteq \kappa_{\xi+1}$. If $\xi$ is a limit, then let $\alpha_\xi = \text{sup}_{\eta < \xi} \alpha_\eta$ and let $\kappa_\xi = \text{sup}_{\eta < \xi} \kappa_\eta$. And let $q_\xi$ be defined so that $\text{sprt} q_\xi = \text{sup}_{\eta < \xi} \text{sprt} q_\eta$ and such that $q_\xi \models [\kappa_\eta, \kappa_{\eta+1}] = q_\eta$ for $\eta < \xi$. Then $q_\xi$ will be an element of $Q[\chi, \mu]$ and it will be the case that $q_\xi \models \alpha_\xi \in \dot{C}$. Furthermore, we can see that $\alpha_\xi, \kappa_\xi, q_\xi$ are in $M$ because they are defined with regard to the parameters from $\mathcal{H}$ and the sequence $\langle \delta_\eta : \eta < \xi \rangle$, which is in $M$.

Once we have $\langle q_\xi : \xi < \aleph_n \rangle$, let $\eta$ be defined such $\text{sprt} \eta = \bigcup_{\xi < \aleph_n} \text{sprt} q_\xi$ (note that $\text{sprt} \eta \subseteq \text{sprt}_{\xi < \aleph_n} \kappa_\xi < \mu$) and such that for all $\xi < \aleph_n$, $\eta \models [\chi, \kappa_\xi] = q_\xi$. Then $\eta$ is a lower bound of $\langle q_\xi : \xi < \aleph_n \rangle$, so $\eta \models \delta = \text{sup}_{\eta < \xi} \alpha_\xi \in \dot{C}$, and thus $\eta \models \dot{C} \cap S \neq \emptyset$. \qed

Now we consider the non-trivial case for singular cardinals in $\mathcal{E}_B$.

**Lemma 24.** Suppose $H$ is $S[\chi, \lambda]$-generic over $W$. If $\lambda$ is singular cardinal of cofinality $\aleph_n$ and $S \subseteq \lambda^+ \cap \text{cof}(\aleph_n)$ is stationary in $W[H]$, then the stationarity of $S$ is preserved by $Q[\chi, \lambda]$.

Proof. Work in $W[H]$ and fix a stationary set $S \subseteq \mu \cap \text{cof}(\aleph_n)$. Let $q_0 \in Q[\chi, \lambda]$, $q_0 \models \lnot \forall C \subseteq \lambda^+$ is a club. Fix a sequence of regular cardinals $\lambda = \langle \lambda_\eta : \eta < \omega \rangle$ converging to $\lambda$, let $\Theta$ be a large enough regular cardinal, and consider the structure $\mathcal{H} := H_\Theta(\in, \in, Q[\chi, \lambda], q_0, \dot{C}, \lambda^+, \dot{\lambda})$. Using the stationarity of $S$, we can find an elementary submodel $M \prec H$ of size $\lambda$ such that $\delta = M \cap \lambda^+$, then $\delta \in S$. If $n = 0$, then we pick any sequence $\langle \delta_n : n < \omega \rangle$ converging to $\delta$. If $n > 0$, then we will appeal to the fact that $\lambda^+ \cap \text{cof}(\aleph_n) \in I[\lambda^+]$ (i.e. there is enough approachability) to choose $M$ (and hence $\delta$) so that there is a sequence $\langle \delta_\xi : \xi < \aleph_n \rangle$ converging to $\delta$ such that $\langle \delta_\xi : \xi < \eta \rangle \subseteq M$ for all $\eta < \aleph_n$.

Fix an increasing and continuous sequence of cardinals $\langle \lambda_\xi : \xi < \aleph_n \rangle$ converging to $\lambda$ such that $\lambda_0 = \chi$, and such that if $\xi$ is a successor ordinal then $\lambda_\xi$ is regular. We will define a decreasing sequence $\langle q_\xi : \xi < \aleph_n \rangle \subseteq M$ of conditions from $Q[\chi, \lambda]$ and an increasing sequence of ordinals $\langle \alpha_\xi : \xi < \aleph_n \rangle$ as follows: If $q_\xi$ and $\alpha_\xi$ have been defined, view $Q[\chi, \lambda]$ as the product $Q[\chi, \lambda_\xi] \times Q[\lambda_\xi^+, \lambda]$ and use the Freezing Lemma to find $q_{\xi+1}$ and $\alpha_{\xi+1} \in (\text{max}\{\alpha_\xi, \delta_\xi\}, \delta)$ such that $q_{\xi+1} \models \lnot \forall \alpha_{\xi+1} \in \dot{C}^\eta$ and $q_{\xi+1}(\kappa) = q_\xi(\kappa)$ for all $\kappa < \lambda_\xi$. If $\xi$ is a limit, then let $\alpha_\xi = \text{sup}_{\eta < \xi} \alpha_\eta$ and let $q_\xi$ be a condition such that $q_\xi \models [\lambda_\eta, \lambda_{\eta+1}] = q_\eta$ for all $\eta < \xi$, and such that if $\kappa \in [\lambda_\xi, \lambda]$, then then $q_\xi(\kappa)$ is a lower bound of $\langle q_\eta(\kappa) : \eta < \xi \rangle$ using the $\aleph_n$-closure of $\mathcal{T}(\kappa, \aleph_n)$. So $q_\xi$ is an actual condition in $Q[\chi, \lambda]$ and $q_\xi$ forces that
such that for all $\mathbf{D}$.

Finally, define $\bar{q}$ such that $q \upharpoonright [\lambda_{\xi}, \lambda_{\xi+1}] = q_{\xi} \cap [\lambda_{\xi}, \lambda_{\xi+1}]$ for all $\xi < \aleph_n$. Then $\bar{q}$ is a lower bound of $\langle q_{\xi} : \xi < \aleph_n \rangle$, so $q$ forces that $\sup_{\xi < \aleph_n} \alpha_{\xi}$, i.e. $\delta$, is in $\check{C}$. Thus $\bar{q} \forces \check{C} \cap S \neq \emptyset$.

\section{The Interleaving Argument for Stationary Preservation}

Now we turn our attention to stationary preservation for the cardinals in $\mathcal{C}_\kappa$. We make use of the following very important property:

\textbf{Proposition 25.} Fix $\lambda^+ \in \mathcal{C}_\kappa$ and let $\check{H}$ be $\mathbb{S}[\chi, \lambda^+]$-generic over $W$. There is a club $D_\lambda \subseteq \lambda^+$ in $W[\check{H}]$ such that for all $\alpha \in D_\lambda \cap \text{cof}(\aleph_n)$, there is some $\tau < \lambda$ such that for all $\kappa \in \text{dom} f_\alpha^\lambda \cap (\tau, \lambda)$, $f_\alpha^\lambda(\kappa) \notin S_\kappa$.

\textbf{Proof.} The club $D_\lambda$ is defined to be $D_\lambda^* \cap \bigcup_{p \in H} p(\lambda^*)$ (recall that $D_\lambda^*$ was defined as a club such that all $\alpha \in \text{lim} D_\lambda^* \cap \text{cof}(\aleph_n)$ are points of continuity for $f_\lambda^\alpha$). The fact that it is closed follows immediately from the definition, and the fact that it is unbounded follows from our ability, given any $\beta < \lambda^+$ to extend any $p \in \mathbb{S}[\chi, \lambda^+]$ to $q \leq p$ such that $\max(q(\lambda^+)) > \beta$. And if $q \in \check{H}$ is such that $\alpha < \text{max dom } q(\lambda^+)$, then the Annullment Property of $\alpha$ witnesses the fact that there is some $\tau < \lambda$ such that $f_\alpha^\lambda(\kappa) \notin S_\kappa$ for all $\kappa \in \text{dom} f_\alpha^\lambda \cap (\tau, \lambda)$.

The crux of this construction uses the fact that continuous points of the wide scales $f_\alpha$ are defined uniquely up to interleaving for large $\kappa < \lambda$. We need a lemma that shows that we can decide elements of a club added in the extension in such a way that allows us to throw away initial segments of the domains of the conditions while assuring that we are still making the correct decisions about the club.

We need to introduce some notation for the following discussion. Recall that elements of $\mathbb{D}[\chi, \nu]$ formally belong to $\mathbb{D}[\chi, \nu]$ and so we can refer to their support.

\textbf{Definition 14.} If $q \in \mathbb{Q}[\chi, \nu]$, then $q[\mu, \nu]$ is a condition such that $q[\mu, \nu](\kappa)$ is the trivial condition for $\kappa \notin [\mu, \nu]$ and $q[\mu, \nu](\kappa) = q(\kappa)$ for $\kappa \in [\mu, \nu]$.

\textbf{Lemma 26.} Suppose $\lambda$ is a singular cardinal such that $\lambda^+ \in \mathcal{C}_\kappa$ and let $\langle \lambda_{\xi} : \xi < \text{cf } \lambda \rangle$ be a sequence of regular cardinals converging to $\lambda$. Suppose that $\models_{\mathbb{S}[\chi, \lambda^+]} \lnot \check{C} \subseteq \lambda^+$ is a club (i.e. this is forced by the empty condition). Then for all $\beta < \lambda^+$ and $q \in \mathbb{Q}[\chi, \lambda^+]$, there is some $\alpha \in (\beta, \lambda^+]$ and some $q' \leq q$ such that for all $\xi < \text{cf } \lambda$, $q'[\lambda_{\xi}^+, \lambda^+] \forces \alpha \in \check{C}$.

Of course, $\mathbb{Q}[\chi, \lambda^+]$ is trivial at $\lambda^+$, but this is the interval we will be considering when we apply this lemma.

\textbf{Proof.} Starting by working in $\mathbb{D}[\chi, \lambda^+]$, consider conditions in $\mathbb{D}[\chi, \lambda^+]$. Let $q = T$ and consider $p^\neg T \in \mathbb{D}[\chi, \lambda^+]$. Let $K$ be a $\mathbb{D}[\chi, \text{cf } \lambda]$-generic containing $(p^\neg T)[\chi, \text{cf } \lambda]$. Note that the cofinality of $\lambda^+$ is preserved in $W[K]$.

Now we work in $W[K]$. For each $i < \omega$ we will define a decreasing sequence $\langle p_i, \varepsilon \neg T_{i, \xi} : \xi < \text{cf } \lambda \rangle$ of conditions in $\mathbb{D}[(\text{cf } \lambda)^+, \lambda^+]$ below $(p^\neg T)[(\text{cf } \lambda)^+, \lambda^+]$ and a (not necessarily increasing) sequence $\langle \alpha_i, \xi : i < \omega, \xi < \text{cf } \lambda \rangle$ of ordinals in the interval $(\beta, \lambda^+)$. Furthermore, we will define $\alpha_0^* := \sup_{\xi < \lambda} \alpha_i$ as we proceed. We will use strong strategic closure of $\mathbb{D}[(\text{cf } \lambda)^+, \lambda^+]$ to keep the construction going, but we will suppress the distinction between even and odd successor ordinals, and we will not repeat the specifics of the strategy.
At successor stages we are given $p_{i,ξ}\dot{\upharpoonright}T_{i,ξ} ∈ D[(cf λ)^+ + λ^+]$ for some $ξ < λ$ and $i < ω$. Since $(p_{i,ξ}\dot{\upharpoonright}T_{i,ξ})[λ^+] \Vdash "\dot{C} ⊆ λ^+ is a club"$, we can use the Freezing Lemma to find $r ≤ p_{i,ξ}\dot{\upharpoonright}T_{i,ξ}$ and $α_{i,ξ} ∈ (α_{i}^+, λ^+]$ such that $r[λ^+] \Vdash α_{i,ξ} ∈ \dot{C}$. Then let $p_{i,ξ+1}\dot{\upharpoonright}T_{i,ξ+1} ≤ r$ be chosen according to the strategy. Lower bounds can be chosen at limit stages as in the proof of strong strategic closure of $D[(cf λ)^+ + λ^+]$ and $α_{i,ξ}$ for a limit $ξ$ can be chosen as in the successor stage of this argument. The case we have defined $p_{i,ξ}\dot{\upharpoonright}T_{i,ξ}$ for all $ξ < λ$ and need to define $p_{i+1,0}\dot{\upharpoonright}T_{i+1,0}$ is another limit case.

Let $p^- T$ be a lower bound for the whole sequence. Let $α = sup_{i<ω} α_i$ and let $q$ be the “$T$-part” of $p^- T$. Then for any $S[(cf λ)^+ + λ^+]$-generic $K'$ containing $p$, we have arranged so that $q[λ^+] \Vdash \dot{C} ∩ [α_i, α_{i+1}) ≠ ∅$ for all $i < ω$, and hence $q[λ^+] \Vdash \alpha ∈ \dot{C}$. Hence $s^- (p^- T)$ witnesses the lemma.

**Lemma 27.** If $λ$ is a singular cardinal such that $λ^+ ∈ ℂ_C$, then $Q[λ, λ^+]$ preserves stationary subsets of $λ^+ ∩ cof(κ_n)$.

Recall that given $p ∈ S$, we defined $g_κ^λ$, which has domain $sp_{p ∩ ℂ_A ∩ λ}$ and maps $κ$ to max dom $p(κ)$.

**Proof.** As in the freezing arguments, we work in $W[H]$. By the Mixing Principle, it is enough to consider a $Q[λ, λ^+]$-name $C$ where $V_δ ⊆ Q[λ, λ^+]$ is a club. Let $Θ$ be a large enough regular cardinal and consider the structure $H := H_Θ(∈, <_θ, Q[λ, λ^+], C, λ^+)$. Using the stationarity of $S$, we can find an elementary submodel $M ⊆ H$ of size $λ$ such that if $δ = M ∩ λ^+ = sup(M ∩ λ^+)$, then—and this is the crux of the whole construction—we have $δ ∈ S ∩ D_λ$. If $n > 0$, we can use approachability to select $M$ and $δ$ such that there is a sequence $q_δ : ξ < λ_n$ such that $q_δ : ξ < η ∈ M$ for all $η < ξ$. Otherwise, if $n = 0$, let $q_δ : ξ < λ_0$ be any sequence converging to $δ$.

We will define a decreasing sequence $q_δ^λ : ξ < λ_n$ ⊆ $M$ of conditions in $Q[λ, λ^+]$ and an increasing and continuous sequence of ordinals $α_ξ : ξ < λ_n$. We will also make use of the function $g_{q_δ}^λ$, where dom $g = sp_{q_δ} ∩ λ$ and $g(κ) = max g_{q_δ}(κ)$. For the successor case, suppose $q_δ$ is already defined. We can choose $q_{δ+1} ≤ q_δ$ such that $f_{δ+1}^λ <^* g_{δ+1}^λ$, and moreover by **Lemma 26** we can choose $q_{δ+1}$ such that there is some $α_ξ ∈ (δ_ξ, δ)$ such that $q_{δ+1}[τ, λ^+] \Vdash α_ξ ∈ \dot{C}$ for cofinally many $τ < λ$. If $ξ$ is a limit, let $q_ξ$ be a lower bound of $(q_η : η < ξ)$ using the $λ_n$-closure of $T(κ, λ_n)$ for each $κ$, and let $α_ξ = sup_{η<ξ} α_η$. These are contained in $M$ because they are defined with the parameters from $H$ and the sequence $q_η : η < ξ ∈ M$.

We claim that $g_{q_δ}^λ : ξ < λ_n$ and $f_{δ+1}^λ : ξ < λ_n$ cofinally interleave each other.

By the construction, we know that $f_{δ+1}^λ <^* g_{δ+2}^λ$. By elementarity of $M$, and the fact that $q_ξ ∈ W$ for any $ξ < λ_n$ (and so $f_λ$ is cofinal in $PID(C_A ∩ λ)$ there is some $η < λ_n$ such that $g_{δ+1} <^* f_{δ+1}^λ$. Hence, if we define $g$ such that dom $g = (λ_n, λ) \cap \bigcup_{ξ<λ_n} g_{q_δ}^λ$, and $g(κ) = sup_{ξ<λ_n} g_{q_δ}^λ(κ)$, then we see that $g =^* f_{δ}^λ$ by **Proposition 8**. In other words, there is some $τ$ be such that dom $g ∩ (τ, λ) = dom f_{δ+1} ∩ (τ, λ)$ and such that $κ ∈ dom g ∩ (τ, λ)$ implies $g(κ) = f_{δ+1}^λ(κ)$, and thus $g(κ) ∉ S_κ$. It follows that there is a lower bound $[q_ξ[τ^+, λ^+] : ξ < λ_n]$ and that $q_ξ[τ^+, λ^+]$ witnesses the lemma. 

□
2.8. Lifting the Embeddings. Since we have been working in the model \( W \). Recall that \( \chi \) is supercompact in \( V \). Recall also that \( W = V[G] \) where \( G \) is \( C \)-generic over \( V \) and \( C \) is defined in one of two ways. Either \( n = 0 \) and \( C = \text{Col}(\aleph_1, \chi) \) or else \( n > 0 \) and \( C = \text{Col}(\aleph_{n-1}, < \psi) \times \text{Col}(\psi^+, < \chi) \) where \( \psi \) is a weakly compact cardinal below \( \chi \). In this section we will finally use the supercompactness of \( \chi \). In particular, \( V[G] \models \chi = \aleph_n \). We will also continue referring to cardinals \( \geq \chi \) as belonging to one of \( C_A, C_B \), or \( C_C \).

Our main tool will be a lifting argument that is due to Silver.

**Fact 10.** If \( j : V \to M \) is an embedding, \( G \) is \( P \)-generic, \( H = j(P) \)-generic, and \( j[H] \subseteq H \), then \( j \) can be lifted to \( j' : V[G] \to M[j(G)] \).

Another key fact is in its original form due to Solovay, and it will allow us to set up more stationary preservation. Its purpose it to make ugly quotients behave nicely.

**Fact 11.** (Absorption Theorem) Suppose that \( P \) is a separative and \( (\kappa + 1) \)-strategically closed poset such that \( |P| < \lambda \). Then there is a complete embedding \( \iota : P \to \text{Col}(\kappa^+, < \lambda) \) such that if \( G \) is \( P \)-generic over \( V \), the \( \text{Col}(\kappa^+, < \lambda) \) is forcing-equivalent to \( \text{Col}(\kappa^+, < \lambda)/\iota(G) \). Moreover, this works if \( \text{Col}(\kappa^+, < \lambda) \) is replaced by \( \text{Col}(\kappa^+, A) \) where \( \text{sup} A = \lambda \).

We have two remarks on this version of the Absorption Theorem, which appears in a few other guises (the best source is James Cummings’ chapter in the Handbook [2]). First, the statement occasionally includes the hypothesis that \( \lambda \) is inaccessible, but this is not necessary—it implies that \( \text{Col}(\kappa^+, < \lambda) \) has the \( \lambda \)-chain condition, which is circumstantially helpful but not required. Second, the statement of the Absorption Theorem usually includes a hypotheses about the closure of \( P \), but here was are using strong strategic closure. This is in fact enough: the core of the proof of the Absorption Theorem is the fact that, \( P \) is forcing-equivalent to \( \text{Col}(\kappa, \lambda) \) if it is separative, \( \kappa \)-closed, has cardinality \( \lambda \), and collapses \( \lambda \) to have size \( \kappa \) [8]. The reader can verify that it is enough to assume that \( P \) is \( \eta \)-strongly strategically closed for all \( \eta < \kappa \). Also, it is worth noting that the strongly strategically closed version of the Absorption Theorem has been used elsewhere [9].

We need another stationary-preservation fact for another component of our lifting argument.

**Fact 12.** If \( P \) is \( (\aleph_n + 1) \)-strategically closed, \( \mu \cap \text{cof}(\aleph_n) \in I[\mu] \), and \( S \subseteq \mu \cap \text{cof}(\aleph_n) \) is stationary, then forcing with \( P \) preserves the stationarity of \( S \).

The proof of this fact is very similar to the proof of the fact that \( \aleph_n + 1 \)-closed posets preserve stationary subsets of \( \mu \cap \text{cof}(\aleph_n) \) if \( \mu \cap \text{cof}(\aleph_n) \in I[\mu] \), which can be found in several good sources [10].

Finally, we will need to apply Fact 12 to a two-step iteration, so we need one more item.

**Proposition 28.** If \( P \) is \( \kappa^+ \)-closed and \( \models_P \langle \dot{Q} \rangle \) is \( \kappa + 1 \)-strategically closed, then \( P * \dot{Q} \) is \( \kappa + 1 \)-strategically closed.

We include the proof for skeptical readers:

**Proof.** Consider a play \( \langle p_\xi, \dot{q}_\xi \rangle : \xi \leq \kappa \) and let \( \dot{\sigma} \) be a \( P \)-name such that \( \models_P \langle \dot{\sigma} \rangle \) is a strategy for \( \dot{Q} \). We describe the strategy of Player II as follows: If \( \xi = \eta + 1 \) is
an even successor, choose \( p_\xi \leq p_\eta \) and \( \hat{q}_\xi \) such that \( p_\xi \forces \lnot \forall \eta < \xi \forall \xi (\hat{q}_\xi : \zeta \leq \eta) \). If \( \xi \) is a limit, then let \( p_\eta \) be a lower bound of \( \{ p_\eta : \eta < \xi \} \). Then \( p_\eta \) forces that for all \( \eta < \xi \) where Player II chooses conditions, \( \hat{q}_\eta = \hat{\sigma}(\langle \hat{q}_\xi : \zeta < \eta \rangle) \), hence there is some \( p_\xi \leq p_\eta \) and some \( \hat{q}_\xi \) such that \( p_\xi \forces \lnot \forall \eta < \xi \forall \xi (\hat{q}_\xi : \zeta \leq \eta) \). And it is evident that this strategy works. \( \square \)

Now we can do the main work of this section.

Lemma 29. Suppose that one of the following holds:

1. \( \mu \in \mathcal{C}_B, \) \( H \) is \( S[\chi, \mu] \)-generic, and \( S \in V[G \ast H] \) is a stationary subset of \( \mu \cap \text{cof}(\aleph_n) \).
2. \( \lambda^+ \in \mathcal{C}_C, \) \( H \) is \( S[\chi, \lambda^+] \)-generic, and \( S \in V[G \ast H] \) is a stationary subset of \( \mu \cap \text{cof}(\aleph_n) \).

Then \( S \) reflects in \( V[G \ast H] \).

Proof. Let \( \nu = \mu \) or \( \nu = \lambda^+ \), depending on the case we are considering. Most of this lemma consists in proving the following:

Claim. If \( j : V \rightarrow M \) is a \( \nu \)-supercompact embedding with critical point \( \chi \) and suppose one of the following holds:

1. \( \nu = \mu \in \mathcal{C}_B, \) \( G \) is \( \mathcal{C} \)-generic over \( V, \) \( \bar{H} \) is \( \mathcal{S}[\chi, \nu] \)-generic over \( V[G] \), and \( S \subseteq \mu \cap \text{cof}(\aleph_n) \) is stationary and is in \( V[G \ast \bar{H}] \).
2. \( \nu = \lambda^+ \in \mathcal{C}_C, \) \( G \) is \( \mathcal{C} \)-generic over \( V, \) \( \bar{H} \) is \( \mathcal{S}[\chi, \nu] \)-generic over \( V[G] \), and \( S \subseteq \nu \cap \text{cof}(\aleph_n) \) is stationary and is in \( V[G \ast \bar{H}] \).

Then there is an extension \( V[G \ast \bar{H} \ast \text{L}] \) in which \( S \) is still stationary and \( j \) can be lifted to \( j^+: V[G \ast \bar{H}] \rightarrow M[j^+(G \ast \bar{H})] \).

Proof of Claim. We will perform the lift in several stages. We will also do the proof for the case where \( n > 0 \), and \( C = \text{Col}(\aleph_{n-1}, < \psi) \times \text{Col}(\psi^+, < \chi) = C_0 \times C_1 \), because this is strictly more difficult than the \( n = 0 \) case.

If \( n > 0 \) then we let \( G \) factor as \( G_0 \times G_1 \) where \( G_0 = \text{Col}(\aleph_{n-1}, < \psi) \)-generic. The first step of the lift comes from the fact that \( \text{Col}(\aleph_{n-1}, < \psi) \subseteq \mathcal{C} \): Because \( \text{Col}(\aleph_{n-1}, < \psi) \in V, j(\text{Col}(\aleph_{n-1}, < \psi)) = \text{Col}(\aleph_{n-1}, < \psi), \) and so we can apply Fact 10 to show that \( j \) lifts to \( j^0 : V[G_0] \rightarrow M[j^0(G_0)] = M[G_0] \). The stationarity of \( S \) is preserved because \( \text{Col}(\aleph_{n-1}, < \psi) \subseteq \text{cof}(\aleph_n) \).

The next step is to lift the embedding through the forcing \( \text{Col}(\aleph_{n+1}, < \chi) \), which is the interpretation of \( \text{Col}(\psi^+, < \chi) \) in \( V[G_0] \) because if \( n > 0 \) then \( \psi^{V[G_0]} = \aleph_n \).

We observe that \( j^0(C_1) = j^0(\text{Col}(\aleph_{n+1}, < \chi)) = \text{Col}(\aleph_{n+1}, < \chi) \times R = C_1 \times R \), where \( R := \prod_{\alpha \in \aleph_\alpha} \text{Col}(\aleph_{n+1}, \alpha) \).

Next we use the quotient forcing.

Case 1: \( \nu = \mu \in \mathcal{C}_B \). Then we let \( I \) be \( \mathcal{Q}[\chi, \mu] \)-generic over \( V[G \ast \bar{H}] \). Then \( \mathcal{Q}[\chi, \mu] \cong \mathcal{Q}[\chi, \mu] \) since \( F(\mu) = 0 \). Hence the stationarity of \( S \) is preserved by Proposition 21 if \( \mu \) is a successor of a regular. Lemma 23 if \( \mu \) is inaccessible, Lemma 24 if \( \mu \) is the successor of a singular of cofinality \( \aleph_n \), and by the fact that \( |\mathcal{Q}[\chi, \mu]| < \mu \) if \( \mu \) is the successor of a singular \( \lambda \) and \( \{ \kappa < \lambda : F(\kappa) = 1 \} \) is bounded in \( \mu \).

Case 2: \( \nu = \lambda^+ \in \mathcal{C}_C \). We let \( I \) be \( \mathcal{Q}[\chi, \lambda^+] \)-generic over \( V[G \ast \bar{H}] \). Then the stationarity of \( S \) is preserved by Lemma 27.

We proceed to work in \( V[G \ast \bar{H} \ast I] \) where \( \bar{H} \) is \( \mathcal{S}[\chi, \nu] \)-generic and \( I \) is \( \mathcal{Q}[\chi, \nu] \)-generic.
Since $S[\chi, \nu] \ast \mathbb{Q}[\chi, \nu]$ is forcing-equivalent to the $\aleph_{n+1}$-strangely strategically closed poset $\mathbb{B}[\chi, \nu]$, we can apply the Absorption Theorem to find a complete embedding $j : S[\chi, \nu] \ast \mathbb{Q}[\chi, \nu] \to \mathbb{R}$ such that $\mathbb{R}/j(\mathbb{H} \ast I)$ is forcing-equivalent to $\mathbb{R}$.

Then let $J$ be $\mathbb{R}$-generic over $V[G \ast \mathbb{H} \ast I \ast J]$. Now the embedding $j^0$ can be lifted to $j'$ in this model because $j^0[V_1] \subseteq G_1 \ast \mathbb{H} \ast I \ast J$ (where conditions in $G_1$ are sent to themselves in the first coordinate). Hence we get a lift $j' : V[G] \to M[j'(G)]$.

Now we are working in $V[G \ast \mathbb{H} \ast I \ast J]$. Recall that $V[G] \models \text{"}\chi = \aleph_{n+2}\text{"}$. We claim that $j'(S[\chi, \nu])$ is $\aleph_{n+1}$-strategically closed in $V[j'(G)]$. We have established that $S[\chi, \nu]$ is $\left(\aleph_{n+1}\right)$-strategically closed in $V[G]$, and so $M[j'(G)] \models j'(S[\chi, \nu])$ is $\left(\aleph_{n+1}\right)$-strategically closed" by elementarity. Because $M^\nu \subseteq M$ and because $j(C_1)$ is $\aleph_{n+1}$-closed (using the fact that $\aleph_n$ is below the critical point of the embedding), we have that $M[j'(G)]$ is closed under sequences of length $\aleph_n$, and so $V[G \ast \mathbb{H} \ast I \ast J]$ believes that $j'(S[\chi, \nu])$ is $\left(\aleph_{n+1}\right)$-strategically closed.

The iteration is $(\mathbb{R} + j')^\nu(S[\chi, \nu])$ preserves the stationarity of $S$ over $V[G \ast \mathbb{H} \ast I \ast J]$: The iteration is $\left(\aleph_{n+1}\right)$-strategically closed by Proposition 28 $\nu \cap \text{cof}(\aleph_n) \in I[\nu]$ in the model $V[G \ast \mathbb{H} \ast I]$ by Proposition 9 and the fact that $\mathbb{B}[\chi, \nu]$ is $\aleph_{n+1}$-distributive over $V[G]$.

If $\nu$ is a successor of a singular cardinal, we can apply Fact 12 and if $\nu$ is inaccessible then we can use an argument in the vein of Lemma 23. Either way we conclude that $S$ remains stationary in $V[G \ast \mathbb{H} \ast I \ast J \ast K]$ if $K$ is $j'(S[\chi, \nu])$-generic over $V[G \ast \mathbb{H} \ast I \ast J]$. But it remains to prove that we can find a generic for $j'(S)$ that allows us to apply Fact 10. For this we use a master condition argument. Define $p$ as follows:

- $\text{sprt} p = \{j'(\kappa) : \kappa \in [\chi, \nu] \setminus C_B\};$
- for all $\kappa \in C_A$, $\text{dom}(p(j'(\kappa))) = \text{sup } j'(\kappa) + 1$ and for all $\alpha \leq \text{sup } j'[S_\alpha]$, $p(j'(\kappa))(\alpha) = 1$ if and only if $\alpha \in j'[S_\alpha];$
- for all $\lambda^+ \in C_C$, $p(j'(\lambda^+))$ is the closure of $j[D_\lambda]$ where $D_\lambda$ comes from Proposition 25.

Claim. $p$ is a condition in $j'(S)$.

Proof of Claim. The domain of $p$ has Easton support from the point of view of $M[j'(G)]$ because for all regulars $\kappa \in [\chi, \nu]$, the fact that $M^\kappa \subseteq M$ implies that $\text{sup } j'(\kappa) < j'(\kappa)$, and hence sup $j'(\kappa) < j'(\kappa)$. For each $\kappa \in \text{sprt} p \cap C_A$, let $T_\kappa$ be the club added by $\mathbb{Q}[\chi, \nu]$ that avoids $S_\kappa$. Since $j'$ is continuous for sequences of ordinals of length $\leq \aleph_n$, and $T_\kappa$ avoids $S_\kappa$, it follows that $j[T_\kappa]$ avoids $j[S_\kappa]$.

We are left to verify the Annulment Property for points of cofinality $\aleph_n$ in the closure $C$ of $j[D_\lambda]$ for $\lambda^+ \in C_C$. Observe that if $\delta := j[D_\lambda] = j'[\lambda^+]$, then $(\text{cf } \delta)^{\mathbb{V}} = \lambda^+$ and $(\text{cf } \delta)^{M[j'(G)]} > \aleph_n$ by the $\aleph_{n+1}$-distributivity of $\mathbb{C} * \mathbb{R}$. Hence, if $\alpha \in j[D_\lambda]$ has cofinality $\aleph_n$, then $\alpha = j'(\beta)$ where $\beta \in D_\lambda \cap \text{cof}(\aleph_n)$, again by continuity of $j'$ for sequences of length $\leq \aleph_n$.

Let $\bar{p} \in \mathbb{H}$ be a condition such that $\beta \in p(\lambda^+)$. Then by elementarity, the following is true in $M[j'(G)]:$

There is some $\tau < j'(\lambda)$ such that $\text{dom}(j'(f)_\alpha^\tau) \cap (\tau, j'(\lambda)) \subseteq \text{sprt } j'(\bar{p})$ and such that for all $\kappa \in \text{dom}(j'(f)_\alpha^\tau) \cap (\tau, j'(\lambda))$, $(j'(f)_\alpha^\tau(j'(\lambda))(\kappa) \in \text{dom } j'(p)(\kappa)$ and $j'(p)(\kappa)((j'(f)_\alpha^\tau(j'(\lambda))(\kappa)) = 0$.

Since $j'(\bar{p})$ satisfies the Annulment Property, the master condition $p$ satisfies the Annulment Property for $\alpha$ because it extends $\bar{p}$ as a function as in Proposition 10. That is, since the Annulment Property was verified with respect to $j'(\bar{p})$.


as witnessed by the interval \( \text{dom}(j'f)_{\alpha}^{j'((\lambda))} \cap (\tau, j'(\lambda)) \), it is does not matter that the support of \( \bar{p} \) is larger. 

Now that we have a master condition, force with a \( j'(S[\chi, \nu]) \)-generic \( K \) that contains \( p \) and let \( L = I * J * K \). This allows us to extend \( j' \) to \( j^+ : V[G * \bar{H}] \to M^+[j^+(G * \bar{H})] \), and so we have proved the claim.  

We work in \( V[G * H * L] \) in which our stationary set \( S \subseteq \nu \cap \text{cof}(\aleph_n) \) remains stationary. We consider \( j(S) \) and \( \rho := \text{sup } j^+[\nu] \), noting that \( \rho < j^+(\nu) \). We can argue that \( M^+[j(G * \bar{H})] \models “j^+(S) \cap \rho \text{ is stationary in } \rho” \) as follows: Suppose \( C \subseteq \rho \) is a club and that \( C \) is stationary in \( M^+[j^+(G * \bar{H})] \).

**Claim.** \( \check{C} := \{ \alpha < \nu : j^+(\alpha) \in C \} \) is unbounded in \( \rho \) and \( \aleph_{n+1} \)-closed.

**Proof of Claim.** The facts that \( C \) is a club and that \( j^+ \) is continuous for sequences of length \( < \aleph_{n+1} \) imply that \( \check{C} \) is \( \aleph_{n+1} \)-closed. For unboundedness, we define \( \langle \alpha_n : n < \omega \rangle \subset C \) and \( \langle \beta_n : n < \omega \rangle \subset \nu \) as follows: Given \( \alpha_n \), find \( \beta_{n+1} < \nu \) such that \( j^+(\beta_{n+1}) > \alpha_n \), and given \( \beta_n \), find \( \alpha_{n+1} \in C \) such that \( j^+(\beta_n) < \alpha_{n+1} \). Let \( \bar{\gamma} := \text{sup }_{n < \omega} \beta_n \). Then \( j^+(\bar{\gamma}) = \text{sup }_{n < \omega} \alpha_n \) by interleaving, so \( j^+(\bar{\gamma}) \in C \) and thus \( \bar{\gamma} \in \check{C} \).

Therefore, \( \check{C} \) intersects \( S \), \( C \) intersects \( j^+(S) \cap \rho \) and \( M[j^+(G * \bar{H})] \models “\exists \rho < j^+(\nu) \text{, } j^+(S) \cap \rho \text{ is stationary”} \). By elementarity, \( V[G * \bar{H}] \models “S \text{ reflects”} \).

2.9. **Finishing the Theorem.** Now we can tie everything together, keeping in mind that \( V[G] \models “\chi = \aleph_{n+1}” \).

**Proof of Theorem 1** We have demonstrated that \( S \) adds non-reflecting stationary sets where directed by \( F \)—that is, \( \kappa \in [\aleph_{n+2}, \text{ON}] \) such that \( F(\kappa) = 1 \). If \( F(\mu) = 0 \), then we consider the factoring \( S|_{\aleph_{n+2}, \text{ON}} = S|_{\aleph_{n+2}, \mu} \times S|_{\mu, \text{ON}} \). For any stationary subset \( S \) of \( \mu \cap \text{cof}(\aleph_n) \), the distributivity of \( S|_{\mu, \text{ON}} \) implies that \( S \) is already contained in \( V[G * \bar{H}] \) where \( \bar{H} \) is \( S|_{\aleph_{n+2}, \mu} \)-generic. And since \( F(\mu) = 0 \), the previous section shows that \( S \) reflects.  

3. **Further Questions**

There are more interesting questions around global compactness properties.

**Question 1.** Can the results of this paper be generalized to an Easton result for \( \text{RP}(\kappa \cap \text{cof}(\aleph_{n+1})) \), or more generally to an Easton result for \( \text{RP}(\kappa \cap \text{cof}(\lambda)) \) given any fixed cofinality \( \lambda \)?

We believe that the answer to this question is positive, and that it is possible to force approachability at all successors of singulars without using the result of Shelah. The idea would be to shoot clubs through the set of approachable points at successors of singulars \( \lambda^+ \) where reflection should be preserved. However, the proof would necessarily be more involved.

The following is a harder question:

**Question 2.** Can the results of this paper be extended to stationary sets concentrating on points of arbitrary (or un-fixed) cofinality? In other words, suppose that \( F \) is a function on the class of regular cardinals to itself. Is it possible to obtain a model such that \( \text{RP}(\kappa \cap \text{cof}(\lambda)) \) holds precisely when \( F(\kappa) = \lambda \)?
We conjecture that the answer to this question is negative, and that there is a Silver’s Theorem for stationary reflection that is waiting to be discovered.

Lastly, we have:

**Question 3.** Does ZFC put any restrictions on the global behavior of $\square_\kappa$? Suppose $F$ is two-valued function on the class of all cardinals. Is it possible to obtain a model such that $\square_\kappa$ holds precisely if $F(\kappa) = 1$? And if there are ZFC restrictions, what exactly are they?

Some progress has been made for this question. Cummings, Foreman, and Magidor constructed a model in which $\square_{\aleph_n}$ holds for all $n < \omega$, but where $\square_{\aleph_\omega}$ fails [3]. However, it appears difficult to generalize this result to singulars of uncountable cofinality. Cummings et al. also showed that the existence of square sequences below a singular cardinal $\kappa$ implies the existence of something resembling but distinct from a $\square_{\kappa}$-sequence [4]. It may be possible to take their argument further.

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