ESI Workshop

ESI = Erwin Schrödinger Institut, Vienna

ESI WORKSHOP ON LARGE CARDINALS AND DESCRIPTIVE SET THEORY
June 14–25, 2009

1st week: June 14–18 Emphasis on Large Cardinals
2nd week: June 21–25 Emphasis on Descriptive Set Theory

All are welcome; no registration fee

For further information:

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$\varphi$ is *internally consistent* iff $\varphi$ is true in some inner model.

Assumption: There are inner models of $V$ with large cardinals.

A new type of (relative) consistency result.

$\text{Con}(\text{ZFC} + \varphi) = \text{ZFC} + \varphi$ is consistent

$I\text{Con}(\text{ZFC} + \varphi) = \text{ZFC} + \varphi$ holds in some inner model.
Internal Consistency

Consistency result:
Con(ZFC + LC) $\rightarrow$ Con(ZFC + $\varphi$),
where LC is a large cardinal axiom

*Internal* consistency result:
$ICon(ZFC + LC) \rightarrow ICon(ZFC + \varphi)$
Internal consistency is stronger than consistency

Example

Con(ZFC) → Con(ZFC + GCH fails at all regular cardinals)
(Force with an Easton product over $L$)

$ICon(ZFC + 0\# \text{ exists}) →$
$ICon(ZFC + GCH fails at all regular cardinals)$
(Force with a reverse Easton iteration over $L$, build a generic using the Silver indiscernibles)

Proving Internal Consistency *demands new techniques*
Two types of internal consistency results:

Type 1. \( \lnot \text{Con}(\text{ZFC} + 0^\# \text{ exists}) \rightarrow \lnot \text{Con}(\text{ZFC} + \varphi) \)

Build generics by cohering partial generics along Silver indiscernibles.

Two techniques:
Easier cases: \textit{Generic modification} (F-Ondrejović)
Harder cases: \textit{Partial master conditions} (F-Thompson)

Key to Type 2 results: Show that the relevant forcings preserve measurability
2 Types of Internal Consistency Results

Type 2. First show:

(*) \( \text{ZFC + LC} \rightarrow \) In some set-forcing extension, \( \varphi \) holds in \( V_\kappa \) for some measurable \( \kappa \)

Then we have:

\[ \text{ICon}(\text{ZFC + LC}) \rightarrow \text{ICon}(\text{ZFC + } \varphi \text{ holds in } V_\kappa, \kappa \text{ measurable}) \]
(Force with a countable p.o. over an inner model)

and also:

\[ \text{ICon}(\text{ZFC + } \varphi \text{ holds in } V_\kappa, \kappa \text{ measurable}) \rightarrow \text{ICon}(\text{ZFC + } \varphi) \]
(Iterate the measure to \( \infty \))

So we conclude:

\[ \text{ICon}(\text{ZFC + LC}) \rightarrow \text{ICon}(\text{ZFC + } \varphi) \]
2 Types of Internal Consistency Results

How do we show (∗)?

(∗) ZFC + LC → In some set-forcing extension, φ holds in $V_\kappa$ for some measurable $\kappa$

In easier cases: Master conditions (Silver), Partial master conditions (Magidor, F-Honzík) or Generic modification (Woodin)
In harder cases: $\kappa$-Tree forcings (F-Thompson for $\kappa$-Sacks products, Dobrinen-F for $\kappa$-Sacks iterations, F-Zdomskyy for $\kappa$-Miller iterations)
Examples of Internal Consistency

Some Internal Consistency Results

Cardinal Exponentiation: F-Ondrejović, F-Honzík

Costationarity of the Ground Model: Dobrinen-F

Global Domination: F-Thompson

Tree Property: Dobrinen-F

Embedding Complexity: F-Thompson

Cofinality of the Symmetric Group: F-Zdomskyy
**Internal Consistency: Cardinal Exponentiation**

*Cardinal Exponentiation*

Easton function:
\[ F : \text{Reg} \rightarrow \text{Card}, \ F \text{ nondecreasing}, \ \text{cof}(F(\kappa)) > \kappa \text{ for all } \kappa \in \text{Reg} \]

Easton: \( F \) a provably definable Easton function. Then
\[ \text{Con}(\text{ZFC}) \rightarrow \text{Con}(\text{ZFC} + 2^\kappa = F(\kappa) \text{ for all regular } \kappa) \]

Easton used an Easton product

This gives *no* internal consistency result
Internal Consistency: Cardinal Exponentiation

F-Ondrejović: Instead use Easton iteration of Easton products and *generic modification*

**Theorem**

*F a provably definable Easton function. Then*

$ICon(ZFC + 0\# \text{ exists}) \rightarrow ICon(ZFC + 2^\kappa = F(\kappa) \text{ for all regular } \kappa)*

Type 2 result (F-Honzík): *F a provably definable Easton function, \( \kappa \) is \( H(F(\kappa)) \)-hypermeasurable witnessed by \( j \) with \( j(F)(\kappa) \geq F(\kappa) \). Then in a set-generic extension, \( \kappa \) is measurable and \( F \) is realised below \( \kappa \) (in fact, everywhere). Sample corollary:*

**Theorem**

$ICon(ZFC + \text{ There is a } P_{2\kappa} \text{ hypermeasurable}) \rightarrow ICon(ZFC + 2^\kappa = \kappa^{++} \text{ for all regular } \kappa + \text{ There is a proper class of Ramsey cardinals})$
Internal Consistency: Global Domination

**Global Domination**

\( \kappa \) an infinite regular cardinal
Suppose \( f, g : \kappa \to \kappa \)
\( f \) dominates \( g \) iff \( f(\alpha) > g(\alpha) \) for sufficiently large \( \alpha < \kappa \)
\( \mathcal{F} \) is a dominating family iff
every \( g : \kappa \to \kappa \) is dominated by some \( f \) in \( \mathcal{F} \)
\( d(\kappa) = \) the smallest cardinality of a dominating family

Fact: \( \kappa < d(\kappa) \leq 2^\kappa \) for all infinite regular \( \kappa \)

**Global Domination:** \( d(\kappa) < 2^\kappa \) for all infinite regular \( \kappa \)
Internal Consistency: Global Domination

Cummings-Shelah: Global Domination is consistent
Proof uses $\kappa$-Cohen and $\kappa$-Hechler forcings
Corollary to their proof:
$I\text{Con}(\text{ZFC} + \text{a supercompact cardinal}) \rightarrow I\text{Con}(\text{ZFC} + \text{Global Domination})$

F-Thompson: Instead use $\kappa$-Sacks product (and tuning forks)

**Theorem**
$I\text{Con}(\text{ZFC} + 0^\# \text{ exists}) \rightarrow I\text{Con}(\text{ZFC} + \text{Global Domination})$

**Theorem**
If $\kappa$ is $P_{2\kappa}$ hypermeasurable then in a set-generic extension, $\kappa$ is measurable and global domination holds below $\kappa$ (in fact, everywhere).
Internal Consistency: Global Domination

In the previous two theorems, \((d(\kappa), 2^\kappa) = (\kappa^+, \kappa^{++})\)

What about other possibilities for \((d(\kappa), 2^\kappa)\)?

*Global Domination Pair* \((d, F)\): For regular \(\kappa\), \(\kappa < d(\kappa) \leq F(\kappa)\), \(d(\kappa)\) regular, \(F\) an Easton function

Realising arbitrary global domination pairs seem to require very large cardinals:

*Definition.* \(\infty\) is *super-Woodin* iff for any class \(A \subseteq \text{Ord}\) there is \(\kappa\) such that for any \(\lambda\), some \(j\) witnessing that \(\kappa\) is \(\lambda\)-supercompact satisfies \(j(A) \cap \lambda = A \cap \lambda\)

(Follows from a stationary class of almost-huge cardinals)
Internal Consistency

Theorem

Assume GCH and $\infty$ super-Woodin. Then if $\varphi$ defines a global domination pair there are inner models $W_0 \subseteq W_1$ such that $\varphi$ defines in $W_0$ the global domination pair realised in $W_1$.

Uses $\kappa$-Cohen and $\kappa$-Hechler forcing, as in Cummings-Shelah, but preserving the measurability of $\kappa$. 
Internal Consistency: The Tree Property

*The Tree Property*

\( \kappa \) regular

A \( \kappa \)-Aronszajn tree is a tree of height \( \kappa \) with no \( \kappa \)-branch

\( \kappa \) has the tree property iff there is no \( \kappa \)-Aronszajn tree

Mitchell: \( \text{Con}(\text{ZFC} + \text{Proper class of weakly compact cardinals}) \rightarrow \text{Con}(\text{ZFC} + \alpha^{++} \text{ has the tree property for all inaccessible } \alpha) \)

Proof uses "Mitchell forcing"

Corollary to proof:

\( \text{ICon}(\text{ZFC} + \text{a supercompact cardinal}) \rightarrow \text{ICon}(\text{ZFC} + \alpha^{++} \text{ has the tree property for all inaccessible } \alpha) \)
Dobrinen-F: Instead use iterated $\kappa$-Sacks forcing

**Theorem**

$ICon(\text{ZFC} + \exists \# \exists) \rightarrow ICon(\text{ZFC} + \exists \# \exists)$

$ICon(\text{ZFC} + \alpha^++ \text{ has the tree property for all inaccessible } \alpha)$

**Theorem**

If $\kappa$ is weakly compact hypermeasurable then in a set-generic extension, $\kappa$ remains measurable and the tree property holds at $\kappa^++$.

**Theorem**

$ICon(\text{ZFC} + \exists \# \exists \text{ There is a weakly compact hypermeasurable}) \rightarrow ICon(\text{ZFC} + \exists \# \exists \text{ The tree property holds at } \alpha^++ \text{ for inaccessible } \alpha \text{ and there is a proper class of Ramsey cardinals})$
A related consistency result:

Foreman:
Con(\text{ZFC} + \text{supercompact} + \text{a larger weak compact}) \rightarrow
Con(\text{ZFC} + \text{Tree Property at } \lambda^{++} \text{ for a singular } \lambda)

Theorem
(F-Halilović-Magidor) Con(\text{ZFC} + \kappa \text{ weakly compact hypermeasurable}) \rightarrow Con(\text{Tree property at } \aleph_{\omega+2})
Internal Consistency: Embedding Complexity

Embedding Complexity

\( \alpha \leq \kappa \) infinite and regular

\( G(\alpha, \kappa) = \) Set of graphs of size \( \kappa \) which omit \( \alpha \)-cliques

Embedding complexity of \( G(\alpha, \kappa) = ECG(\alpha, \kappa) \):
Smallest size of a \( U \subseteq G(\alpha, \kappa) \) such that every graph in \( G(\alpha, \kappa) \) embeds into some element of \( U \) (as a subgraph)

What are the possibilities for \( ECG(\alpha, \kappa) \) as a function of \( \alpha \) and \( \kappa \)?
Internal Consistency: Embedding Complexity

Complexity triple \((a, c, F)\):
- \(a, c, F : \text{Reg} \rightarrow \text{Card}\)
- F is an Easton function
- \(a(\kappa) \leq \kappa < c(\kappa) \leq F(\kappa)\) for all \(\kappa\)

**Theorem**

\((Džamonja-F-Thompson)\) Suppose that \((a, c, F)\) is a provably definable complexity triple. Then \(\text{Con}(\text{ZFC}) \rightarrow \text{Con}(\text{ZFC} + \text{ECG}(a(\kappa), \kappa) = c(\kappa)\) and \(2^\kappa = F(\kappa)\) for all \(\kappa \in \text{Reg}\)
Theorem

(F-Thompson) Suppose that \((a, c, F)\) is a provably definable complexity triple. Then \(\text{ICon}(\text{ZFC} + 0^\# \text{ exists}) \rightarrow \text{ICon}(\text{ZFC} + \text{ECG}(a(\kappa), \kappa) = c(\kappa) \text{ and } 2^\kappa = F(\kappa) \text{ for all } \kappa \in \text{Reg})\)

The generic is built using *partial master conditions*

Consistency with measurability: Looks difficult. Need a “tree-like” forcing to control embedding complexity
Cofinality of the Symmetric Group

\( \kappa \) regular.

\( \text{Sym}(\kappa) = \text{the symmetric group on } \kappa \)

\( \text{cof}(\text{Sym}(\kappa)) = \text{the length of the shortest chain of proper subgroups of } \text{Sym}(\kappa) \text{ whose union is all of } \text{Sym}(\kappa) \)
Internal Consistency: Cofinality of the Symmetric Group

**Theorem**

(F-Zdomskyy) $\text{Con}(\text{ZFC } + \kappa \text{ is } P_{2\kappa} \text{ hypermeasurable}) \rightarrow \text{Con}(\text{ZFC } + \text{cof}(\text{Sym}(\kappa))) = \kappa^{++} \text{ for a measurable } \kappa$.

Uses an iteration of a special version of $\kappa$-Miller forcing.

**Theorem**

(F-Zdomskyy) $\text{ICon}(\text{ZFC } + 0\# \text{ exists}) \rightarrow \text{ICon}(\text{ZFC } + \text{cof}(\text{Sym}(\alpha))) = \alpha^{++} \text{ for all inaccessible } \alpha$.

Uses the *partial master conditions* of F-Thompson.
Internal Consistency: Open Problems

Type 1 results

What global patterns can be realised in inner models of $L[0^\#]$ for the following characteristics?
Easton functions with parameters
Dominating pairs $(d, F)$
Sym$(\kappa)$
Tree Property $(\kappa)$
Stationary reflection at $\kappa$
$\square_\kappa$

Type 2 results

General open problem: How can one preserve measurability with iteration of “$\kappa$-Cohen like” forcings? Is there a general method for converting these into “$\kappa$-Tree like” forcings?