Abstract In this paper we prove the equiconsistency of the assumption that there is a measurable cardinal \( \kappa \) with the Mitchell order \( o(\kappa) = \kappa^{++} \) with the statement that there exists a measurable cardinal \( \kappa \) violating GCH such that the continuum function on regular cardinals below \( \kappa \) can be anything not outright inconsistent with the measurability of \( \kappa \) (see Corollary 3.2).

This settles the question of the optimal large cardinal strength needed to not only violate GCH at a measurable cardinal \( \kappa \), but also determine the continuum function below \( \kappa \), generalizing [3] to a large-cardinal setting.

Keywords: Easton’s theorem, large cardinals, Mitchell order.

AMS subject code classification: 03E35,03E55.

1 Introduction

As shown by W. Mitchell and M. Gitik, see [8] and [6], the following holds:

(*) The failure of GCH at a measurable cardinal is equiconsistent with the existence of a measurable cardinal of Mitchell order \( \kappa^{++} \); \( o(\kappa) = \kappa^{++} \).

It seems to be a reasonable assumption that (*) can be generalized for example to:

(**) The existence of \( \kappa \) such that \( o(\kappa) = \kappa^{++} \) is equiconsistent with the existence of \( \kappa \) such that \( 2^{\kappa} = \kappa^{++} \), \( \kappa \) is measurable, and for all regular cardinals \( \alpha < \kappa \), \( 2^{\alpha} = \alpha^{++} \).

Though (**) may seem to be only a minor generalization of (*), it is not so, and as a matter of fact (**) was not resolved in [6], nor [4]. (**) is a paradigmatical case of a still more general question which studies the
possible behaviours of the continuum function with respect to large cardinals (large cardinals restrict the continuum function beyond the ZFC conditions identified in [3]). See the paper [4] for details.

The interesting direction in the equivalence (**), which does not follow from (*), is from the left to the right. We have shown the analogous implication in [4] from the slightly stronger assumption of \( \kappa \) being \( \kappa^{++} \)-strong (see Section 1.1 for definitions).

In Theorem 3.1, we show that (***) is true; in Corollary 3.2, we generalize (***) along the lines of [3]. The proof is structured as follows. By the construction in [6], the assumption \( o(\kappa) = \kappa^{++} \) together with GCH implies that there exists a generic extension \( V^* \) satisfying GCH and an elementary embedding \( j : V^* \rightarrow M \) with the critical point \( \kappa \), which satisfies:

(i) \( M \) is closed under \( \kappa \)-sequences in \( V^* \);
(ii) \( (\kappa^{++})^M = \kappa^{++} \).

In this paper, we will call such a \( j \) a \( \kappa^{++} \)-correct embedding (to our knowledge, no terminology is as yet fixed for embeddings satisfying (i) and (ii)).

We start with GCH and a \( \kappa^{++} \)-correct embedding \( j \) and define a cofinality-preserving forcing \( R \) such that in \( V^R \):

(i) \( 2^{\alpha} = \alpha^{++} \) for every regular cardinal \( \alpha \leq \kappa \), and
(ii) \( \kappa \) is measurable.

We achieve (ii) by lifting the embedding \( j \) to \( V^R \). The crucial step in the lifting argument, see Theorem 2.1, where the weaker assumption of \( \kappa^{++} \)-correctness needs to be taken into account, concerns the \( V \)-regular cardinal \( (\kappa^{++})^M = \kappa^{++} \). Since \( \kappa^{++} \) is regular in \( M \), the forcing iteration \( j(R) \) is non-trivial at \( \kappa^{++} \). Since the entire \( H(\kappa^{++}) \) may not be included in \( M \), the forcing at \( \kappa^{++} \) in \( j(R) \) (which typically uses conditions in \( H(\kappa^{++}) \)) tends to behave erratically in the universe.

Inspired by U. Abraham’s paper [1], we show that if we include in \( R \) some preparatory forcing, we can “force” \( j(R) \) to behave properly at \( \kappa^{++} \). In some sense, this preparatory forcing makes a \( \kappa^{++} \)-correct embedding look more like a \( \kappa^{++} \)-strong embedding, as far as the proximity of \( H(\kappa^{++})^M \) to the real \( H(\kappa^{++}) \) is concerned. Once the problematic step of \( \kappa^{++} \) is resolved, the rest of the lifting to \( V^R \) is standard.

### 1.1 Terminology

We say that \( \kappa \) is a \( \theta \)-strong cardinal, where \( \kappa < \theta \) and \( \theta \) is a cardinal, if there exists an elementary embedding \( j \) from \( V \) into some transitive class \( M \) with the critical point \( \kappa \) such that \( j(\kappa) > \theta \), and \( H(\theta) \) is included in \( M \). If GCH is assumed, and \( \theta \) is regular (this is sufficient for our purposes here),
then the elementary embedding witnessing the $\theta$-strength of $\kappa$ can be taken to have the additional property that $M = \{ j(f)(\alpha) \mid f : \kappa \to V, \alpha < \theta \}$, $\theta < j(\kappa) < \theta^+$, and $M$ is closed under $\kappa$-sequences in $V$ (such a $j$ is called an extender ultrapower embedding). See [2] for details.

We will focus in this paper on the case when $\theta = \kappa^{++}$.

If we omit the condition that $H(\theta)$ is included in $M$, we obtain a weaker notion.

**Definition 1.1** Assume GCH. We say that $j : V \to M$ with the critical point $\kappa$ is a $\kappa^{++}$-correct embedding if $j$ satisfies:

(i) $\kappa^{++} = (\kappa^{++})^M$,

(ii) $M$ is closed under $\kappa$-sequences in $V$.

If $j$ is $\kappa^{++}$-correct, one can use the usual extender ultrapower construction to get an even better embedding. We call $j$ a $\kappa^{++}$-correct extender embedding if $j$ satisfies conditions (i)–(iii) above, and moreover satisfies:

(iv) $M = \{ j(f)(\alpha) \mid f : \kappa \to V, \alpha < \kappa^{++} \}$.

We say that $\kappa$ is $\kappa^{++}$-correct if there is a $\kappa^{++}$-correct embedding with the critical point $\kappa$.

Fact 1.2 shows that in some sense a $\kappa^{++}$-strong embedding is substantially stronger than a $\kappa^{++}$-correct embedding.

**Fact 1.2** (GCH) While there are $\kappa$-many measurable cardinals below every $\kappa^{++}$-strong cardinal $\kappa$, a $\kappa^{++}$-correct cardinal $\kappa$ may be the least measurable cardinal.

Let us note that Fact 1.2 is implicit in the proof of the main theorem in [6].

The weaker notion of a $\kappa^{++}$-correct embedding is important because it has the same strength (in terms of consistency) as the assumption that $\kappa$ has the Mitchell order $\kappa^{++} : o(\kappa) = \kappa^{++}$ (see [6]).

We now provide a quick review of the notion of lifting of embeddings.

**Fact 1.3** Let $\mathbb{P}$ be a forcing notion and $j : V \to M$ an embedding with the critical point $\kappa$. Then the following holds (for proofs, see [2]):

(i) (Silver) Assume $G$ is $\mathbb{P}$-generic over $V$ and $H$ is $j(\mathbb{P})$-generic over $M$ such that $j[G] \subseteq H$. Then there exists an elementary embedding $j^* : V[G] \to M[H]$ such that $j^*[V] = j$, and $H = j^*(G)$. We say that $j$ lifts to $V^\mathbb{P}$.
(ii) If \( j \) is moreover an extender ultrapower embedding then if \( P \) is a \( \kappa^+ \)-distributive forcing notion and \( G \) is \( P \)-generic over \( V \), then the filter \( G^* \) in \( j(P) \) defined as
\[
G^* = \{ q \mid \exists p \in G, j(p) \leq q \}
\]
is \( j(P) \)-generic over \( M \).

(iii) If \( j^* : V[G] \to M[H] \) is the lifting of \( j \), then if \( j \) was an extender ultrapower embedding, so is \( j^* \).

2 The crucial step: \( \kappa^{++} \)

Theorem 2.1 captures the main idea of this paper. Theorem 3.1 and Corollary 3.2 are direct applications of Theorem 2.1 based on results in [5] and [4].

**Theorem 2.1** Assume GCH and let \( j : V \to M \) be a \( \kappa^{++} \)-correct extender embedding with the critical point \( \kappa \). Then there exists a cofinality-preserving forcing notion \( P \) such that if \( G \) is \( P \)-generic, the following holds:

(i) \( 2^\alpha = \alpha^{++} \) for every regular cardinal \( \alpha < \kappa \) which is a double successor of an inaccessible cardinal \( \beta < \kappa \), where \( \alpha \) is a double successor of \( \beta \) if \( \alpha = \beta^{++} \).

(ii) The embedding \( j \) lifts to \( j^* : V[G] \to M[j^*(G)] \), and \( j^* \) is a \( \kappa^{++} \)-correct extender embedding in \( V[G] \).

**Proof.** The proof of the theorem will be given in a sequence of lemmas.

For a regular cardinal \( \alpha \), we write \( \text{Add}(\alpha, 1) \) for the Cohen forcing which adds a single Cohen subset of \( \alpha \): a condition \( p \) is in \( \text{Add}(\alpha, 1) \) if and only if \( p \) is a function of size \( < \alpha \) from \( \alpha \) to 2. We also write \( \text{Add}(\alpha, 1) \) for the Cohen forcing viewed as adding a single Cohen function from \( \alpha \) to \( \alpha \): a condition \( p \) is in \( \text{Add}(\alpha, 1) \) if and only if \( p \) is a function of size \( < \alpha \) from \( \alpha \) to \( \alpha \). These two descriptions are equivalent as far as forcing goes; we will use the one which is best suited for our purposes (we will indicate which representation we choose). If \( \alpha \) is a regular cardinal and \( \beta \) an ordinal greater than 0, we write \( \text{Add}(\alpha, \beta) \) to denote the standard Cohen forcing which adds \( \beta \)-many subsets of \( \alpha \). A condition \( p \) belongs to \( \text{Add}(\alpha, \beta) \) if and only if \( p \) is a function from \( \alpha \times \beta \) to 2 of size \( < \alpha \). At times, we view \( \text{Add}(\alpha, \beta) \) as a \( \beta \)-product with \( < \alpha \)-support of the forcing \( \text{Add}(\alpha, 1) \). Again, these two descriptions of \( \text{Add}(\alpha, \beta) \) equivalent, and we will therefore use the one which best suits the given context (we will indicate which representation we have in mind).

Let us now define the forcing \( P \). \( P \) will be a two stage iteration \( P^0 \ast \dot{P}^1 \), where:
Lemma 2.2

The forcing $\mathbb{P}^0$ is cofinality-preserving.

Proof. First note that this lemma is non-trivial: if the original $M$ missed some subsets of $\kappa^+$ in $V$, then $Q^*$ is a proper subset of $\text{Add}(\kappa^+, 1)$ of $V[G_\kappa * g]$, and hence $Q^*$ is certainly not $\kappa^{++}$-closed over $V[G_\kappa * g]$. We will argue that the preparatory forcing $\text{Add}(\kappa^+, \kappa^{++})$ ensures that $Q^*$ is still $\kappa^{++}$-distributive over $V[G_\kappa * g]$.

Let us work in $V[G_\kappa * g]$. Assume that $p \in Q^*$ is a condition and $\dot{f}$ is a name for a function from $\kappa^+$ to ordinals:

$$(2.3) \quad p \Vdash \dot{f} : \kappa^+ \to \text{ORD}.$$ 

We will show that there exists $q \leq p$ which decides all values of $\dot{f}$. 

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(1) $\mathbb{P}^0$ is an iteration of length $\kappa$ with Easton support, $\mathbb{P}^0 = \langle (\mathbb{P}_\xi, \dot{Q}_\xi) \mid \xi < \kappa \rangle$, where $\dot{Q}_\xi$ is a name for a trivial forcing unless $\xi$ is an inaccessible cardinal $< \kappa$, in which case

$$(2.1) \quad \mathbb{P}_\xi^0 \Vdash "\dot{Q}_\xi \text{ is the forcing } \text{Add}(\xi^+, \xi^{++}) * \text{Add}(\xi^{++}, \xi^{+4}),"$$

where $\text{Add}(\xi^+, \xi^{++})$ is viewed as a product forcing which adds $\xi^{++}$-many Cohen functions from $\xi^+$ to $\xi^+$, and $\text{Add}(\xi^{++}, \xi^{+4})$ is viewed as adding $\xi^{+4}$-many Cohen subsets of $\xi^{++}$.

(2) Notice that $\mathbb{P}^0$ is an element of $M$. $\mathbb{P}^1$ is defined in $M$ to be a $\mathbb{P}^0$-name which satisfies:

$$(2.2) \quad M \models "\mathbb{P}^0 \Vdash "\mathbb{P}^1 \text{ is the forcing } \text{Add}(\kappa^+, \kappa^{++}) * \text{Add}(\kappa^{++}, 1),."$$

where $\text{Add}(\kappa^+, \kappa^{++})$ is viewed as a product forcing which adds $\kappa^{++}$-many Cohen functions from $\kappa^+$ to $\kappa^+$, and $\text{Add}(\kappa^{++}, 1)$ is viewed as adding a single Cohen subset of $\kappa^{++}$.

$\mathbb{P}$ is cofinality-preserving.

The forcing $\mathbb{P}^0$ is clearly cofinality-preserving by a standard argument. Let $G_\kappa$ be a $\mathbb{P}^0$-generic filter over $V$. By $\kappa^{++}$-correctness of $j$, $\mathbb{P}^0$ is in $M$ and $G_\kappa$ is $\mathbb{P}^0$-generic over $M$. In order to verify that $\mathbb{P}$ is cofinality-preserving, it suffices to check that the forcing $(\mathbb{P}^1)^{M[G_\kappa]}$ defined in $M[G_\kappa]$ preserves cofinalities when forced over $V[G_\kappa]$. Notice first that $\text{Add}(\kappa^+, \kappa^{++})$ of $M[G_\kappa]$ is the same set as $\text{Add}(\kappa^+, \kappa^{++})$ of $V[G_\kappa]$: this is because $\mathbb{P}^0$ has $\kappa$-cc, and hence by standard arguments $M[G_\kappa]$ is still closed under $\kappa$-sequences in $V[G_\kappa]$. Let $g$ be a $\text{Add}(\kappa^+, \kappa^{++})^{V[G_\kappa]}$-generic over $V[G_\kappa]$. Then by the previous sentence, $g$ is also $\text{Add}(\kappa^+, \kappa^{++})^{M[G_\kappa]}$-generic over $M[G_\kappa]$. Work in $M[G_\kappa * g]$ and let $Q^*$ denote the forcing $\text{Add}(\kappa^{++}, 1)$ of $M[G_\kappa * g]$. Then the claim that $\mathbb{P}$ preserves cofinalities follows from the following lemma:

Lemma 2.2 The forcing $Q^*$ is still $\kappa^{++}$-distributive over $V[G_\kappa * g]$.

Proof. First note that this lemma in non-trivial: if the original $M$ missed some subsets of $\kappa^+$ in $V$, then $Q^*$ is a proper subset of $\text{Add}(\kappa^{++}, 1)$ of $V[G_\kappa * g]$, and hence $Q^*$ is certainly not $\kappa^{++}$-closed over $V[G_\kappa * g]$. We will argue that the preparatory forcing $\text{Add}(\kappa^+, \kappa^{++})$ ensures that $Q^*$ is still $\kappa^{++}$-distributive over $V[G_\kappa * g]$.
Write $H(\kappa^{++})$ of $M[G_\kappa \ast g]$ as $L_{\kappa^{++}}[B]$ for some subset $B$ of $\kappa^{++}$, $B$ in $M[G_\kappa \ast g]$. This is possible because $H(\kappa^{++})$ of $M[G_\kappa \ast g]$ has size $\kappa^{++}$ in $M[G_\kappa \ast g]$. Fix an elementary submodel $N$ of some large enough $H(\theta)^{V[G_\kappa \ast g]}$ which has size $\kappa^+$, is transitive below $\kappa^{++}$, is closed under $\kappa$-sequences and contains as elements $B$, $Q^*$, $p$ and $f$. We will show that $p$ has an extension $q \leq p$ which hits all dense subsets of $Q^*$ which belong to $N$; this will imply that $q$ decides all values of $f$ as required.

Let $\beta$ be the the ordinal $N \cap \kappa^{++}$ and let $\pi$ be a transitive collapse of $N$ to $\bar{N}$. Then $\pi(Q^*)$, which is equal to $Q^* \cap N$, belongs to $M[G_\kappa \ast g]$ because $Q^*$ is definable in $L_{\kappa^{++}}[B]$, and so by $\pi$ being an isomorphism, $\pi(Q^*)$ is definable in $L_{\pi(\kappa^{++})}[\pi(B)] = L_\beta[B \cap \beta]$. It suffices to extend $\pi(p) = p$ to a condition $q$ which hits all dense subsets of $\pi(Q^*)$ which belong to $N$.

For $\gamma < \kappa^{++}$, let us denote $g \upharpoonright \gamma = \{q \in g \mid q \upharpoonright \gamma = q\}$. Pick some $\gamma < \kappa^{++}$ such that $\bar{N}$ is in $V[G_\kappa \ast g \upharpoonright \gamma]$, and $\pi(Q^*)$ and some enumeration $\langle p^*_\xi \mid \xi < \kappa^+ \rangle$ of $\pi(Q^*)$ are in $M[G_\kappa \ast g \upharpoonright \gamma]$. Such $\gamma$ exists by $\kappa^+$-cc of the forcing $\text{Add}(\kappa^+)$ and the fact that $\bar{N}$ is a transitive set of size $\kappa^+$. Let $h$ be the generic function $\kappa^+ \rightarrow \kappa^+$ at the coordinate $\gamma$ in $g$: $h = \langle \{q(\gamma) \mid q \in g, \gamma \in \text{dom}(q)\}\rangle$. So $h$ is $\text{Add}(\kappa^+, 1)$-generic over $V[G_\kappa \ast g \upharpoonright \gamma]$. Note that $h \in M[G_\kappa \ast g]$.

Define inductively in $M[G_\kappa \ast g]$ a decreasing sequence of conditions $\langle p^*_\xi \mid \xi < \kappa^+ \rangle$ with $p_0 = p$, $p_\lambda = \bigcup_{\xi<\lambda} p_\xi$ for $\lambda$ a limit ordinal $< \kappa^+$, and:

$$p_{\xi+1} = \begin{cases} 
  p^*_h(\xi) & \text{if } p^*_h(\xi) \text{ extends } p_\xi, \\
  p_\xi & \text{otherwise.}
\end{cases}$$

Since all the parameters used in this construction, i.e. the sequence $\langle p^*_\xi \mid \xi < \kappa^+ \rangle$, and $h, \pi(Q^*), p$, are in $M[G_\kappa \ast g]$, so is the whole sequence $\langle p_\xi \mid \xi < \kappa^+ \rangle$. Let $q$ be the greatest lower bound of this sequence, $q = \bigcup_{\xi<\lambda} p_\xi$. Since $\langle p_\xi \mid \xi < \kappa^+ \rangle$ is in $M[G_\kappa \ast g]$, $q \in Q^*$.

We will show in $V[G_\kappa \ast g \upharpoonright \gamma][h]$ that the sequence $\langle p_\xi \mid \xi < \kappa^+ \rangle$ is $(\bar{N}, \pi(Q^*))$-generic. This already implies that $q$ decides all the values of $f$: For each $\xi < \kappa^+$, the set

$$D_\xi = \{p \in \pi(Q^*) \mid p \text{ decides } \pi(\bar{f})(\xi)\}$$

is a dense open set in $\pi(Q^*)$, which is an element of $\bar{N}$. If $p_\xi$ for some $\xi < \kappa^+$ meets $D_\xi$, then $p_\xi = \pi^{-1}(p_\xi)$ decides the value of $\bar{f}(\alpha)$, and so does $q \leq p_\xi$. The $(\bar{N}, \pi(Q^*))$-genericity is proved by using the generic $h$. Let $D$ be a dense open set in $\pi(Q^*)$ which is an element of $\bar{N}$. We will show in $V[G_\kappa \ast g \upharpoonright \gamma][h]$ that there is some $p_\xi$ which meets $D$. To this end, it suffices to show that $q | q \Vdash "\exists \xi < \kappa^+\ p_\xi \in D^*\}$. 

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is dense in \( \text{Add}(\kappa^+, 1) \) in \( V[G_\kappa \ast g | \gamma] \). Given a condition \( q \), extend \( q \) first into some \( q' \) such that \( \text{dom}(q') = \delta \) for some \( \delta < \kappa^+ \); then \( q' \) decides the construction of \( (p_\xi | \xi < \kappa^+) \) up to \( \delta \) (because it decides \( h \) up to \( \delta \)): for some \( p' \in \pi(Q) \), \( q' \vdash p_\delta = p' \). Pick \( p'' \leq p' \) in \( D \). In the enumeration \( (p''_\xi | \xi < \kappa^+) \), \( p'' \) is some condition \( p''_\eta \). Set \( q'' = q' \cup \{ \langle \delta, \eta \rangle \} \). Then \( q'' \vdash \langle \eta, q \rangle \vDash \langle q'' \rangle \). p_\delta+1 extends \( p_\delta \) and meets \( D'' \), and so \( q'' \leq q \) is in \( D'' \). It follows that \( D'' \) is dense and the proof is finished.

This shows that \( \mathbb{P} \) is cofinality-preserving over \( V \). We now show that the embedding \( j \) can be lifted to \( V^\mathbb{P} \).

**The embedding \( j \) lifts to \( j^* \).**

Let \( G = G_\kappa \ast g \ast g' \) be a \( \mathbb{P} \)-generic over \( V \), where \( G_\kappa \) is \( \mathbb{P}^0 \)-generic, \( g \) is \( \text{Add}(\kappa^+, \kappa^{++}) \)\( M[G_\kappa] \)-generic over \( V[G_\kappa] \), and \( g' \) is \( \text{Add}(\kappa^{++}, 1) \)\( M[G_\kappa * g] \)-generic over \( V[G_\kappa * g] \). We need to find a \( j(\mathbb{P}) \)-generic \( H \) over \( M \) such that \( j[G] \subseteq H \).

By \( \kappa^{++} \)-correctness of \( j \), \( j(\mathbb{P})_\kappa = \mathbb{P}^0 \), and so we start building \( H \) by plugging in \( G_\kappa \) as the \( j(\mathbb{P})_\kappa \)-generic over \( M \).

The next forcing in \( j(\mathbb{P}) \) above \( \kappa \) is \( Q = \text{Add}(\kappa^+, \kappa^{++}) \ast \text{Add}(\kappa^{++}, \kappa^{++}) \) as defined in \( M[G_\kappa] \). We need to find in \( V[G] \) a \( Q \)-generic over \( M[G_\kappa] \). It is here, where we make use of the preparatory forcing \( \hat{\mathbb{P}}^1 \). By definition of \( \hat{\mathbb{P}}^1 \), \( g \) is \( \text{Add}(\kappa^+, \kappa^{++}) \)\( M[G_\kappa] \)-generic over \( V[G_\kappa] \) (and hence over \( M[G_\kappa] \)). To complete the construction of a \( Q \)-generic, it remains to find some \( h \), which will be \( \text{Add}(\kappa^{++}, \kappa^{++}) \)\( M[G_\kappa * g] \)-generic over \( M[G_\kappa * g] \).

When we look at the generics at our disposal, the natural candidate for \( h \) is the generic filter \( g' \). Clearly, \( g' \) will need to be tweaked a little because it is only \( \text{Add}(\kappa^{++}, 1) \)\( M[G_\kappa * g] \)-generic over \( V[G_\kappa * g] \), but not \( \text{Add}(\kappa^{++}, \kappa^{++}) \)\( M[G_\kappa * g] \)-generic over \( V[G_\kappa * g] \). Note that there is a good reason for this apparent deficiency of \( g' \): While Lemma 2.2 shows that \( \text{Add}(\kappa^{++}, 1) \)\( M[G_\kappa * g] \) is sufficiently distributive over \( V[G_\kappa * g] \), the forcing \( \text{Add}(\kappa^{++}, \kappa^{++}) \)\( M[G_\kappa * g] \) never is, in fact it collapses \( \kappa^{++} \):

**Observation 2.3** Let \( \gamma \) be an ordinal \( < j(\kappa) \) which has \( V \)-cofinality \( \kappa^+ \), and its cofinality in \( M \) is \( > \kappa^+ \). Then the forcing \( \text{Add}(\kappa^{++}, \gamma) \)\( M[G_\kappa * g] \) collapses \( \kappa^{++} \) to \( \kappa^+ \) if forced over \( V[G_\kappa * g] \).

**Proof.** Notice that every \( M \)-regular cardinal in the interval \( (\kappa^{++}, j(\kappa)] \) has \( V \)-cofinality \( \kappa^+ \) (this uses GCH in \( V \), and the extender representation of \( M \)), and so setting \( \gamma = (\kappa^{++}) \)\( M \) achieves the desired claim. Fix \( X \) to be a cofinal subset of \( \gamma \) of order type \( \kappa^+ \). Now, for each \( \zeta \in \kappa^{++} \) and every \( p \in \text{Add}(\kappa^{++}, \gamma) \)\( M[G_\kappa * g] \), one can find \( q \leq p \) and \( \xi \in X \) such that \( q \) at the coordinate \( \xi \) codes \( \zeta \) in the sense that it contains \( \zeta \)-many 1’s followed by 0. Hence it is dense that every \( \zeta \in \kappa^{++} \) is coded at some element \( \xi \in X \). □
Thus we have no choice but to try to use $g'$ to obtain the desired generic filter $h$. A priori, this strategy may work because $\text{Add}(\kappa^{++}, \kappa^{+4})^{M[G_\star g]}$ is small in $V[G_\kappa * g * g']$: it has size $\kappa^{++}$ here. The following Lemma suggests that perhaps $\text{Add}(\kappa^{++}, 1)^{M[G_\kappa * g]}$ can be “stretched” to $\text{Add}(\kappa^{++}, \kappa^{+4})^{M[G_\kappa * g]}$.

**Lemma 2.4** Let $j : V \to M$ be a $\kappa^{++}$-correct extender ultrapower embedding. Let $\gamma = (\kappa^{+4})^M$. Then:

(i) There exists in $V$ a bijection $\pi : \gamma \to \kappa^{++}$ which is locally $M$-correct in the sense that whenever $X \subseteq \gamma$ is in $M$ and has in $M$ size $\leq \kappa^{++}$, the restriction $\pi | X$ is also in $M$.

(ii) And more generally, if $\mathbb{R}$ is a forcing notion in $M$ and $\mathbb{R}$ has $\kappa^{+3}$-cc in $M$, then the bijection $\pi$ in (i) is $M_\mathbb{R}$-locally correct.

**Proof.** Ad (i). List all $f : \kappa \to \kappa$ in $V$ as $\{f_i | i < \kappa^+\}$. For $\beta < \kappa^+$ let $S_\beta$ denote the set of all $j(f_i)(\alpha)$, where $i < \beta$ and $\alpha < \kappa^{++}$, such that $j(f_i)(\alpha) < \gamma$:

$$S_\beta = \{ j(f_i)(\alpha) | i < \beta, \alpha < \kappa^{++}, j(f_i)(\alpha) < \gamma \}.$$ 

Then each $S_\beta$ belongs to $M$, the $S_\beta$’s form an increasing chain under inclusion, and the union $\bigcup \{ S_\beta | \beta < \kappa^+ \}$ is equal to $\gamma$. Moreover, if $X \in M$ is a subset of $\gamma$ of size $\leq \kappa^{++}$ in $M$, then $X$ is contained in some $S_\beta$. The sequence of $S_\beta$’s can be used to construct a bijection $\pi' : \gamma \to \kappa^+ \times \kappa^{++}$ with the property that $\pi' | X$ is in $M$ for any $X$ as above. If $f$ is a bijection in $M$ between $\kappa^+ \times \kappa^{++}$ and $\kappa^{++}$, then the composition $\pi = \pi' \circ f$ is the desired bijection.

Ad (ii). Let $F$ be $\mathbb{R}$-generic over $M$. If $X$ is a subset of $\gamma$ in $M[F]$ which has size $\leq \kappa^{++}$ in $M[F]$, then by $\kappa^{+3}$-cc of $\mathbb{R}$ there is some $X' \supseteq X$ in $M$ which has size $\leq \kappa^{++}$ in $M$. Then the claim follows by application of (i). \[\square\]

Note that the inverse function $\pi^{-1}$ may not be “locally $M$-correct” in the sense of the previous Lemma. Indeed, if $\langle \xi | \xi < \kappa^+ \rangle$ is cofinal in $(\kappa^{+4})^M$, then for $X = \{ \xi | \xi < \kappa^+ \}$, the set $\pi[X]$ may be in $M$ (for instance when $\kappa$ is $\kappa^{++}$-strong), while $\pi^{-1}[\pi[X]] = X$ is certainly not in $M$.

We know show that the previous Lemma can be used to stretch the $\text{Add}(\kappa^{++}, 1)$-generic $g'$ over $V[G_\kappa * g]$ to a $\text{Add}(\kappa^{++}, \kappa^{+4})^{M[G_\kappa * g]}$-generic over $M[G_\kappa * g]$.

Let $Q^* = \text{Add}(\kappa^{++}, 1)^{M[G_\kappa * g]}$, and $\tilde{Q} = \text{Add}(\kappa^{++}, \kappa^{+4})^{M[G_\kappa * g]}$.

**Lemma 2.5** There exists in $V[G_\kappa * g * g']$ a $\tilde{Q}$-generic over $M[G_\kappa * g]$. Let us denote this generic as $h$.

**Proof.** Let $\pi^* : \kappa^{++} \times (\kappa^{+4})^M \to \kappa^{++}$ be a bijection obtained by composing the bijection $\pi$ from Lemma 2.4 with any bijection in $M$ between $\kappa^{++} \times$
(\kappa^{+4})^M \text{ and } (\kappa^{+4})^M. \text{ Then } \pi^* \text{ is locally } M[G_\kappa * g]-\text{correct in the sense of Lemma 2.4(ii). For } p \in \bar{Q}, \text{ write } p^* \text{ to denote the image of } p \text{ under } \pi^*: \text{ dom}(p^*) = \pi^*[\text{dom}(p)], \text{ and for each } (\xi, \zeta) \text{ in the domain of } p, p^*(\pi(\xi, \zeta)) = p(\xi, \zeta). \text{ By the local } M[G_\kappa * g]-\text{correctness of } \pi^*, \text{ each } p^* \text{ is in } M[G_\kappa * g], \text{ and hence it is a condition in } Q^*:

\{p^* \mid p \in \bar{Q}\} \subseteq Q^*.

Note that the inclusion is proper because } Q^* \text{ is } \kappa^{++}-\text{distributive over } V[G_\kappa * g], \text{ while } \bar{Q} \text{ is not (see Observation 2.3).}

Let us set

\[ h = \{p \mid p^* \in g'\}. \]

We show that } h \text{ is as required. Assume } A \text{ lies in } M[G_\kappa * g] \text{ and it is a maximal antichain in } \bar{Q}, \text{ and so in particular } A \text{ has size } \leq \kappa^{++} \text{ in } M[G_\kappa * g]. \text{ Let us denote } \text{dom}(A) = \bigcup\{ \text{dom}(p) \mid p \in A \}. \text{ Let us write } A^* = \{p^* \mid p \in A\} \text{ and } \text{dom}(A^*) = \bigcup\{\text{dom}(p^*) \mid p^* \in A^*\}; \text{ then } A^* \text{ is an antichain in } Q^* \text{ and } \pi^* \restriction \text{dom}(A) \text{ is in } M[G_\kappa * g] \text{ by the local } M[G_\kappa * g]-\text{correctness of } \pi^*. \text{ To show that } h \text{ is as required, it suffices to show that } A^* \text{ is a maximal antichain in } Q^*. \text{ Let } q \text{ be any condition in } Q^*; \text{ since } q \text{ is in } M[G_\kappa * g], \text{ the intersection } \text{dom}(q) \cap \text{dom}(A^*) \text{ is in } M[G_\kappa * g]. \text{ Since } \pi^* \restriction \text{dom}(A) \text{ is in } M[G_\kappa * g], \text{ the set } (\pi^*)^{-1}[\text{dom}(q) \cap \text{dom}(A^*)] \text{ is also in } M[G_\kappa * g]. \text{ If } q' \text{ denotes the condition in } \bar{Q} \text{ with the domain } (\pi^*)^{-1}[\text{dom}(q) \cap \text{dom}(A^*)] \text{ defined by } q'(\xi, \zeta) = q(\pi^*(\xi, \zeta)), \text{ then there exists by the maximality of } A \text{ some } p \in A \text{ compatible with } q'. \text{ It follows that } p^* \in A^* \text{ is compatible with } q \text{ because it is compatible with } q \text{ on } \text{dom}(p^*) \cap \text{dom}(q). \text{ Thus } A^* \text{ indeed maximal, and } h \text{ meets } A \text{ as required.}

\[
\]

Based on the previous Lemma, we see that } G_\kappa * g * h \text{ is } j(\mathbb{P}^0)_{\kappa + 1}-\text{generic over } M. \text{ The iteration } j(\mathbb{P}^0) \text{ in the interval } (\kappa + 1, j(\kappa)) \text{ is } \kappa^{++}-\text{distributive in } M[G_\kappa * g * h], \text{ and so all the relevant dense open sets in } M[G_\kappa * g * h] \text{ can be met in } \kappa^+-\text{many steps, using the extender representation of } M \text{ (see [4] for details). Let the resulting generic be denoted as } \tilde{h}. \text{ Then } G_\kappa * g * h * \tilde{h} \text{ is } j(\mathbb{P}^0)-\text{generic over } M, \text{ and we can partially lift to }

\[
\]

It remains to lift } j' \text{ to } \mathbb{P}^1 = \text{Add}(\kappa^+, \kappa^{++}) * \text{Add}(\kappa^{++}, 1) \text{ of } M[G_\kappa * g]. \text{ By Lemma 2.2, } \mathbb{P}^1 \text{ is } \kappa^+-\text{distributive over } V[G_\kappa], \text{ and therefore by Fact 1.3, the filter } \tilde{h} \text{ generated by the } j' \text{ image of } g * g' \text{ is } M[G_\kappa * g * h * \tilde{h}]-\text{generic over } j'(\mathbb{P}^1):

\[
\tilde{h} = \{q \mid \exists p \in g * g', j'(p) \leq q\}.
\]

If we define } H = G_\kappa * g * h * \tilde{h}, \text{ then } H \text{ is as required:}

\[
\]

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This finishes the proof of Theorem 2.1.

Remark 2.6 Note that the forcing $\mathbb{P}$ is trivial at $\kappa$, and hence the proof of Theorem 2.1 did not need to take special care to ensure the coherence condition $j[\mathcal{G}] \subseteq H$ (except at $G_\kappa$, but this is satisfied automatically). See next section where the forcing at $\kappa$ is non-trivial.

3 Easton’s theorem and large cardinals from the optimal hypothesis

Theorem 3.1 Assume GCH and let $j : V \rightarrow M$ be a $\kappa^{++}$-correct extender embedding with the critical point $\kappa$. Then there exists a cofinality-preserving forcing notion $\mathbb{R}$ such that if $G$ is $\mathbb{R}$-generic, the following holds:

(i) $2^\alpha = \alpha^{++}$ for every regular cardinal $\alpha \leq \kappa$.

(ii) The embedding $j$ lifts to $j^* : V[G] \rightarrow M[j^*(G)]$, and $j^*$ is a $\kappa^{++}$-correct extender embedding in $V[G]$. In particular, $\kappa$ is still measurable.

Proof. Let $I(\kappa)$ denote the set of all inaccessible cardinals $< \kappa$, and $R(\kappa)$ the set of all regular cardinals $< \kappa$. Set $B = \{ \alpha \in R(\kappa) \mid \exists \beta \in I(\kappa), \alpha = \beta \text{ or } \alpha = \beta^+ \} \cup \{ \kappa \}$, and $A = R(\kappa) \setminus B$. Then $A \cup B$ is the set of all regular cardinals $\leq \kappa$.

We define $\mathbb{R}$ as a two-stage iteration $\mathbb{R}_A * \mathbb{R}_B$. $\mathbb{R}_A$ will be a cofinality-preserving forcing which will force the failure of GCH at every element in $A$. In $V^{\mathbb{R}_A}$, $\mathbb{R}_B$ will be a cofinality-preserving forcing which will violate GCH at the remaining regular cardinals $\leq \kappa$, i.e. at the elements in $B$.

The definition of $\mathbb{R}_A$ is a modification of $\mathbb{P}$, as defined in Theorem 2.1. $\mathbb{R}_A$ is a two stage iteration $\mathbb{R}_0^A * \mathbb{R}_1^A$, where:

(1) $\mathbb{R}_0^A$ is an iteration of length $\kappa$ with Easton support, $\mathbb{R}_0^A = \langle (\mathbb{R}_0^A)_\xi, \dot{Q}_\xi \rangle | \xi < \kappa \rangle$, where $\dot{Q}_\xi$ is a name for a trivial forcing unless $\xi$ is a limit cardinal $< \kappa$, in which case there are two possibilities:

(a) If $\xi$ is regular (and hence inaccessible), then

\begin{equation}
(3.4) \quad (\mathbb{R}_0^A)_\xi \Vdash \text{ \textsc{"Q} is the forcing}
\end{equation}
\[
\text{Add}(\xi^+, \xi^{++}) \times \prod_{\xi^++ < \gamma < \xi^{+\omega}} \text{Add}(\gamma, \gamma^{++})
\]

where $\text{Add}(\xi^+, \xi^{++})$ is viewed as a product forcing which adds $\xi^{++}$-many Cohen functions from $\xi^+$ to $\xi^+$, $\text{Add}(\xi^{++}, \xi^{++})$ is viewed as adding $\xi^{++}$-many Cohen subsets of $\xi^{++}$, and $\prod_{\xi^{++} < \gamma < \xi^{+\omega}} \text{Add}(\gamma, \gamma^{++})$ is the standard Easton product, which adds $\gamma^{++}$-many Cohen subsets to each regular cardinal $\gamma$ such that $\xi^{++} < \gamma < \xi^{+\omega}$ (where $\xi^{+\omega}$ is the least limit cardinal above $\xi$).
(b) If $\xi$ is singular, then

\[
(3.5) \quad (R_0^\xi) \models "\dot{Q}_\xi \text{ is the forcing } \prod_{\xi < \gamma < \xi^{++}} \text{Add}(\gamma, \gamma^{++}),"
\]

where $\prod_{\xi^{++} < \gamma < \xi^{++}} \text{Add}(\gamma, \gamma^{++})$ is the standard Easton product, where $\gamma$ ranges over regular cardinals.

(2) Notice that $R_0^0$ is an element of $M$. $R_1^A$ is defined in $M$ to be a $R_0^A$-name which satisfies:

\[
(3.6) \quad M \models "\dot{R}_1^A \text{ is the forcing } \text{Add}(\kappa^+, \kappa^{++}) \ast \text{Add}(\kappa^+, 1),"
\]

where $\text{Add}(\kappa^+, \kappa^{++})$ is viewed as a product forcing which adds $\kappa^{++}$-many Cohen functions from $\kappa^+$ to $\kappa^+$, and $\text{Add}(\kappa^+, 1)$ is viewed as adding a single Cohen subset of $\kappa^+$.

By standards argument, see [4], and Lemma 2.2 applied in the present context, the forcing $R_A$ is cofinality-preserving. By [4], and an easy modification of Theorem 2.1, $j$ lifts to a $\kappa^{++}$-correct extender embedding $j'$ in $V^{R_A}$: in the proof generalizing the proof of Theorem 2.1, one just needs to take into account the product $\prod_{\kappa^{++} < \gamma < \kappa^{++}} \text{Add}(\gamma, \gamma^{++})$ at the stage $\kappa$ of the iteration $j(R_0^A)$. However, since in $M^{j(R_0^A)}$, $\text{Add}(\kappa^+, \kappa^{++}) \ast \text{Add}(\kappa^+, \kappa^{++})$ has $\kappa^{+3}$-cc and the product $\prod_{\kappa^{++} < \gamma < \kappa^{++}} \text{Add}(\gamma, \gamma^{++})$ is $\kappa^{+3}$-closed, it follows by Easton’s lemma that these two forcing are mutually generic. Accordingly, an $\text{Add}(\kappa^+, \kappa^{++}) \ast \text{Add}(\kappa^+, \kappa^{++})$-generic over $M^{j(R_0^A)}$ is obtained as in Theorem 2.1, while a $\prod_{\kappa^{++} < \gamma < \kappa^{++}} \text{Add}(\gamma, \gamma^{++})$-generic is obtained by a standard construction using the $\kappa^{+3}$-distributivity of the forcing.

Let $G_A$ denote a $R_A$-generic, then the following holds in $V[G_A]$:

(i) GCH holds in $V[G_A]$ at every inaccessible cardinal $\alpha \leq \kappa$ and at the successors of these inaccessible cardinals.

(ii) $2^\alpha = \alpha^{++}$ for every regular cardinal $< \kappa$ other than in (i) in the previous line.

(iii) There exists in $V[G_A]$ a $\kappa^{++}$-correct extender embedding $j': V[G_A] \rightarrow M[j'(G_A)]$ which is a lifting of the original $j$.

In $V[G_A]$, we define $R_B$ as follows.

$R_B$ is an iteration of length $\kappa+1$ with Easton support, $R_B = \langle (R_B)_{\xi}, \dot{Q}_\xi \rangle | \xi < \kappa + 1\rangle$, where $\dot{Q}_\xi$ is a name for a trivial forcing unless $\xi$ is an inaccessible cardinal $\leq \kappa$, in which case there are two cases:

(a) If $\xi < \kappa$, then

\[
(3.7) \quad (R_B)_{\xi} \models "\dot{Q}_\xi \text{ is the forcing Sacks}(\xi, \xi^{++}) \times \text{Add}(\xi^+, \xi^{+3}),"
\]

where Sacks$(\xi^+, \xi^{++})$ is the generalized Sacks product forcing at $\xi$ which adds $\xi^{++}$-many new subsets of $\xi$ (see [7], and [5] for details), and $\text{Add}(\xi^+, \xi^{++})$ is viewed as adding $\xi^{+3}$-many Cohen subsets of $\xi^+$. 

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(b) If $\xi = \kappa$, then

$$(3.8) \quad (\mathbb{R}_B)^{\kappa} \models \text{“}\dot{Q}_\xi \text{is the forcing Sacks}(\xi, \xi^{++}) \times \text{Add}(\xi^+, 1)\text{.”}$$

By standard results, see [4], $\mathbb{R}_B$ is cofinality-preserving over $V[G_A]$ (here, it is important that $\text{Add}(\xi^+, \xi^{+3})$ is still $\xi^+$-distributive over $\text{Sacks}(\xi, \xi^{++})$).

Let $G_B$ be a $\mathbb{R}_B$-generic over $V[G_A]$. Using the “tuning-fork” argument in the original paper [5], together with [4], one can show that $j'$ lifts to $V[G_A][G_B]$. Notice here that it is sufficient to add just one Cohen subset of $\kappa^+$, cf. (3.8), in order to lift, and so GCH holds in $V[G_A][G_B]$ above $\kappa$ (if so desired).

If we set $G = G_A \ast G_B$, then $V[G]$ is as required. \hfill $\square$

We can achieve even more generality, along the lines [3] and [4]. We say that a proper-class function $F$ from regular cardinals into cardinals is an \textit{Easton function}, if for all regular cardinals $\kappa, \lambda$:

(i) $\kappa < \lambda \rightarrow F(\kappa) \leq F(\lambda)$,

(ii) $\text{cf}(F(\kappa)) > \kappa$.

A cardinal $\mu$ is said to be a closure point of $F$ if $F(\nu) < \mu$ for every regular cardinal $\nu < \mu$.

We say that $F$ is realised in some cofinality-preserving extension $V^\mathbb{R}$ if $F$ is the continuum function in $V^\mathbb{R}$ on regular cardinals.

\textbf{Corollary 3.2} Assume GCH and let $\kappa$ be $\kappa^{++}$-correct cardinal. If an Easton function $F$ satisfies:

(i) $\kappa$ is a closure point of $F$, $F(\kappa) = \kappa^{++}$, and

(ii) There exists a $\kappa^{++}$-correct embedding $j : V \rightarrow M$ with the critical point $\kappa$ such that $j(F)(\kappa) \geq F(\kappa)$,

then there exists a cofinality-preserving forcing $\mathbb{R}$ such that the Easton function $F$ is realised in $V^\mathbb{R}$, and $j$ lifts to $V^\mathbb{R}$; in particular $\kappa$ is still measurable in $V^\mathbb{R}$.

\textit{Proof.} This is just like the relevant part of [4], with the arguments in Theorems 2.1 and 3.1 added to be able to prove this result from the optimal hypothesis of a $\kappa^{++}$-correct embedding (Lemma 2.4 must be generalized to $j(F)(\kappa^{++})$ instead to $(\kappa^{++})^M$; this is straightforward since the cofinality of $j(F)(\kappa^{++})$ is strictly greater than $\kappa^{++}$). \hfill $\square$
References


