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TALL $\alpha$-RECURSIVE STRUCTURES

SY D. FRIEDMAN$^1$ AND SAHARON SHELAH$^2$

Abstract. The Scott rank of a structure $M$, $\text{sr}(M)$, is a useful measure of its model-theoretic complexity. Another useful invariant is $\text{o}(M)$, the ordinal height of the least admissible set above $M$, defined by Barwise. Nadel showed that $\text{sr}(M) \leq \text{o}(M)$ and defined $M$ to be tall if equality holds. For any admissible ordinal $\alpha$ there exists a tall structure $M$ such that $\text{o}(M) = \alpha$. We show that if $\alpha = \beta^+$, the least admissible ordinal greater than $\beta$, then $M$ can be chosen to have a $\beta$-recursive presentation. A natural example of such a structure is given when $\beta = \omega_1^1$ and then using similar ideas we compute the supremum of the levels at which $\Pi_1(L_{\omega_1^1})$ singletons appear in $L$.

The results in this paper concern structures which are complicated model-theoretically, yet recursion-theoretically simple. Fix a structure $M$ for a language $\mathcal{L}$ of finite similarity type. The Scott rank of $M$ is defined as follows: Let $\bar{x}, \bar{y}, \bar{x}', \bar{y}', \ldots$ range over $|M|^{<\omega}$. By induction define a sequence of relations $\sim$ on members of $|M|^{<\omega}$ or the same length:

- $\bar{x} \sim_0 \bar{y}$ iff $\bar{x}, \bar{y}$ realize the same atomic type in $M$,
- $\bar{x} \sim_{\alpha+1} \bar{y}$ iff $\forall \bar{x}' \exists \bar{y}' (\bar{x} * \bar{x}' \sim \bar{y} * \bar{y}')$ and $\forall \bar{y}' \exists \bar{x}' (\bar{x} * \bar{x}' \sim \bar{y} * \bar{y}')$
- $\bar{x} \sim_\lambda \bar{y}$ iff $\bar{x} \sim \bar{y}$ for all $\beta < \lambda$, $\lambda$ limit.

In the above, $*$ denotes concatenation of sequences. Finally, Scott rank $(M)$ is the least $\alpha$ such that $\forall \bar{x} \forall \bar{y} (\bar{x} \sim_\alpha \bar{y} \rightarrow \bar{x} \sim_{\alpha+1} \bar{y})$. Scott rank $(M)$ is a useful measure of the model-theoretic complexity of $M$.

Nadel [74] provides a bound on the Scott rank of a structure $M$ in terms of admissible set theory: Scott rank $(M) \leq \text{o}(M)$ where $\text{o}(M)$ is the ordinal height of the least admissible set above $M$ (see Barwise [69]). $M$ is tall if equality holds. This bound is best possible in that for any admissible ordinal $\alpha$ there is a tall structure $M$ such that Scott rank $(M) = \alpha$.

Let $\beta$ be a limit ordinal. $M$ is $\beta$-recursive if $|M| = \beta$ and all of the relations, functions of $M$, are $\beta$-recursive. (For a definition of $\beta$-recursive, see Friedman [78].) In this paper we need only consider those $\beta$ which are either admissible or the limit of admissible ordinals, in which case $\beta$-recursive coincides with $\Delta_1(L_{\beta, \epsilon})$. It is

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shown in Nadel [74] that there is an $\omega$-recursive (= recursive) structure of Scott rank $\omega_1^{ck}$. (The example is a recursive linear ordering of ordertype $\omega_1^{ck} + \omega_1^{ck} \cdot \eta_1 \eta = \text{ordertype of the rationals}.) §1 of the present paper shows that for every limit ordinal $\beta$ there is a $\beta$-recursive structure of Scott rank $\beta^+$, the least admissible ordinal greater than $\beta$. Such a structure $M_\beta$ is tall since it belongs to $L_\beta^+$ and hence $o(M_\beta) = \beta^+$. Define $L_{\omega_1^\omega}$-rank $(M)$ in exactly the same way as Scott rank $(M)$ except where $\bar{\varepsilon}, \bar{\eta}, \bar{x}, \bar{y}, \ldots$ now range over $|M|^{<\omega_1}$. §2 focuses on the special case: $\beta = \omega_1$. Using entirely different methods than in §1 a natural example of an $\omega_1$-recursive structure of $L_{\omega_1^\omega}$-rank $\omega_1^+$ is presented (from this an $\omega_1$-recursive structure of Scott rank $\omega_1^+$ is easily obtained). Similar techniques are then used to show that $\Pi_1(L_{\omega_1})$-singletons appear cofinally inside $L_\sigma$, where $\sigma$ is the least stable ordinal greater than $\omega_1$.

1. Game rank versus Scott rank. The goal of this section is to prove

**Theorem 1.** For any limit ordinal $\beta$ there is a $\beta$-recursive structure of Scott rank $\beta^+ = \text{least admissible ordinal greater than } \beta$.

It clearly suffices to treat the case where $\beta$ is either admissible or the limit of admissible ordinals. It will also be convenient to assume that $\beta$ is greater than $\omega$ (otherwise the result is known).

The proof of Theorem 1 can be outlined as follows: We first show that there is a $\beta$-recursive open game with a winning strategy for the “closed player”, but none inside $L_\beta^+$. This allows one to build a $\beta$-recursive tree $T$ of “game rank” $\beta^+$. Then a $\beta$-recursive structure $M$ of Scott rank $\beta^+$ is obtained by building $M$ so that its Scott analysis is very similar to the “game analysis” of $T$.

We must first describe the “game rank” of a tree. All trees are subtrees of $\beta^{<\omega} = \text{all finite sequences of ordinals less than } \beta$. Our definition here is rather nonstandard but is designed to allow the transition from game rank to Scott rank to go smoothly.

Let $T$ be a tree. If $\eta = \langle \eta(0), \eta(1), \ldots \rangle \in T$ has even length we let $(\eta)_{\text{even}} = \langle \eta(0), \eta(2), \ldots \rangle$. Let $A_k = \{ (\eta)_{\text{even}} \mid \eta \in T, l(\eta) = 2k \}$. For $v \in A_k$ let $B_k = \{ \eta \in T \mid (\eta)_{\text{even}} = v \}$. If $\eta \in T$ has even length we define $\text{Rk}(\eta)$ by

$$Rk(\eta) = 0 \Leftrightarrow \text{there is } v \geq (\eta)_{\text{even}} \text{ such that } \eta \text{ has no extension in } B_v, v \in \bigcup_k A_k;$$

$$Rk(\eta) = \alpha > 0 \leftrightarrow 0 \neq Rk(\eta) \neq \beta \text{ for all } \beta < \alpha \text{ and there is } v \geq (\eta)_{\text{even}} \text{ such that }$$

$$\eta' \supsetneq \eta, \quad \eta' \in B_v \rightarrow Rk(\eta') = \beta \text{ for some } \beta < \alpha;$$

$$Rk(\eta) = \infty \leftrightarrow \forall \alpha Rk(\eta) \neq \alpha, Rk(T) = \sup \{ Rk(\eta) \mid \eta \in T \text{ and } Rk(\eta) \neq \infty \}.$$ 

Thus $\text{Rk}(\eta)$ measures how good a position player I is in after $\eta$ has been played in the following game: Players I and II alternately choose $v_0, v_0, v_1, \ldots$ with the restrictions that $v_0 \subseteq v_1 \subseteq \ldots, \eta_0 \subseteq \eta_1 \subseteq \ldots, \eta_i \in B_{v_i}, v_i \in \bigcup_k A_k$. Player I wins if at some stage player II can make no legal move. Otherwise player II wins.

**Lemma 2.** There is a $\beta$-recursive tree $T$ such that $\text{Rk}(T) = \beta^+$.

**Proof.** We use some ideas from $\beta$-logic. Enlarge the language of set theory by adjoining (Henkin) constants $c_0, c_1, \ldots$ and a name $\beta'$ for each ordinal $\beta' \leq \beta$. 

Formulas in this language can be easily coded by ordinals less than $\beta$. Let $S$ consist of the following sentences in this language:

(a) Axioms for admissibility,
(b) $\beta'$ is an ordinal, $\beta_1, \beta_2 \in \beta_2$ (whenever $\beta_1 < \beta_2 \leq \beta$).

Then the tree $T$ consists of all sequences of sentences $\langle \phi_0, \phi_1, \ldots \rangle$ such that

(i) if $\phi_{2n} = \psi$ then $\phi_{2n+1} = \psi$ or $\psi$,
(ii) if $\phi_{2n} = \exists x \psi$ then $\phi_{2n+1} = (\psi(x))$ some $k$ or $\phi_{2n}$,
(iii) if $\phi_{2n+1} = \psi_1 \lor \psi_2$ then $\phi_{2n+1} = \psi_1$ or $\psi_2$ or $\phi_{2n}$,
(iv) if $\phi_{2n} = "c_k \in \beta"$ then $\phi_{2n+1} = "c_k = \beta"$ some $\beta' < \beta$ or $\phi_{2n}$,
(v) $\phi_1 \land \phi_3 \land \phi_5 \land \cdots$ is consistent with $S$.

Since $\beta > \omega$ condition (v) is $\beta$-recursive.

**CLAIM.** $R_k(T) = \beta^+$. 

**PROOF OF CLAIM.** As the inductive definition of $R_k$ can be carried out in $L_{\beta^+}$ it is clear that $R_k(T) \leq \beta^+$. By absoluteness we can assume that $\beta$ is countable.

As $S$ has a model where $\beta$ is standard, $R_k(\phi) = \infty$. Now suppose $R_k(T) = \gamma < \beta^+$. Let $\psi_0, \psi_1, \ldots$ be a listing of the sentences in this language. Define $\phi_0, \phi_1, \ldots$ by

\[ \phi_{2n} = \psi_n, \]
\[ \phi_{2n+1} = \text{least } \phi \text{ such that } \langle \phi_0, \ldots, \phi_{2n}, \phi \rangle \text{ has } R_k \geq \gamma. \]

As $\{ \eta \in T \mid R_k(\eta) \geq \gamma \} \in L_{\beta^+}$ the sequence $\langle \phi_0, \phi_1, \ldots \rangle \in L_{\beta^+}$. But $\{ \phi_{2n+1} \mid n \in \omega \}$ describes the complete Henkin theory of an end extension of $L_{\beta^+}$. This is a contradiction. Q.E.D.

We can now describe the structure $M$ to satisfy Theorem 1. Let $T$ be as in Lemma 2. Define $A_k, B_v$ for $v \in \bigcup_k A_k = A$ as before. Let $P_v$ be all finite subsets of $B_v$, for $v \in A$. Endow each $P_v$ with a distinct $0$, so that $v_1 \neq v_2 \Rightarrow P_{v_1} \cap P_{v_2} = \emptyset$. The universe of $M = |M| = \bigcup\{P_v \mid v \in A\}$. Introduce predicates for each $P_v$.

We now provide $P_v$ with an “affine” group structure; that is, a group structure without a distinguished identity. Note that $P_v$ is a group under the operation $\Delta$ of symmetric difference. For $w \in P_v$ let $S_{v,w} = \{ (w_1, w_2) \mid w_1 \Delta w_2 = w \}$.

Notice that with these relations, any automorphism of $P_v$ is determined by its action at a single argument.

Finally, we introduce functions connecting the different $P_v$‘s. If $v \ast (\alpha) \in A_n$ then $f_{v \ast (\alpha)}$ is defined by: $f_{v \ast (\alpha)}(w) = \{ \eta \mid 2n - 2 \mid \eta \in w \}$ for $w \in P_{v \ast (\alpha)}$; $f_{v \ast (\alpha)}(w) = w$ otherwise. Thus any automorphism of $P_{v \ast (\alpha)}$ has a unique extension to $P_v$ preserving the function $f_{v \ast (\alpha)}$.

Thus the desired structure is $M = \langle |M|, P_v, S_{v,w}, f_{v \ast (\alpha)} \rangle$, $v \in A$, $w \in P_v$. It remains to compute the Scott rank of $M$.

For any collection $G$ of partial functions from $M$ to $M$ define $G$-$R_k(g)$ for $g \in G$ by

\[ G$-$R_k(g) \geq 0 \iff g \in G; \]
\[ G$-$R_k(g) \geq \alpha + 1 \iff \forall m \in |M| \exists h \in G(g \subseteq h, m \in \text{Dom}(h), G$-$R_k(h) > \alpha) \text{ and } \forall m \in |M| \exists h \in G(g \subseteq h, m \in \text{Range}(h), G$-$R_k(h) > \alpha); \]
\[ G$-$R_k(g) \geq \lambda \iff \forall \alpha < \lambda G$-$R_k(g) \geq \alpha \text{ for limit } \lambda; \]
\[ G$-$R_k(g) = \infty \iff G$-$R_k(g) \geq \alpha \text{ for all } \alpha. \]

Also let $R_k(G) = \sup\{G$-$R_k(g) \mid g \in G, G$-$R_k(g) < \infty\}$. Thus we are interested in showing that $R_k(G_0) = \beta^+$ where $G_0$ is all finite partial isomorphisms of $M$.

For any $D \subseteq |M|$ let $D = \text{closure}(D) = \bigcup\{P_v \mid \text{ For some } v' \supseteq v, D \cap P_{v'} \neq \emptyset\}$. As remarked earlier any partial isomorphism of $M$ with domain $D$ has a unique
extension to a partial isomorphism with domain (and range) $D$. Thus it suffices to show that $\text{Rk}(G_1) = \beta^+$ where $G_1 = \{g \in G_0 \mid \text{Dom}(g) = \text{Dom}(g)\}$.

Now if $g \in G_1$, then $g$ is uniquely determined by $g^*$ which is defined by Domain $(g^*) = \{v \mid P_v \subseteq \text{Dom}(g)\}$, $g^*(v) = g(\cap v)$. Moreover, $g^*$ satisfies

\[(*) \quad f_{\nu + (\alpha)}(g^*(v \cdot (\alpha))) = g^*(v).\]

Conversely, any function $h$ with domain a finite $t \subseteq A$ closed under initial segments, obeying $(*)$ must be of the form $g^*$ for some $g$. Let $H = \{g^* \mid g \in G_1\}$. Then $\text{Rk}(G_1) = \text{Deg}(H)$ which is defined by

\[
\text{Deg}(h) > 0 + \text{h \in H};
\]

\[
\text{Deg}(h) > \alpha + 1 \iff \forall \nu \in A \exists h_1 \supseteq h(v \in \text{Dom}(h_1)), \text{Deg}(h_1) \geq \alpha;\]

\[
\text{Deg}(h) > \lambda \iff \forall \alpha < \lambda \text{Deg}(h) \geq \alpha \text{ for limit } \lambda;\]

\[
\text{Deg}(h) = \infty \iff \text{Deg}(h) \geq \alpha \text{ for all } \alpha, \text{Deg}(H) = \text{sup}\{\text{Deg}(h) \mid \text{Deg}(h) < \infty\}.
\]

Thus it suffices to show that $\text{Deg}(H) = \beta^+$.

Our final claim establishes the theorem by relating $\text{Deg}$ (defined on $H$) to $\text{Rk}$ (defined on $\eta \in T$, length$(\eta)$ even).

**Claim.** For $h \in H$, $\text{Deg}(H) = \text{min}\{\text{Rk}(\eta) \mid \eta \in h(v) \text{ for some } v\}$.

**Proof.** By induction on $\alpha$ we show that $\text{Deg}(h) \geq \alpha$ iff $\text{Rk}(h) \geq \alpha$ iff $\text{Rk}(\eta) \geq \alpha$ for all $\eta \in \bigcup \text{Range}(h)$. This is trivial for $\alpha = 0$ or for limit $\alpha$ (by induction).

Let $\alpha = \gamma + 1$. Suppose $\text{Rk}(\eta) \geq \gamma + 1$ for all $\eta \in \bigcup \text{Range}(h)$ and $v \in A$. We show that $\exists h_1 \supseteq h(v \in \text{Dom}(h_1))$ and $\text{Rk}(\eta) \geq \gamma$ for all $\eta \in \bigcup \text{Range}(h_1)$. Let $\nu_0 \subseteq v$ be maximal, $\nu_0 \in \text{Dom}(h)$. For each $\eta \in h(\nu_0)$ choose $\eta' \supseteq \eta$, $\eta' \in B_v$ so that $\text{Rk}(\eta') \geq \gamma$ (this is possible since $\text{Rk}(\eta) \geq \gamma + 1$). Then set $h_1(v') = h(v')$ for $v' \in \text{Dom}(h)$, $h_1(v \cdot k) = \{\eta' \cdot 2k \mid \eta \in h(\nu_0)\}$ for $k \leq \text{length}(v)$.

Conversely suppose $\text{Deg}(h) \geq \gamma + 1$, $\eta \in \bigcup \text{Range}(h)$. We show that for all $v \supseteq (\eta)_{\text{even}}$ there is $\eta' \supseteq \eta$ such that $\eta' \in B_v$, $\text{Rk}(\eta') \geq \gamma$. For, given $v \supseteq (\eta)_{\text{even}}$, let $h_1 \supseteq h, v \in \text{Dom}(h_1), \text{Deg}(h_1) \geq \gamma$. By induction, $\text{Rk}(\eta') \geq \gamma$ for all $\eta' \in h_1(v)$. But $\eta$ has an extension $\eta' \in h_1(v)$ as $h_1 \in H$. Q.E.D.

Finally as $\text{Rk}(T) = \beta^+$ we conclude $\text{Deg}(H) = \beta^+$ and hence the theorem.

2. $\omega_1$-recursive trees. We use here G"odel condensation methods to build an $\omega_1$-recursive tree $T$ of $L_{\omega_1 \omega}$-rank $\omega_1^+ = \text{least admissible ordinal greater than } \omega_1$. For simplicity assume $\omega_1 = \omega_1^\omega$. The general case follows from the fact that the proof given below can be easily adapted to any $L$-cardinal $\kappa$ such that $\kappa$ is regular in $L$, $\alpha = \text{least admissible greater than } \kappa$.

Let $S = \{\alpha < \omega_1 \mid \alpha \text{ admissible, } L_\alpha = \omega_1 \text{ exists and is the largest admissible}\}$. A typical member of $S$ is $\alpha$ where $L_\alpha$ is the transitive collapse of a countable elementary submodel of $L_{\omega_1^\omega}$.

We first define the tree $T' = \{(\alpha_0, \ldots, \alpha_n) \mid \text{For all } i, \alpha_i \in S, \alpha_i < \alpha_{i+1} \text{ and there exists } \Pi: L_{\alpha_i} \cong L_{\alpha_{i+1}}\}$. Note that $\Pi$ as above must be the identity on $\omega_1^{L_{\alpha_i}}$ and every element of $L_{\alpha_i}$ is definable over $L_{\alpha_i}$ from ordinals $\leq \omega_1^{L_{\alpha_i}}$. Thus if $\Pi$ exists in the definition of $T'$ then $\Pi^{-1}$ must be the transitive collapse of $H = \text{Skolem hull of } \omega_1^{L_{\alpha_i}}$ inside $L_{\alpha_{i+1}}$. This proves that $T'$ is $\omega_1$-recursive.

The desired tree $T$ is obtained via a minor modification of $T'$. This modification is needed to eliminate certain inhomogeneities on $T'$. Define $T = \{((\alpha_0, i_0), \ldots, (\alpha_n, i_n)) \mid \text{For all } k, \alpha_k \in S, i_k \in \omega, \alpha_k \leq \alpha_{k+1} \text{ and there exists } \Pi: L_{\alpha_k} \cong L_{\alpha_{k+1}}\}$. Thus an
ordinal $\alpha \in S$ can be “repeated” countably often.) As before $T$ is $\omega_1$-recursive. Our goal is to show that $T$ has $L_{\omega_1 \omega_1}$-rank $\omega_1^+$. (We shall in fact show that $T$ is isomorphic to the tree $T$ in §1 of Friedman [81].)

We begin by analyzing the structure of $T$. We show that the structure of $T$ below $((\alpha_0, i_0), \ldots, (\alpha_n, i_n))$ is determined by the $S$-rank $(\alpha_n)$. This is defined by

$S$-rk$(\alpha) \geq 0 \iff \alpha \in S$;
$S$-rk$(\alpha) \geq \gamma + 1 \iff \exists \alpha' \exists \Pi : L_\alpha \models L_\alpha'$, $S$-rk$(\alpha') \geq \gamma$;
$S$-rk$(\alpha) \geq \lambda \iff S$-rk$(\alpha) \geq \gamma$ for all $\gamma < \lambda$, for limit $\lambda$;
$S$-rk$(\alpha) = \infty \iff S$-rk$(\alpha) \geq \gamma$ for all $\gamma$.

Also set $\text{Rank}(S) = \sup \{ S$-rk$(\alpha) | \alpha \in S, S$-rk$(\alpha) < \infty \}$.

We can also define $\text{rk}((\alpha_0, i_0), \ldots, (\alpha_n, i_n)) = S$-rk$(\alpha_n)$, when $((\alpha_0, i_0), \ldots, (\alpha_n, i_n)) \in T$. Then a node on $T$ of $\text{rk} \in T$ has exactly $\omega$-many immediate extensions on $T$. A node on $T$ of $\text{rk} \gamma > 0$ has exactly $\omega$-many immediate extensions of $\text{rk} \gamma$ and $\omega_1$-many immediate extensions of $\text{rk} \delta$ for $\delta < \gamma$. A node on $T$ of $\text{rk} \infty$ has $\omega_1$-many immediate extensions of $\text{rk} \infty$.

Our main goal is to show that for each $\alpha_0 \in T$, $\text{rk} \alpha_0 = \infty$ or $\alpha_0 = \omega_1$. From this it follows that $L_{\omega_1 \omega_1}$-rank of $T$ is $\omega_1^+$. Note that the inductive definition of $\text{rk}$ as well as the inductive analysis of the $L_{\omega_1 \omega_1}$-rank of $T$ can be carried out in $L_{\omega_1^+}$. If $\sigma_0 \in T$, $\text{rk} \sigma_0 = \infty$ then $\sigma_0$ must have immediate extensions of $\text{rk} \gamma$ for each $\gamma < \omega_1^+$ as otherwise $\{ \sigma \in T | \sigma \geq \sigma_0 \text{ and } \text{rk} \sigma = \infty \} = \{ \sigma \in T | \sigma \geq \sigma_0 \text{ and } \text{rk} \sigma \geq \gamma \}$ for some $\gamma < \omega_1^+$ and this latter set is a member of $L_{\omega_1^+}$. Thus we can conclude that if two nodes on $T$ lie on the same level and have the same $\text{rk}$, they can be mapped to each other by an automorphism of $T$. Thus determining the $L_{\omega_1 \omega_1}$-type of nodes on $T$ is nothing more than determining their $\text{rk}$ and the level of $T$ on which they lie. If $L_{\omega_1 \omega_1}$-rank of $T$ is less than $\omega_1^+$ then $\{ \sigma \in T | \text{rk} \sigma = \infty \} = \{ \sigma \in T | \text{rk} \sigma \geq \gamma \}$ for some $\gamma < \omega_1^+$ and this latter set belongs to $L_{\omega_1^+}$. This contradicts our main claim.

CLAIM. $S$-rk$(\alpha) = \infty \iff \alpha < \omega_1$ and $\exists \Pi : L_\alpha \models L_{\omega_1^+}$.

From this claim it is clear that $\{ \sigma \in T | \sigma \geq \sigma_0 \text{ and } \text{rk} \sigma = \infty \} \notin L_{\omega_1^+}$ when $\text{rk} \sigma_0 = \infty$ or $\sigma_0 = \emptyset$, as otherwise $\{ \alpha < \omega_1 | \exists \Pi : L_\alpha \models L_{\omega_1^+} \} \in L_{\omega_1^+}$ which is impossible.

PROOF OF CLAIM. Clearly if $\alpha < \omega_1$ and $\exists \Pi : L_\alpha \models L_{\omega_1^+}$ then $S$-rk$(\alpha) = \infty$ as if $X$ is the set of all such $\alpha$’s then $X$ is uncountable and each element of $X$ can be elementarily embedded in all larger elements of $X$. For the converse suppose $\alpha \in S$, $S$-rk$(\alpha) = \infty$. Choose $\beta > \alpha$, $\exists \Pi : L_\alpha \models L_\beta \models L_{\omega_1^+}$. Now inductively define $L_\alpha \models L_\alpha_1 \models L_\alpha_2 \models \cdots$ and $L_\beta \models L_\beta_1 \models L_\beta_2 \models \cdots$ such that $S$-rk $\alpha_i = S$-rk $\beta_i = \infty$ for each $i$ and $\beta_i < \alpha_i < \beta_{i+1}$. (This is possible by the definition of $S$-rk.) If $\text{Direct Lim}(L_\alpha, i < \omega)$ is well-founded then it is isomorphic to some $L_{\alpha'}$. If $\text{Direct Lim}(L_\beta, i < \omega)$ is well-founded then it is isomorphic to some $L_{\beta'}$. But $\omega_1^{L_{\alpha'}} = \omega_1^{L_{\beta'}}$ so $\alpha' = \beta'$ since $\alpha', \beta' \in S$. We conclude that $\exists \Pi \alpha : L_\alpha \models L_{\alpha'}$, $\Pi \beta : L_\beta \models L_{\alpha'}$, so $\Pi \alpha \circ \Pi \beta : L_\alpha \models L_\beta$ (since $\Pi \alpha$, $\Pi \beta$ is just the inverse of the transitive collapse of the Skolem hull of $L_\alpha, L_\beta$ in $L_{\alpha'}$). So $\exists \Pi : L_\alpha \models L_{\omega_1^+}$.

It remains to justify the well-foundedness of the direct limits. This is provided by our final subclaim.

SUBCLAIM. Direct $\text{Lim}(L_\alpha, i < \omega)$ is well-founded if $L_\alpha \models L_\alpha_1 \models \cdots$ with $\alpha_1 < \alpha_2 < \cdots$ in $S$. 

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**PROOF.** Let \( M = \text{Direct Limit}(L_\alpha, \ | \ i < \omega) \) and we identify \( \text{sp}(M) = \text{standard part of } M \) with some \( L_\gamma \). Note that \( \omega^M_1 = \sup\{\omega^L_\alpha, \ | \ i < \omega\} < \gamma \). But \( \gamma \) is admissible as either \( L_\gamma = M \) or \( L_\gamma \) is the standard part of a model of \( KP \). As \( M \models \omega_1 \) is the largest admissible, we can conclude that \( \gamma = (\omega^M_1)^+ \).

Now suppose \( L_\gamma \not\models M \) and choose \( i \) and \( \Pi : L_\alpha \models M \) so that \( \text{Range}(\Pi) \not\models L_\gamma \). Let \( \lambda < \alpha \), be so that \( \Pi(\lambda) \not\models L_\gamma \). Then \( \omega_1^{L_\alpha} < \lambda \). \( L_\alpha, \models \omega_1 \) is the largest admissible, we may choose \( \eta \in T \) such that \( Rk(\eta) = \lambda \) where \( T \) is the \( \omega_1 \)-\( \alpha \)-recursive tree constructed in Lemma 2 (where \( \beta = \omega_1^L \)). Note that for arbitrary \( \eta' \in T \), \( Rk(\eta') < \infty \) if and only if player I has a winning strategy at position \( \eta' \) for the game described immediately before Lemma 2.

If \( T' = \text{tree obtained from Lemma 2 when } D = w_i w \) then \( \Pi(T) = T' \) and \( \Pi(\eta) = \eta \) has nonstandard \( Rk' \) (\( = Rk \) for \( T' \)). But then player II has a winning strategy in the \( T' \)-game. This easily yields a winning strategy for player II in the \( T \)-game, contradicting \( Rk(\eta) < \infty \) Q.E.D.

Thus we have established

**THEOREM 3.** \( T \) is an \( \omega_1 \)-recursive tree of \( L_{\infty \omega_1} \)-rank \( \omega_1^+ \).

An \( \omega_1 \)-recursive structure of Scott rank \( \omega_1^+ \) can now be obtained by considering \( T^\omega = \text{infinite direct product of } \omega \)-many copies of \( T \). For then the analysis of \( L_{\infty \omega_1} \)-rank for \( T \) reduces to the Scott analysis of \( T^\omega \).

We end with an observation concerning \( \Pi_1(L_{\omega_1}) \)-singletons. Assume \( V = L \). A function \( f : L_{\omega_1} \rightarrow L_{\omega_1} \) is an \( \Pi_1(L_{\omega_1}) \)-singleton if it is the unique solution to a \( \Pi_1(L_{\omega_1}) \) formula \( \phi(f) \) with a single variable for a total function. An \( \omega_1 \)-recursive tree with a unique branch of length \( \omega_1 \) yields a \( \Pi_1(L_{\omega_1}) \)-singleton. We will show that for any \( \beta < \sigma = \text{least stable } \omega_1 > \omega_1 \) there is an \( \omega_1 \)-recursive tree with a unique branch of length \( \omega_1 \), which is constructed in \( L \) past \( \beta \). Note that any \( \Pi_1(L_{\omega_1}) \)-singleton must be a member of \( L_\sigma \).

Note that \( L_\sigma = \Sigma_1 \text{ Skolem hull } (L_{\omega_1} \cup \{L_{\omega_1}\}) \). Thus we can choose a \( \Sigma_1 \) formula \( \phi(x, y, z) \) and \( p \in L_{\omega_1} \) such that \( \beta \) is the unique solution to \( \phi(x, \omega_1, p) \). Let \( \alpha \) be the least admissible such that \( \beta < \alpha, L_\alpha \models \phi(\beta, \omega_1, p) \) and \( \alpha^* = \Sigma_1 \text{ projection of } \alpha = \omega_1 \).

We describe now an \( \omega_1 \)-recursive tree \( T \) whose unique path \( f \) consists of an \( \omega_1 \)-sequence of elementary submodels of \( L_\alpha \). This will suffice as clearly \( f \not\in L_\beta \). \( S \) consists of all \( \bar{\alpha} < \omega_1 \) such that

(a) \( L_{\bar{\alpha}} \models KP + \omega \exists \), \( \alpha^* = \omega_1^L_{\bar{\alpha}} \);
(b) \( p \in L_{\bar{\alpha}} \) where \( \bar{\alpha} = \omega_1^L_{\bar{\alpha}}, L_{\bar{\alpha}} \models \phi(\bar{\beta}, \bar{\alpha}, p) \) for some \( \bar{\beta} < \bar{\alpha} \);
(c) \( L_{\bar{\alpha}} \models \text{There are no admissible } \beta < \bar{\alpha} \text{ s.t. } \delta^* = \omega_1 \).

Then the tree \( T = \{ \langle \bar{\alpha}_0, \bar{\alpha}_1, \ldots \rangle \in \omega^{< \omega_1} \ | \ \bar{\alpha}_\delta \in S \text{ for all } \delta, \bar{\alpha}_\delta = \text{greatest } \bar{\alpha} < \bar{\alpha}_{\delta + 1} \text{ s.t. } \exists \Pi : L_{\bar{\alpha}_{\delta + 1}} \models L_{\bar{\alpha}_\delta}, \omega_1^L_{\bar{\alpha}_\delta} = \bigcup\{ \omega_1^L_{\bar{\alpha}_\delta} \ | \ \delta < \lambda \}, \lambda \text{ limit, } \sim \exists \bar{\alpha}_0 \exists \bar{j} : L_{\bar{\alpha}_0} \models L_{\bar{\alpha}_0} \} \). It is not hard to check that II as above is uniquely determined as every element of \( L_{\bar{\alpha}} \) is definable in \( L_{\bar{\alpha}} \) from \( \bar{\beta} \) together with ordinals \( \leq \omega_1^{\bar{L}_{\alpha}} \), for \( \bar{\alpha} \in S \).

So \( T \) is \( \omega_1 \)-recursive.

Now define an \( \omega_1 \)-sequence of elementary submodels \( M_0 < M_1 < \cdots \) of \( L_\alpha \) by:

\( M_0 = \text{Skolem hull of } \{ p, \omega_1, \beta \} \text{ in } L_\alpha, \gamma_0 = M_0 \cap \omega_1; M_{\delta + 1} = \text{Skolem hull of } \gamma_\delta \cup \{ p, \omega_1, \beta \} \text{ inside } L_\alpha, \gamma_\delta + 1 = M_{\delta + 1} \cap \omega_1; M_\lambda = \bigcup\{ M_\delta \ | \ \delta < \lambda \}, \gamma_\lambda = \bigcup\{ \gamma_\delta \ | \ \delta < \lambda \} \) for limit \( \lambda \). Then \( \langle \bar{\alpha}_0, \bar{\alpha}_1, \ldots \rangle \) forms an \( \omega_1 \)-branch through \( T \) where \( \bar{\alpha}_\delta = \text{transitive collapse } (M_\delta) \).
If $f$ is an $\omega_1$-branch through $T$ then there are elementary embeddings $L_{f(0)} \to L_{f(1)} \to \cdots$ and we can form the direct limit $L_{\alpha'}$. Now $\alpha'$ must be the least $\mu$ such that $\mu$ is admissible, $\mu^* = \gamma'$, $\mu > \beta'$, $L_\mu \models \phi(\beta', \gamma', p)$ for some $\beta' < \alpha'$, $\gamma' = \omega_1^{\omega_1^{\omega_1}}$. But $\gamma' = \omega_1$. So $\beta' = \beta$ since $\beta$ is the unique solution to $\phi(x, \omega_1, p)$. It follows that $\alpha' = \alpha$ and hence $f(\delta) = \bar{\alpha}_\delta$ for all $\delta$. Thus $T$ has a unique $\omega_1$-branch.

We have shown that $\Pi_1(\omega_1)$-singletons are constructed in $L$ cofinally in the least stable ordinal $\sigma > \omega_1$. By way of contrast all $\Pi_1(\omega)$-singletons are constructed in $L$ before $\omega^+ = \omega_1^{\omega_1}$. The disparity here is due to the fact that well-foundedness is easily expressible over $L_{\omega_1}$.

**Final Note.** The second author has found a way to modify the construction in §1 to produce an $\omega_1$-recursive structure of $L_{\infty \omega_1}$-rank $\omega_1^+$. The key to the argument is in establishing the existence of an $\omega_1$-recursive tree of $\omega_1$-rank $\omega_1^+$, where $\omega_1$-Rk is defined in analogy to our earlier definition of Rk. Then the appropriate structure is obtained from such a tree much as the structure $M$ was obtained from $T$ in §1.

**References**


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