The Appeal of $0^\#$

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Set-theorists have had great success in solving problems under the hypothesis $V = L$. Under this assumption, Gödel proved the Generalised Continuum Hypothesis and also precisely determined the behaviour of projective sets of reals in $L$, with regard to regularity properties such as Lebesgue measurability and the perfect set property. Further important work on $L$ was accomplished by Jensen.

But the fact remains that $V = L$ is not a theorem of ZFC: The forcing method allows us to consistently enlarge $L$ to models $L[G]$ where $G$ is a set or class that is generic over $L$ with respect to some forcing notion $P$. Thus it is reasonable to suggest that $V$, rather than equal to $L$, should be in fact a generic extension $L[G]$ of $L$. But this then gives rise to the new and difficult question: Which generic extension is it? Usually, a forcing notion $P$ gives rise not to one, but to many different generics $G$. Some hope is provided by:

**Theorem 1** Assume some weak large cardinal axioms, consistent with $V = L$ (precisely: an $n$-ineffable cardinal for each $n$). Then there is an $L$-definable forcing notion $P$ with a unique generic.

(Unless otherwise stated, we take “definable” to mean “definable without parameters”.) But this does not solve our problem, for $P$ is not the only forcing notion satisfying this Theorem and there does not seem to be a canonical choice for $P$. Moreover, the hypothesis that generics exist for all definable forcing notions is inconsistent:

**Theorem 2** There exist forcing notions $P_0$, $P_1$ which are definable over $L$ and which preserve ZFC, such that there cannot be generics for $P_0$ and $P_1$ simultaneously.

So if we want $V$ to not be $L$ we must decide for which forcing notions $P$ to allow generics. The needed criterion arises naturally through the consideration of **CUB-absoluteness**: 

**Definition**. A class $C$ of ordinals is CUB iff it is closed and unbounded. $V$ is **CUB-absolute over $L$** iff every $L$-definable class of ordinals which has a CUB subclass definable with parameters in a generic extension of $V$ has one definable with parameters in $V$.  

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Theorem 3 $V$ is CUB-absolute over $L$ iff $0^\#$ exists.

$0^\#$ is a special set of integers discovered by Silver and Solovay, whose existence is a “transcendence principle for $L$” in the sense that it implies that $V$ is not a generic extension of $L$. The existence of $0^\#$ is equivalent to the existence of a nontrivial elementary embedding of $L$ into itself. If $0^\#$ exists then there is a smallest inner model which satisfies “$0^\#$ exists”, namely the canonical model $L[0^\#]$.

Thus as an alternative to $V = L$ we could propose the hypothesis: $0^\#$ exists and $V = L[0^\#]$. This allows for the existence of generic extensions of $L$. Moreover we can now provide a generic existence criterion for $L$-definable forcings, by declaring an $L$-definable forcing to have a generic iff it has one definable in $L[0^\#]$.

An open problem is to provide a more convincing characterisation of $0^\#$ in terms of forcing. One possibility is suggested by the following

Theorem 4 Assume a weak large cardinal axiom, consistent with $V = L$ (precisely: an $\omega + \omega$-Erdős cardinal). If $0^\#$ exists and an $L$-definable forcing has a generic, then it has one definable in $L[0^\#]$.

Thus $L[0^\#]$ is “saturated” with respect to $L$-definable forcings. A nice result would be the converse to this, giving that $0^\#$ exists iff $V$ is saturated with respect to $L$-definable forcings.

A second possibility would be to define a new concept of forcing and prove that the existence of $0^\#$ is equivalent to the statement that $V$ is not “generic” over $L$ in this new sense. One cannot simply use the usual notion of class forcing for this purpose; indeed there exist reals $R$ in $L[0^\#]$ which are not class-generic over $L$ and from which $0^\#$ is not constructible.

Cardinal-Preserving Extensions

By not assuming $V = L$ we can compare set-theoretic problems according to their degree of nonconstructibility. We shall now examine a wide class of such problems, under the assumption that $0^\#$ exists.

Definition A subset $X$ of $L$ is $\Sigma^C_1$ iff $X$ can be written in the form

$a \in X$ iff $\varphi(a)$ holds in a cardinal-preserving extension of $L$
for some $\Sigma_1$ formula $\varphi$. (We intend our cardinal-preserving extensions of $L$ to satisfy AC and to be contained in a set-generic extension of $V$.)

**Example:** A classic result of Baumgartner-Harrington-Kleinberg [1] implies that assuming CH a stationary subset of $\omega_1$ has a CUB subset in a cardinal-preserving set-generic extension of $V$. This implies that the set

$$\{X \in L \mid X \subseteq \omega_1^L \text{ and } X \text{ has a CUB subset in a cardinal-preserving extension of } L\}$$

is constructible, as it equals the set of constructible subsets of $\omega_1^L$ which in $L$ are stationary.

Is there a similar such result for subsets of $\omega_2^L$? Building on work of M. Stanley [9], we show that there is not. We shall also consider a number of related problems, examining the extent to which they are “solvable” in the above sense, as well as defining a notion of reduction between them. We assume throughout that $0^\#$ exists.

**Theorem 5** If $X$ is $\Sigma_1^{CP}$ then $X$ is constructible from $0^\#$.

**Theorem 6** $0^\#$ is $\Sigma_1^{CP}$. And there are $\Sigma_1^{CP}$ sets of constructibility degree strictly between 0 and $0^\#$.

**Theorem 7** The following $\Sigma_1^{CP}$ sets are equiconstructible with $0^\#$:

1. $\{T \mid T \in L \text{ and } T \text{ is a tree on } \kappa \text{ of height } \kappa \text{ with a cofinal branch in a cardinal-preserving extension of } L\}$, for $\kappa$ an uncountable successor cardinal of $L$.
2. $\{X \subseteq \kappa \mid X \in L \text{ and } X \text{ contains a CUB subset in a cardinal-preserving extension of } L\}$, for $\kappa$ regular in $L$, $\kappa > \omega_1^L$.
3. $\{X \subseteq \kappa \mid X \in L \text{ and } X \text{ is the set of ordinals } < \kappa \text{ which are admissible relative to some real in a cardinal-preserving extension of } L\}$, for $\kappa$ uncountable in $L$.
4. $\{X \subseteq \kappa \mid X \in L \text{ and } X \text{ is the intersection with } \kappa \text{ of a class which is } \Delta_1\text{-definable over } L[R] \text{ without parameters, for some real } R \text{ in a cardinal-preserving extension of } L\}$, where $\kappa$ is at least $\omega_3^L$.

Theorem 7 is proved by “reducing” $0^\#$ to the sets mentioned. In fact we shall need the following more general notion of “reduction”.
Definition Suppose that \((X_0, X_1)\) and \((Y_0, Y_1)\) are pairs of disjoint subsets of \(L\). Then we write
\[(X_0, X_1) \longrightarrow_L (Y_0, Y_1)\]
iff there is a function \(F\) in \(L\) such that
\[a \in X_0 \rightarrow F(a) \in Y_0\]
\[a \in X_1 \rightarrow F(a) \in Y_1.\]

We write \(X\) instead of \((X_0, X_1)\) in case \(X = X_0\) is the complement (within some constructible set) of \(X_1\), and similarly for the \(Y\)'s. It is clear that if \((X_0, X_1)\) is nonconstructible and \((X_0, X_1) \longrightarrow_L (Y_0, Y_1)\), then \((Y_0, Y_1)\) is also nonconstructible. In the proof of Theorem 7 we shall obtain reductions in this sense of \(0^\#\) to the sets mentioned.

Theorem 7 suggests that the Baumgartner-Harrington-Kleinberg result should be viewed as a rare example of a nontrivial “solvable” \(\Sigma_1^{CP}\) problem. However it is not the only such example:

**Theorem 8** If \(\kappa\) is \(\omega^L_k\) in the set described in Theorem 7 (d), then the resulting set is constructible.

About the Proofs of Theorems 5-8

To prove Theorem 5, one shows the following: If \(\varphi\) is a \(\Sigma_1\) formula, \(a\) is a constructible set and \(\varphi(a)\) is true in a cardinal-preserving extension of \(L\), then this cardinal-preserving extension of \(L\) can be chosen as a set-generic extension of \(L[0^\#]\). The proof is based on ideas used to prove the Martin-Solovay Basis Theorem.

Theorem 6 is proved using the techniques used to prove the \(\Pi_1^1\)-singleton conjecture (see [3]).

The proof of Theorem 7 is based heavily on the notion of reduction introduced above: The first step is to reduce \(0^\#\) to the tree problem. Let \(T(\kappa)\) denote the set of constructible trees on \(\kappa\) of height \(\kappa\) with a cofinal, cardinal-preserving branch; i.e., a cofinal branch \(b\) such that \(L\) and \(L[b]\) have the same cardinals. To each \(n\) we associate a tree \(T_n\) on \(\kappa\) (\(\kappa\) an uncountable successor \(L\)-cardinal) in such a way that \(n\) belongs to \(0^\#\) iff \(T_n\) has a cofinal,
cardinal-preserving branch. Moreover, the sequence of trees $T_n$, $n \in \omega$ is constructible, so this proves $0^\# \longrightarrow T(\kappa)$ for uncountable successor $L$-cardinals $\kappa$.

To reduce $0^\#$ to the CUB subset problem for $L$-regular cardinals greater than $\omega^L_2$ it would suffice to reduce the tree problem $T(\kappa)$ to the CUB subset problem $C(\kappa)$. However we do not know how to do this. Instead we work with a modified version of the tree problem. Define

$$T^*(\kappa^+) = \{ T \in T(\kappa^+) \mid T \text{ is } \Delta_1 \text{-definable over } L_{\kappa^+} \text{ from the parameter } \kappa $$

and $T$ has a cardinal-preserving, stationary, $\mathcal{P}(\kappa)$-preserving cofinal branch $\}$

$$T^{**}(\kappa^+) = \{ T \in T(\kappa^+) \mid T \text{ is } \Delta_1 \text{-definable over } L_{\kappa^+} \text{ from the parameter } \kappa $$

and $T$ has a cardinal-preserving cofinal branch $\}$

where $b$ is stationary if in $L[b]$ its intersection with cof $\kappa$ is stationary, and $b$ is $\mathcal{P}(\kappa)$-preserving if $L$ and $L[b]$ have the same subsets of $\kappa$. We show: $0^\# \longrightarrow_L (T^*(\omega^L_2), \sim T^{**}(\omega^L_2))$, and $(T^*(\omega^L_2), \sim T^{**}(\omega^L_2)) \longrightarrow_L C(\omega^L_2)$. Finally, using the combinatorial principle $\Box$, which is true in $L$, we show that for any $L$-regular cardinal greater than $\omega^L_2$, $C(\omega^L_2) \longrightarrow_L C(\kappa)$.

A similar modification of the CUB subset problem is then reduced both to the admissibility spectrum problem at uncountable $L$-cardinals, and to the $\Delta_1$-definability problem at $L$-cardinals greater than $\omega^L_2$.

Theorem 8 is proved using a special type of coding construction.

**Open Questions**

Very simple questions concerning the notion of reduction $\longrightarrow_L$ remain unanswered. For example, is the tree problem at $\omega^L_2$ reducible to the CUB subset problem at $\omega^L_2$? Explicitly:

Is there a constructible function that associates to each constructible tree $T$ on $\omega^L_2$ a subset $X$ of $\omega^L_2$ such that $T$ has a cofinal, cardinal-preserving branch iff $X$ has a cardinal-preserving CUB subset?

If so, the indirect arguments sketched could be avoided. Is the CUB subset problem reducible to the problem of finding a cardinal-preserving homogenous set for a given partition? One can easily formulate a host of similar such open problems.
Other open questions concern a weakening of cardinal-preservation. An example concerns the CUB subset problem: Is the following set constructible?

\[ C'(\omega^L_3) = \{ X \in L \mid X \subseteq \omega^L_3 \text{ and } X \text{ has a CUB subset in an extension of } L \text{ which preserves } \omega^L_1, \omega^L_3 \} \]

It is possible that the solution to this problem will require genuine use of a gap 2 morass.

**Literatur**


