The Higher Descriptive Set Theory of Isomorphism 2010

1.-2. Vorlesungen

Introduction

These lectures relate two approaches to classifying countable first-order theories.

The “standard” way of doing this is through Shelah’s stability theory, which divides theories into the classifiable ones, for which a structure theory exists for the uncountable models, and the unclassifiable ones, for which chaos takes over. Roughly speaking, the classifiable theories are the ones for which there are fewer than the maximum number $2^\kappa$ of models of size $\kappa$ for some uncountable $\kappa$. (This is not quite true; the classifiable deep theories still have the maximum number of models in all uncountable cardinals.) Shelah’s “Main Gap” analysis shows that classifiability is equivalent to a conjunction of very absolute model-theoretic conditions: superstable without the DOP (dimensional order property) and without the OTOP (omitting types order property).

An alternative classification arises from Higher Descriptive Set Theory. Assume GCH and suppose that $\kappa$ is uncountable and regular. Then the Baire space $\omega^\omega$ generalises nicely to $\kappa^\kappa$, where points are functions $f : \kappa \to \kappa$ and basic open sets are of the form $N_p = \{f \mid p \subseteq f\}$ for some $p$ in $\kappa^{<\kappa}$. The (generalised) Borel sets are obtained from the basic open sets by closing under unions of size $\kappa$ and complements. A function $f : X_0 \to X_1$ between Borel sets is Borel iff the pre-image under $f$ of each Borel set is Borel. As in the classical case, one can also introduce a Borel-reducibility $\leq_B$ between equivalence relations $E_0, E_1$ on $\kappa^\kappa$:

$$E_0 \leq_B E_1 \text{ iff for some Borel } f, x E_0 y \iff f(x) E_1 f(y) \text{ for all } x, y$$

For a first-order theory $T$ let $\text{Mod}_T^\kappa$ denote the set of models of $T$ with universe $\kappa$; this can be viewed as a Borel set. Also let $\text{Isom}_T^\kappa$ denote the isomorphism relation on $\text{Mod}_T^\kappa$; this is a $\Sigma^1_1$ equivalence relation. Our second way of classifying theories $T$ is to look at the complexity of the equivalence relation $\text{Isom}_T^\kappa$ under Borel reducibility $\leq_B$. 

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Our main goal is to establish the following result, which relates the model-theoretic and set-theoretic methods of classification to each other. Missing definitions will be supplied later.

**Theorem 1 (F-Hyttinen-Kulikov)** Assume GCH and let $\kappa$ be uncountable and regular.

1. $T$ is classifiable and shallow (in Shelah’s sense) iff Isom$^\kappa_T$ is Borel.
2. In $L$, $T$ is classifiable iff Isom$^\kappa_T$ is $\Delta^1_1$.
3. $T$ is classifiable iff for all regular $\lambda < \kappa$, $E^\kappa_\lambda \not\leq_B$ Isom$^\kappa_T$, where $E^\kappa_\lambda$ is the equivalence relation of equality modulo $\lambda$-nonstationarity.

It is important that $\kappa$ be uncountable in this theorem. For example, the theory DLO (dense linear orders without endpoints) is unclassifiable, but the equivalence relation Isom$^\omega_T$ has only one equivalence class and therefore is trivially Borel! Martin Koerwien found an example in the other direction: There is a theory which is very nice model-theoretically ($\omega$-stable of “depth” 2, and therefore very classifiable) such that Isom$^\omega_T$ is not Borel. So from now on, $\kappa$ is always assumed to denote an uncountable regular cardinal.

**Higher Descriptive Set Theory**

Before bringing in the model theory, we’ll first take a look at descriptive set theory on the generalised Baire space $\kappa^\kappa$ for an uncountable regular cardinal $\kappa$. There are strong differences between this theory and classical descriptive set theory on the classical Baire space $\omega^\omega$. Also a number of simple questions are undecidable in ZFC and there remain many open problems. We assume GCH throughout.

We have defined the basic open sets in $\kappa^\kappa$ to be the sets of the form $N_p = \{f \in \kappa^\kappa \mid p \subseteq f\}$, where $p \in \kappa^{<\kappa}$. The Borel sets are obtained by closing the class of basic open sets under complements and unions of size $\kappa$.

A nice way to think of Borel sets is as follows: By a tree we mean a partial order in which the predecessors of any point are wellordered. Let $T$ be a tree of size $\kappa$ which is wellfounded (i.e., has no infinite branch) and let $l$ be a function which assigns a basic open set to each terminal node of $T$. Now for any $x \in \kappa^\kappa$ consider the game $G(T, l, x)$ with two players I and II, in which player I begins by choosing a minimal node of $T$ and then the players take turns choosing immediate successors of the nodes previously chosen until a
terminal node $s$ is reached. Then Player $II$ wins iff $x$ belongs to the basic open set $l(s)$. A set $B$ is Borel iff for some wellfounded tree $T$ of size $\kappa$, we have: $x \in B$ iff $II$ has a winning strategy in the game $G(T, l, x)$. Indeed, if $B$ is of this form then $B$ is Borel, as is shown by induction on the rank of the tree $T$. Conversely, the collection of sets $B$ of this form contains all basic open sets, is closed under complementation (add a new bottom node, replace each basic open set by its complement, an open set, and switch players) and is closed under intersections of size $\kappa$.

The product spaces $(\kappa^n)$ are defined in the usual way, using the corresponding product topologies. The (“boldface”) $\Sigma^1_1$ sets are the projections of closed sets, i.e., the sets of the form $\{f \in \kappa^n \mid (f, g) \in C \text{ for some } g\}$ where $C$ is a closed subset of $(\kappa^n)^2$. A set is $\Pi^1_1$ iff its complement is $\Sigma^1_1$ and is $\Delta^1_1$ iff it is both $\Sigma^1_1$ and $\Pi^1_1$.

**Proposition 2** Borel sets are $\Delta^1_1$, but not conversely.

*Proof.* If $B$ is Borel then for some wellfounded tree $T$ of size $\kappa$ and labelling $l$ of its terminal nodes by basic open sets, $x$ belongs to $B$ iff there exists a winning strategy for Player $II$ in the game $G(T, l, x)$. It is not difficult to write the latter condition on $x$ as the projection of a closed set. This shows that Borel sets are $\Sigma^1_1$ and therefore also $\Delta^1_1$.

The converse fails because there is a $\Delta^1_1$ set $D \subseteq (\kappa^n)^2$ which is *universal* for the Borel subsets of $\kappa^n$, i.e., the Borel sets are exactly the sets of the form

$$D_\xi = \{\eta \mid (\eta, \xi) \in D\}$$

for $\xi$ ranging over $\kappa^n$. Given such a universal set, we obtain a $\Delta^1_1$ non-Borel set $D^0$ through diagonalisation: $D^0 = \{\eta \mid (\eta, \eta) \notin D\}$.

To obtain $D$, we choose a coding $((T_\xi, l_\xi) \mid \xi \in C)$, where $C \subseteq \kappa^n$ is a Borel set of “codes”, of all pairs $(T, l)$ where $T$ is a wellfounded tree of size $\kappa$ and $l$ is a labelling of its terminal nodes by basic open sets. Without the requirement of wellfoundedness this is easy to do, even for $\kappa = \omega$. The fact that $\kappa$ is uncountable is needed to know that the set of $\xi \in \kappa^n$ which code a *wellfoundal* binary relation on $\kappa$ forms the set of branches of length $\kappa$ through a subtree of $\kappa^n$ and is therefore a closed set. Then we define:
\((\eta, \xi) \in D \text{ iff} \)
\[ \eta \in C \text{ and } \eta \text{ belongs to the Borel set determined by the pair } (T_\xi, l_\xi) \text{ iff} \]
\[ \xi \in C \text{ and Player II has a winning strategy in } G(T_\xi, l_\xi, \eta) \text{ iff} \]
\[ \xi \in C \text{ and Player I does not have a winning strategy in } G(T_\xi, l_\xi, \eta). \]

As the set of (codes for) winning strategies is a closed set, \(D\) is the desired \(\Delta^1_1\) set which is universal for Borel sets. □

**Proposition 3** For \(A \subseteq \kappa^\kappa\) the following are equivalent:
1. \(A\) is \(\Sigma^1_1\), i.e. the projection of a closed set in \((\kappa^\kappa)^2\).
2. \(A\) is the projection of a Borel set.
3. \(A\) is the projection of a \(\Sigma^1_1\) set.
4. \(A\) is the continuous image of a closed set.

*Proof.* As \(\text{Closed} \subseteq \text{Borel} \subseteq \Sigma^1_1\) and the projection map is continuous, we need only check \(4 \rightarrow 1\). But the image of a continuous map \(f\) on a closed set is the projection of the graph of \(f\) and the graph of \(f\) is closed. □

The collection of \(\Sigma^1_1\) sets is very rich, as is illustrated by the next proposition. Again the key fact is that wellfoundedness is a Borel and hence \(\Delta^1_1\) property for uncountable \(\kappa\).

**Proposition 4** A subset of \(\kappa^\kappa\) is \(\Sigma^1_1\) iff it is definable over \(H(\kappa^+)\) by a \(\Sigma_1\) formula (with parameters).

*Proof.* The direction from left to right is clear. Conversely, suppose that \(\varphi(\xi)\) is a \(\Sigma_1\) formula with parameters in \(H(\kappa^+)\) and let \(\xi_0 \in \kappa^\kappa\) code the parameters. Then \(\varphi(\xi)\) holds iff there exists \(\eta \in \kappa^\kappa\) which codes a wellfounded transitive model \(M\) of \(\text{ZF}^-\) containing \(\xi, \xi_0\) such that \(M\) satisfies \(\varphi(\xi)\). The latter condition on \(\xi\) is the projection of a Borel set and therefore is \(\Sigma^1_1\). □

*Remark.* It can also be shown that \(B\) is Borel iff for some fixed transitive \(\text{ZF}^-\) model \(M\) containing \(\kappa\) and formula \(\varphi(\xi)\) with parameters from \(M\), \(\xi \in B\) iff \(M[\xi] \models \varphi(\xi)\).

3.-4.Vorlesungen

*Regularity Properties*
The three classical regularity properties are the BP (Baire Property), the PSP (Perfect Set Property) and LM (Lebesgue Measurability). As we know of no good analogue of LM, we consider just BP and PSP.

**The Baire Property**

A set is *nowhere dense* iff its closure has no interior (i.e., has no nonempty open subset). A set is *meager* iff it is the union of $\kappa$-many nowhere dense sets. A *comeager* set is the complement of a meager set.

**Proposition 5** (Baire Category Theorem) Comeager sets are dense.

*Proof.* Chasing the definitions, this says that the intersection of $\kappa$-many open dense sets is dense. This is easy to prove: List the open dense sets as $(U_i \mid i < \kappa)$ and below a given $p \in \kappa^{<\kappa}$ successively extend to $p_i \in \kappa^{<\kappa}$ so that $N_{p_i}$ is contained in $U_i$ and the $p_i$’s converge to an $x \in \kappa^{\kappa}$. Then $x$ belongs to all of the $U_i$’s. □

A set $X$ has the BP iff its symmetric difference with some open set is meager.

**Proposition 6** Borel sets have the BP.

*Proof.* The collection of sets with the BP contains all basic open sets, is closed under complement and is closed under unions of size $\kappa$. □

**Theorem 7** (Halko-Shelah) There are $\Sigma^1_1$ sets without the BP.

*Proof.* Let CUB denote the filter of closed unbounded sets, which can be viewed as a subset of $\kappa^\kappa$ by considering the set of $\xi$ such that $X_\xi = \{ \alpha < \kappa \mid \xi(\alpha) = 0 \}$ contains a closed unbounded set. Clearly CUB is $\Sigma^1_1$. If CUB had the BP then it would be either meager or comeager on some basic open set $N_p$. But consider any $\kappa$-intersection $\bigcap_{i<\kappa} D_i$ of dense open subset $D_i$ of $N_p$.

Extend $p \kappa$ times to $(p_i \mid i < \kappa)$ of lengths $(\alpha_i \mid i < \kappa)$ where $p_0 = p$, $N_{p_{i+1}}$ is contained in $D_i$ and $p_{i+1}$ extends $p_i$, and for limit $\lambda < \kappa$, $p_\lambda$ extends the $p_i$, $i < \lambda$ and $p_\lambda(\alpha_\lambda) = 1$. Then $\xi$ = the union of the $p_i$’s belongs to each $D_i$ yet $X_\xi$ is nonstationary. This shows that CUB is not comeager on $N_p$. Similarly, by replacing 1 by 0 we have that CUB is not meager on $N_p$. □
The above argument also shows that for any regular \( \lambda < \kappa \), \( \text{CUB}_\lambda \) does not have the BP where the latter denotes all subsets of \( \kappa \) which contain \( \lambda \)-closed unbounded subsets.

Do \( \Delta^1_2 \) sets have the BP?

**Theorem 8** (a) In \( L \), there is a \( \Delta^1_2 \) set without the BP.
(b) It is consistent that all \( \Delta^1_2 \) sets have the BP.

**Proof.** (a) In \( L \) there is a \( \Delta^1_2 \) wellorder of \( \kappa^\omega \): \( \xi <_L \eta \) iff \( \xi \) appears before \( \eta \) in the \( L \)-hierarchy iff there exists a transitive model \( M \) of \( ZF^- \) of with \( \xi, \eta \in M \) such that \( M \vDash \xi <_L \eta \). Now using this wellorder it is easy by diagonalisation to construct in \( \kappa^+ \) steps a \( \Delta^1_2 \) set \( A \) such that for each \( p \) and comeager subset \( C \) of \( N_p \), both \( C \cap A \) and \( C \setminus A \) are nonempty. It follows that \( A \) does not have the BP.

(b) Add \( \kappa^+ \)-many \( \kappa \)-Cohen’s to \( L \) with a size \( < \kappa \) supported product. Then in this model BP holds for \( \Delta^1_2 \): Let the extension be \( L[G] \). Now suppose that \( \varphi_0, \varphi_1 \) are \( \Sigma^1_1 \) formulas with parameters in \( L[G(<i)] \) which in \( L[G] \) define a partition of \( \kappa^\omega \) into two disjoint pieces, \( i < \kappa^+ \). Note that \( G \) factors as \( G(<i) \times G[i, \kappa^+] \). Let \( x \in G(i) \), which is \( \kappa \)-Cohen over \( L[G(<i)] \). Then for some \( j < \kappa^+ \), either \( \varphi_0(x) \) or \( \varphi_1(x) \) holds in \( L[G(<i)][x][y] \) where \( y = G[i+1, j] \) is equivalent to a \( \kappa \)-Cohen over \( L[G(<i)][x] \). Assume that it is \( \varphi_0(x) \) that holds. But \( L[G(<i)][x] \) is \( \Sigma^1_1 \) elementary in \( L[G(<i)][x][y] \); if \( p \) forces \( \sigma \) to name a witness to a \( \Sigma^1_1 \) formula \( \varphi(x) \) in \( L[G(<i)][x][y] \) then \( L[G(<i)][x] \) contains a \( y' \) extending \( p \) which is \( \kappa \)-Cohen over an \( L_\alpha[G(<i)][x] \) which is large enough to contain the name \( \sigma \). So \( \varphi(x) \) holds in \( L[G(<i)][x] \). By the same argument, \( \varphi_0(x') \) holds in \( L[G(<i)][x'] \) for all \( x' \) extending some condition \( p \) which are sufficiently \( \kappa \)-Cohen generic over \( L[G(<i)] \), and this forms a comeager set below \( p \) in \( L[G] \). By the persistence of \( \Sigma^1_1 \) formulas, \( \varphi_0(x') \) holds in \( L[G] \) for all such \( x' \). So the original \( \Delta^1_2 \) set defined by \( \varphi_0, \varphi_1 \) is comeager on some open set and therefore has the BP. □

**Remark.** In \( L \), \( \text{CUB}_\lambda \) is not \( \Delta^1_2 \) for any regular \( \lambda < \kappa \). But it is consistent that \( \text{CUB}_\lambda \) is \( \Delta^1_2 \) for all regular \( \lambda < \kappa \), giving another proof of (b) above.

**The Perfect Set Property**

\( T \) is a perfect subtree of \( 2^{<\kappa} \) iff each node of \( T \) splits and \( T \) is closed under increasing sequences of length \( < \kappa \). Equivalently: \( T \) is the downward closure
of the range of an injective, order-preserving and continuous function from the full tree $2^{<\kappa}$ into itself.

**Lemma 9** Suppose $T$ is perfect. Then for CUB-many $\alpha < \kappa$, $T$ has $\text{card}(2^\alpha)$ nodes of length $\alpha$.

*Proof.* Let $T$ be the downward closure of the range of $f : 2^{<\kappa} \rightarrow 2^{<\kappa}$ where $f$ is injective, order-preserving and continuous. If $\kappa$ is inaccessible then for CUB-many $\alpha < \kappa$, $f(\sigma)$ has length less than $\alpha$ whenever $\sigma$ has length less than $\alpha$ and therefore by continuity $f(\sigma)$ has length $\alpha$ whenever $\sigma$ has length $\alpha$. This gives $\text{card}(2^\alpha)$ many nodes in $T$ of length $\alpha$.

If $\kappa = \lambda^+$ then we must show that there are $2^\lambda = \kappa$ nodes of $T$ of some fixed length $\alpha < \kappa$. If not, then let $\alpha_0 < \alpha_1 < \cdots$ be a $\lambda$-sequence such that every node of $T$ of length $\alpha_i$ has incompatible extensions of length $\alpha_{i+1}$. If $\alpha$ is the supremum of the $\alpha_i$'s then $T$ has $2^\lambda$ nodes of length $\alpha$. □

For any subtree $T$ of $2^{<\kappa}$ we let $[T]$ denote the set of $\kappa$-branches of $T$, a closed subset of $2^\kappa$ (where $2^\kappa$ is viewed as a subspace of $\kappa^\kappa$). A subset of $2^\kappa$ is *perfect* iff it is of the form $[T]$ for some perfect subtree $T$ of $2^{<\kappa}$.

Perfect sets are nonempty, closed and contain no isolated points. However, as we see below, the converse can fail.

$X \subseteq 2^\kappa$ has the *perfect set property (PSP)* iff it either has size at most $\kappa$ or contains a perfect set, i.e. a set of the form $[T]$ for some perfect tree $T$. Trivially, open sets have the PSP, as they are either empty or contain a basic open set, and basic open sets are perfect.

The PSP can fail for closed sets. A *quasi Kurepa tree* on $\kappa$ is a subtree $T$ of $2^{<\kappa}$ with the following properties:

1. Every node of $T$ splits and can be extended to a $\kappa$-branch.
2. $T$ has more than $\kappa$ many $\kappa$-branches.
3. For stationary many $\alpha < \kappa$, $T$ has at most $\alpha$-many nodes of length $\alpha$.

If $T$ is quasi Kurepa then $[T]$ does not have the PSP by Lemma 9.

**Theorem 10** In $L$, there is a quasi Kurepa tree on $\kappa$ for every regular uncountable $\kappa$. 7
Proof. Assume $V = L$. For each singular ordinal $\alpha < \kappa$ let $\beta(\alpha)$ denote the least limit ordinal $\beta > \alpha$ such that $\alpha$ is singular in $L_\beta$. Then $\beta(\alpha)$ is defined and less than $\alpha^+$ for singular $\alpha$. Now let $T$ be the tree of all $s \in 2^{<\kappa}$ such that:

1. $C(s) = \{ \alpha < |s| \mid s(\alpha) = 1 \}$ is closed in $|s|$.
2. If $\alpha < |s|$ is a singular limit point of $C(s)$ then $s \upharpoonright \alpha$ belongs to $L_{\beta(\alpha)}$.
3. If $\kappa = \lambda^+$ for some cardinal $\lambda$, then $s(\alpha) = 0$ for all $\alpha < \lambda$.

Then $[T]$ consists of all (characteristic functions of) closed subsets $C$ of $\kappa$ such that $C \cap \alpha$ belongs to $L_{\beta(\alpha)}$ for all singular limit points $\alpha$ of $C$ and if $\kappa = \lambda^+$ for some cardinal $\lambda$, then $C \cap \lambda = \emptyset$.

Clearly each node of $T$ splits and can be extended to a $\kappa$-branch. If $\kappa$ is inaccessible then for all singular cardinals $\alpha < \kappa$, $T$ has at most $\text{card}(\alpha)$ nodes of length $\alpha$, as each of these nodes belongs to $L_{\beta(\alpha)}$. If $\kappa = \lambda^+$ for some cardinal $\lambda$ then for all $\alpha < \kappa$, $T$ has at most $\text{card}(\alpha)$ nodes of length $\alpha$, as for $\alpha \leq \lambda$ there is at most one such node and for $\alpha$ in $(\lambda, \kappa)$ all such nodes belong to $L_{\beta(\alpha)}$ by induction.

So to verify that $T$ is a quasi Kurepa tree we need only show that $T$ has $\kappa^+$ many branches.

For any $\alpha$ in $(\kappa, \kappa^+]$ let $C_\alpha$ be the CUB set of $\bar{\kappa} < \kappa$ such that $\bar{\kappa} = H(\bar{\kappa}) \cap \kappa$, where $H(\bar{\kappa})$ is the $\Sigma_1$ Skolem hull of $\bar{\kappa} \cup \{ \kappa, \alpha \}$ in $L_{\beta(\alpha)}$. If $\bar{\kappa}$ is a singular limit point of $C_\alpha$ then $\bar{\kappa}$ is regular in the transitive collapse of $H(\bar{\kappa})$ and $C_\alpha \cap \bar{\kappa}$ is definable over that transitive collapse; it follows that $C_\alpha \cap \bar{\kappa}$ belongs to $L_{\beta(\bar{\kappa})}$. So $C_\alpha$ is a branch through $T$. As each CUB set in $L_{\beta(\alpha)}$ almost contains $C_\alpha$, it follows that $C_\alpha$ does not belong to $L_{\beta(\alpha)}$ and therefore there are $\kappa^+$ many distinct such $C_\alpha$'s. □

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Kurepa trees are defined just like quasi Kurepa trees but with the stronger requirement that $\{ \alpha < \kappa \mid T \text{ has at most } \alpha \text{-many nodes of length } \alpha \}$ contain all infinite ordinals less than $\kappa$. Silver showed long ago that to kill Kurepa trees on $\kappa$ one needs an inaccessible, so to get the PSP for closed sets one needs an inaccessible.
\textbf{Theorem 11} Starting with an inaccessible above \( \kappa \), one can force the PSP to hold for all \( \Sigma_1^1 \) subsets of \( 2^\kappa \).

\textit{Proof.} We use the same model that Silver used to kill Kurepa trees. Start with \( L \) where \( \kappa < \lambda \) is inaccessible and let \( P = \text{Coll} (\kappa, < \lambda) \) denote the forcing that makes \( \lambda \) into \( \kappa^+ \) with conditions of size \( \kappa \). Thus a condition is a function \( p : \text{Dom} (p) \rightarrow \lambda \) where \( \text{Dom} (p) \) is a size \( \kappa \) subset of \( \lambda \times \kappa \) and \( p(\alpha, \gamma) < \alpha \) for each \( (\alpha, \gamma) \) in \( \text{Dom} (p) \).

Suppose now that \( X \subseteq 2^\kappa \) in the resulting model \( L[G] \) is \( \Sigma_1^1 \) with parameters coming from the inner model \( L[G|\alpha] \), \( \alpha < \lambda \), \( \alpha \) regular in \( L \). Factor \( L[G] \) as \( L[G|\alpha][G|\alpha, \lambda] \) where \( G|\alpha \) is generic for \( P|\alpha \) and \( G|\alpha, \lambda \) is generic over \( L[G|\alpha] \) for \( P|\alpha, \lambda \). We may assume that \( X \) has more than \( \kappa \) many elements and therefore can choose an element \( a \) of \( X \) in \( L[G] \setminus L[G|\alpha] \). Also choose an \( L \)-regular \( \beta \) between \( \alpha \) and \( \lambda \) so that the membership of \( a \) in \( X \) is witnessed inside \( L[G|\beta] \). Let \( \dot{a} \) be a \( P(\alpha, \beta) \)-name for \( a \) in the ground model \( L[G|\alpha] \) and \( p \) a condition in \( P(\alpha, \beta) \) which forces that \( \dot{a} \) is an element of \( X \) not in \( L[G|\alpha] \).

Now in \( L[G] \) build a perfect tree \( U \) of conditions below \( p \) in the \( \kappa \)-closed forcing \( P(\alpha, \beta) \) such that each \( \kappa \)-branch of \( U \) hits each maximal antichain of \( P(\alpha, \beta) \) which belongs to \( L[G|\alpha] \) (there are only \( \kappa \)-many) and distinct branches of \( U \) force different interpretations of the name \( \dot{a} \). Thus each \( \kappa \)-branch \( x \) through \( U \) yields a pair \((g_x, a_x)\) where \( g_x \) is \( P(\alpha, \beta) \)-generic over \( L[G|\alpha] \), \( a_x \) is the interpretation of \( \dot{a} \) given by \( g_x \) and the membership of \( a_x \) in \( X \) is witnessed in \( L[G|\alpha][g_x] \). The set of such \( a_x \)'s forms a perfect subset of \( X \). \( \square \)

Can the PSP hold for all \( \Pi_1^1 \) sets? The answer is not known.

\textit{Borel Reducibility}

As mentioned earlier, to obtain a provable characterisation of classifiable theories in terms of the complexity of their isomorphism relations on uncountable models we must use a generalisation of the theory of Borel reducibility. I'll introduce now such a generalisation and prove some basic facts about it.

Suppose that \( X_0, X_1 \) are Borel subsets of \( \kappa^\kappa \). Then \( f : X_0 \rightarrow X_1 \) is a \textit{Borel function} iff \( f^{-1}[Y] \) is Borel whenever \( Y \) is Borel. This implies that the graph of \( f \) is Borel, as \((x, y)\) belongs to the graph of \( f \) iff for all \( s \in \kappa^{<\kappa} \), either \( y \) does not belong to \( N_s \) or \( x \) belongs to \( f^{-1}[N_s] \).
If $E_0, E_1$ are equivalence relations on Borel sets $X_0, X_1$ respectively then we say that $E_0$ is Borel reducible to $E_1$, written $E_0 \leq_B E_1$, iff for some Borel $f : X_0 \to X_1$:

$$x_0E_0y_0 \text{ iff } f(x_0)E_1f(x_1).$$

Now recall the following picture from the classical case:

$$1 <_B 2 <_B \cdots <_B \omega <_B \text{id} <_B E_0$$

forms an initial segment of the Borel equivalence relations under $\leq_B$ where $n$ denotes an equivalence relation with $n$ classes for $n \leq \omega$, id denotes equality on $\omega$ and $E_0$ denotes equality modulo finite on $\omega$.

At $\kappa$ we easily get the initial segment

$$1 <_B 2 <_B \cdots <_B \omega <_B \omega_1 <_B \cdots <_B \kappa$$

where for each nonzero cardinal $\lambda \leq \kappa$ we identify $\lambda$ with the $\equiv_B$ class of Borel equivalence relations with exactly $\lambda$-many classes. What happens above these equivalence relations? We might hope for:

**Silver Dichotomy** The equivalence relation $\text{id}$ (equality on $\kappa^\omega$) is the strong successor of $\kappa$ under $\leq_B$, i.e., if a Borel equivalence relation $E$ has more than $\kappa$ classes then $\text{id}$ is Borel-reducible to $E$.

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**Theorem 12** (a) The Silver Dichotomy implies the PSP for Borel sets. Therefore it fails in $L$ and its consistency requires at least an inaccessible cardinal.

(b) The Silver Dichotomy is false with Borel replaced by $\Delta^1_1$.

**Proof.** (a) Assume the Silver Dichotomy and suppose that $B$ is Borel with more than $\kappa$ elements. Consider the equivalence relation $xEy$ iff $x = y$ or $x, y \notin B$. Applying the Silver Dichotomy we get a Borel reduction of $\text{id}$ to $E$; as all but one point in the range of this reduction belongs to $B$, we get a Borel, injective map from $\kappa^\omega$ into $B$ and therefore also a Borel, injective map from $2^\kappa$ into $B$. If this map were continuous then its range would be a perfect subset of $B$, verifying the PSP for $B$. We finish the proof as follows:
Lemma 13  (a) Any Borel function $f : \kappa^\kappa \to \kappa^\kappa$ is continuous on a comeager set.
(b) Any comeager set contains a perfect subset.

Proof. (a) Recall that each Borel set has a Borel code $(T, l)$, where $T$ is a wellfounded tree of size $\kappa$ and $l$ is a labelling of the terminal nodes of $T$ by basic open sets. As $f$ is Borel it has a Borel graph and so there is a Borel code for its graph. Now let $M$ be a transitive $\text{ZF}^-$ model of size $\kappa$ containing $H(\kappa)$ and this Borel code. Let $X$ be the set of $x \in \kappa^\kappa$ which are generic over $M$ for the forcing $\kappa^{<\kappa}$ (equivalent to $\kappa$-Cohen forcing). Then $X$ is comeager as it is the intersection of the $\kappa$-many open dense sets coded in $M$. And the function $f$ is continuous on $X$ as for any $s \in \kappa^{<\kappa}$, any true statement $s \subseteq f(x)$ is forced by a condition $t \subseteq x$ and therefore $s \subseteq f(y)$ holds for all $y \in X$ extending $t$, as such $y$ are generic over $M$.
(b) This is just like the proof that comeager sets are dense. Let $(D_i \mid i < \kappa)$ be open dense and build a perfect tree $T$ so that if $\sigma$ is on the $i$-th splitting level of $T$ then $N_\sigma$ is contained in $D_j$ for all $j < i$. This is possible using the density of the $D_i$’s. Then every cofinal branch through $T$ belongs to the intersection of the $D_i$’s. □

This completes the proof of (a) of the Theorem.

(b) Consider the equivalence relation $xEy$ iff $x, y$ code the same ordinal or $x, y$ do not code ordinals. Then $E$ has $\kappa^+$ many equivalence classes. Also $E$ is $\Delta^1_1$: We have seen (using $\kappa > \omega$) that wellfoundedness is Borel; for $x, y$ coding ordinals we have $xEy$ iff there exists order-preserving maps between the ordinals coded by $x, y$ iff there is no order-preserving map from the ordinal coded by $x$ into a proper initial segment of the ordinal coded by $y$ (or vice-versa).

If $\text{id}$ Borel-reduces to $E$ then as above we get a perfect set $[T]$ such that distinct elements of $[T]$ code distinct ordinals. Let $M$ be a transitive $\text{ZF}^-$ model of size $\kappa$ containing $H(\kappa)$ and the perfect tree $T$ and suppose that $x$ is a branch through $T$ which is generic over $M$ (for the forcing whose conditions are the nodes of $T$). Then the ordinal coded by $x$ belongs to $M$ as $M[x]$ satisfies $\text{ZF}^-$. But there are $2^\kappa$ many such $x$’s, contradicting the fact that different such $x$’s must code different ordinals. □
Is the Silver Dichotomy consistent? This question remains open.

We can also consider what happens above id. In the case $\kappa = \omega$ we have:

*Classical Glimm-Effros Dichotomy* $E_0 = (\text{equality mod finite})$ is the strong successor of id, i.e., if a Borel equivalence relation $E$ is not Borel-reducible to id (i.e., $E$ is not smooth) then $E_0$ Borel-reduces to $E$.

At $\kappa$, what shall we take $E_0$ to be? For infinite regular $\lambda \leq \kappa$, define $E_{\leq \lambda}^0 = \text{equality for subsets of } \kappa \text{ modulo sets of size } < \lambda$.

**Proposition 14** For $\lambda < \kappa$, $E_{\leq \lambda}^0$ is Borel bireducible with id.

So we can forget about $E_{\leq \lambda}^0$ for $\lambda < \kappa$ and set $E_0 = E_0^\kappa$, equality modulo bounded sets.

**Proof.** For each ordinal $\alpha < \kappa$ consider the equivalence relation on subsets of $\alpha$ given by: $x \equiv_\alpha y$ iff $x \triangle y$ has size < $\lambda$ (where $\triangle$ denotes symmetric difference). For each $\alpha < \kappa$ choose a selector $f_\alpha$ for $\equiv_\alpha$, i.e. a (noneffective) function reducing $\equiv_\alpha$ to equality on $2^\alpha$. So for $x, y$ subsets of $\alpha$, $f_\alpha(x)$ equals $f_\alpha(y)$ iff $x \equiv_\alpha y$. This is possible by the axiom of choice. The sequence $(f_\alpha | \alpha < \kappa)$ is just one big parameter, i.e. can be coded by a fixed subset of $\kappa$.

Now define a reduction of $E_{\leq \lambda}^0$ to id (= equality on $2^\kappa$) by sending $x$ to a subset of $\kappa$ coding the sequence $f_\alpha(x) = (f_\alpha(x \cap \alpha) | \alpha < \kappa)$. If $x \triangle y$ has size < $\lambda$ then $f(x) = f(y)$. If $x \triangle y$ has size at least $\lambda$ then there is $\alpha < \kappa$ such that $(x \cap \alpha) \triangle (y \cap \alpha)$ has size at least $\lambda$ and hence $f(x)$ does not equal $f(y)$. The reduction is clearly continuous.

Conversely, id is continuously reducible to $E_{\leq \lambda}^0$: Choose an injection $i : 2^{<\kappa} \to \kappa$ and reduce id to $E_{\leq \lambda}^0$ by: $f(x) = \{i(x \cap \alpha) | \alpha < \kappa\}$. □

**7.-8.Vorlesungen**

As in the classical case we have:

**Proposition 15** $E_0 = E_{\leq \kappa}^0$ is not Borel-reducible to id.
Proof. Suppose that \( f \) were a Borel reduction of \( E_0 \) to equality. Let \( M \) be a transitive ZF\(^-\) model of size \( \kappa \) containing \( H(\kappa) \) and a Borel-code for \( f \). Consider \( \kappa \)-Cohen forcing over \( M \) and let \( G \) denote the \( \kappa \)-Cohen generic. Choose a condition \( p \) forcing that \( f(G) \) is defined. Then for each \( \alpha < \kappa \), any two \( \kappa \)-Cohen conditions extending \( p \) must force the same value for \( f(G)(\alpha) \): Otherwise, choose two conditions \( p_0, p_1 \) of equal length extending \( p \) and forcing different values of \( f(G)(\alpha) \). Let \( x \) be \( \kappa \)-Cohen generic over \( M \) extending \( p_0 \) and let \( y \) be obtained from \( x \) by replacing \( p_0 \) by \( p_1 \). Then \( xE_0 y \) but \( f(x)(\alpha) \neq f(y)(\alpha) \), contradicting the assumption that \( f \) is a reduction. So \( f(x) \) is constant on all \( x \) which are \( \kappa \)-Cohen generic and extend \( p \). But that is impossible, because there are \( \kappa \)-Cohen generics \( x, y \) extending \( p \) which disagree on an unbounded subset of \( \kappa \). \( \square \)

Other versions of \( E_0 \): For regular \( \lambda < \kappa \) define \( E_\lambda^\kappa = \text{equality modulo the ideal of \( \lambda \)-nonstationary sets.} \)

These equivalence relations are key for connecting Shelah Classification with Higher Descriptive Set Theory.

How do the relations \( E_\lambda^\kappa \) compare to each other under Borel reducibility for different \( \lambda \)? For simplicity, consider the special case \( \kappa = \omega_2 \).

**Theorem 16** (SDF-Hyttinen-Kulikov) (a) It is consistent that \( E_{\omega_2}^{\omega_2} \) and \( E_{\omega_1}^{\omega_2} \) are incomparable under Borel reducibility. (b) Relative to a weak compact it is consistent that \( E_{\omega_2}^{\omega_2} \) is Borel-reducible to \( E_{\omega_1}^{\omega_2} \).

It is not known if it is consistent for \( E_{\omega_1}^{\omega_2} \) to be Borel-reducible to \( E_{\omega_2}^{\omega_2} \).

**Proof of (a).** We begin with the following general result. As always assume GCH. Fix \( \kappa \) uncountable and regular, not the successor of a singular cardinal (to handle this case we would need Shelah’s approachability property on \( \kappa \)). For regular \( \lambda < \kappa \), \( S_\kappa^\lambda \) denotes the set of ordinals less than \( \kappa \) of cofinality \( \lambda \). For arbitrary stationary \( X \subseteq \kappa \), \( E_X \) denotes the equivalence relation defined by \( A E_X B \iff (A \triangle B) \cap X \text{ is nonstationary.} \)

**Lemma 17** Suppose that \( \mu_1, \mu_2 < \kappa \) are regular, \( X \subseteq S_{\mu_1}^\kappa \) is stationary, \( Y \subseteq S_{\mu_2}^\kappa \) is stationary and \( X \cap \alpha \) is not stationary in \( \alpha \) for \( \alpha \in Y \) (automatic if \( \mu_1 \geq \mu_2 \)). Also suppose that \( X, Y \) are disjoint (automatic if \( \mu_1 \neq \mu_2 \)). Suppose that \( (T, l) \) is a code for a Borel reduction from \( E_X \) to \( E_Y \). Then \( (T, l) \) is not such a code in some generic extension by a \( \kappa \)-closed forcing of size \( \kappa \).
Proof of Lemma. First add a \( \kappa \)-Cohen set \( G \). If \( (T,l) \) does not code a Borel reduction from \( E_X \) to \( E_Y \) in \( V[G] \) then we are done. Otherwise let \( f \) be the Borel reduction coded by \( (T,l) \) in \( V[G] \). Note that \( G \cap X \) is stationary, as \( \kappa \)-Cohen sets have stationary intersection with all ground model stationary sets (this already uses the hypothesis that \( \kappa \) is not the successor of a singular, or alternatively the approachability property on \( \kappa \)). So \( (f(G \cap X) \triangle f(\emptyset)) \cap Y = Z \) is stationary. Now consider the forcing \( Q \) for killing the stationarity of \( G \cap X \):

A condition is a closed, bounded subset \( c \) of \( \kappa \) such that \( c \cap (G \cap X) \) is empty.

Claim. \( Q \) preserves the stationarity of \( Z \).

Proof. First we need:

Subclaim. Choose a large \( H(\theta) \), \( \theta \) regular and also a parameter \( x \) in \( H(\theta) \). Then there is a club \( C \) in \( \kappa \) such that for all \( \alpha \in C \) of cofinality \( \mu_2 \) there exists an elementary submodel \( M \) of \( H(\theta) \) containing the parameter \( x \) which has size \( \mu_2 \), is \( \mu_2 \)-closed and satisfies \( \sup(M \cap \kappa) = \alpha \).

Proof of Subclaim. First choose a continuous increasing chain \( (M_i \mid i < \kappa) \) of elementary submodels of \( H(\theta) \) such that \( x \) belongs to \( M_0 \) and for each \( i < \kappa \), \( M_{i+1} \) is \( \mu_2 \)-closed (this is possible as we have assumed GCH and \( \kappa \) is not the successor of a singular cardinal). Let \( C \) consist of all \( M_\lambda \cap \kappa \) for limit \( \lambda < \kappa \), a club in \( \kappa \). Then if \( \lambda \) has cofinality \( \mu_2 \) the model \( M_\lambda \) is \( \mu_2 \)-closed and therefore it is easy to build an increasing chain \( (N_j \mid j < \mu_2) \) of elementary submodels of \( M_\lambda \) where \( x \) belongs to \( N_0 \), each \( N_j \) has size less than \( \mu_2 \) and the union \( M \) of the \( N_j \)'s is \( \mu_2 \)-closed and has cofinal intersection with \( M_\lambda \cap \kappa \).

\( \square \) (Subclaim)

Now suppose that the condition \( q \in Q \) forces that the \( Q \) name \( \dot{C} \) is a club in \( \kappa \) and choose a large \( H(\theta) \), containing \( q, \dot{C}, \kappa, G \cap X \) as elements. Using the Subclaim choose an elementary submodel \( M \) of \( H(\theta) \) which has size \( \mu_2 \) and is \( \mu_2 \)-closed, and is such that \( \alpha = \sup(M \cap \kappa) \) belongs to \( Z \). This is possible because \( Z \) is stationary and the set of possibilities for \( \sup(M \cap \kappa) \) contains all ordinals of cofinality \( \mu_2 \) in some closed unbounded set. As \( \alpha \) belongs to \( Z \subseteq Y, X \cap \alpha \) is not stationary in \( \alpha \) and therefore \( G \cap X \cap \alpha \) is not stationary in \( \alpha \). Let \( D \) be closed unbounded in \( \alpha \) and disjoint from \( G \cap X \). Now we can build a continuous sequence of conditions \( q = q_0 \geq q_1 \geq \cdots \) of length \( \mu_2 + 1 \) so that for \( i < \mu_2 \), \( q_i \) belongs to \( M \), \( q_{i+1} \) forces an ordinal greater than \( \max(q_i) \) to belong to \( \dot{C} \) and there is an element of \( D \) between \( \max(q_i) \) and
max(q_{i+1}). As α does not belong to X it follows that q^* = q_p \cup \{α\} is a condition extending q with max(q^*) = α \in Z and q^* forces α to belong to \(\hat{C}\). So Z remains stationary. □ (Claim)

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Now to finish the proof of the Lemma, note that we have shown that \(C \ast Q\) forces that the Borel reduction \(f\) coded by \((T,l)\) does not reduce \(E_X\) to \(E_Y\) because it forces \(G \cap X\) to be nonstationary on \(X\) and its image \(f(G \cap X)\) to be stationary on \(Y\). It remains to show that \(Z = C \ast Q\) is equivalent to a \(\kappa\)-closed forcing of size \(\kappa\). But this forcing is equivalent to forcing with pairs \((p,q)\) where \(p\) is a \(\kappa\)-Cohen condition and \(q\) is a \(\kappa\)-Cohen condition of the same length \(\alpha\) such that \(\{β \mid q(β) = 1\}\) is closed in \(\alpha\) and disjoint from \(\{β \mid p(β) = 1\}\). This forcing is \(\kappa\)-closed because if \((p_i,q_i)\), \(i < \lambda\), is a descending sequence for some limit \(λ < \kappa\) we obtain a lower bound by taking \(p,q\) to be the unions of the \(p_i,q_i\)'s. □

Now to prove (a) of the Theorem, choose a model in which some stationary subset \(X\) of \(S^ω_{ω^1}\) has nonstationary intersection with each ordinal in \(S^ω_{ω^1}\). For example, take a model in which \(\Box_{ω^1}\) holds. Then by the Lemma applied to the pair \(X,S^ω_{ω^1}\), we can force with a \(\kappa\)-closed forcing of size \(\kappa\) to kill any code for a Borel reduction of \(E_X\) to \(E^ω_{ω^1}\). The same holds for the pair \(S^ω_{ω^1},S^ω_{ω^2}\). Now perform a \(\kappa\)-closed iteration of length \(\kappa^*\), successively killing codes for Borel reductions of \(E_X\) to \(E^ω_{ω^1}\) and of \(E^ω_{ω^1}\) to \(E^ω_{ω^2}\). The result is a model in which \(E_X\) is not Borel reducible to \(E^ω_{ω^1}\) and \(E^ω_{ω^1}\) is not Borel reducible to \(E^ω_{ω^2}\). So the proof of (a) ends with the following

Fact. Suppose \(X \subseteq Y\). Then \(E_X\) is Borel reducible to \(E_Y\).

Proof of Fact. Send \(A\) to \(A \cap X\). If \((A \triangle B) \cap X\) is nonstationary then so is \(((A \cap X) \triangle (B \cap X)) \cap Y\) because it equals \((A \triangle B) \cap X\). Conversely, if \((A \triangle B) \cap X\) is stationary then so is \(((A \cap X) \triangle (B \cap X)) \cap Y\) for the same reason. □

Now we begin the proof of (b), which says that relative to a weak compact, it is consistent that \(E^ω_{ω^2}\) is Borel reducible to \(E^ω_{ω^1}\).

We obtain Borel reductions using “\(\Diamond\) reflection”. Suppose that \(X,Y\) are subsets of a regular \(\kappa\) and \(Y\) consists of ordinals of uncountable cofinality. We say that \(X\) \(\Diamond\)-reflects to \(Y\) if there is a sequence \((D_\alpha \mid \alpha \in Y)\) such that:
1. \( D_\alpha \) is a stationary subset of \( \alpha \) for each \( \alpha \in Y \).
2. If \( Z \subseteq X \) is stationary then \( \{ \alpha \in Y \mid D_\alpha = Z \cap \alpha \} \) is stationary.

**Lemma 18** Suppose that \( X \lhd \)-reflects to \( Y \). Then \( E_X \) is Borel reducible to \( E_Y \).

**Proof.** Let \( (D_\alpha \mid \alpha \in Y) \) witness the hypothesis. For \( A \subseteq \kappa \) set \( f(A) = \{ \alpha \in Y \mid A \cap X \cap D_\alpha \text{ is stationary in } \alpha \} \). Note that \( f \) is continuous. We claim that \( f \) is the desired reduction.

Suppose that \((A \Delta B) \cap X\) is nonstationary and choose a club \( C \) disjoint from it. Then for \( \alpha \) in \( Y \cap \text{Lim} \ C \), \( C \cap \alpha \) is club in \( \alpha \) and so \( A \cap X \cap D_\alpha \cap C \) is stationary in \( \alpha \) iff \( B \cap X \cap D_\alpha \cap C \) is stationary in \( \alpha \). Thus \( (f(A) \setminus f(B)) \cap Y \) is nonstationary, as desired.

Conversely, suppose that \((A \Delta B) \cap X\) is stationary and without loss of generality assume that \((A \setminus B) \cap X\) is stationary. By hypothesis, \( S = \{ \alpha \in Y \mid (A \setminus B) \cap X \cap \alpha = D_\alpha \} \) is stationary. For \( \alpha \) in \( S \) we have that \( A \cap X \cap D_\alpha \) is stationary in \( \alpha \) and and \( B \cap X \cap D_\alpha \) is empty. So \( (f(A) \setminus f(B)) \cap Y \) is stationary, as desired. \( \Box \)

Suppose that \( \kappa \) is weak compact. Then \( \kappa \) satisfies \( \Pi^1_1 \) reflection: If \((H(\kappa), A)\) satisfies the \( \Pi^1_1 \) sentence \( \varphi(A) \) then \((H(\alpha), A \cap H(\alpha))\) satisfies \( \varphi(A \cap H(\alpha)) \) for some \( \alpha < \kappa \).

**Lemma 19** Suppose that \( \kappa \) is weak compact and \( X \subseteq \kappa \) is stationary. Then for stationary many regular \( \alpha < \kappa \), \( X \cap \alpha \) is stationary in \( \alpha \).

**Proof.** If \( C \) is a club in \( \kappa \) then \((H(\kappa), X, C)\) satisfies the \( \Pi^1_1 \) sentence saying that \( \kappa \) is regular (there is no cofinal function from an ordinal into the ordinals), \( X \) is stationary and \( C \) is club. By reflection there is a regular \( \alpha \) so that \( C \cap \alpha \) is club in \( \alpha \) and \( X \cap \alpha \) is stationary in \( \alpha \). As \( C \) is closed, \( \alpha \) belongs to \( C \). \( \Box \)

**Lemma 20** Suppose that \( \kappa \) is weak compact and \( V = L \). Then all stationary subsets of \( \kappa \lhd \) reflect to \( \text{Reg} \cap \kappa = \text{the set of regular cardinals less than } \kappa \). In particular, \( E_X \) is Borel reducible to \( E_{\text{Reg} \cap \kappa} \) for all stationary \( X \subseteq \kappa \).
Proof. Let $Y$ denote $\text{Reg} \cap \kappa$. We define the desired $D_\alpha$'s inductively as follows. Given $\alpha$ in $Y$, let $(Z, C)$ be the $L$-least pair of subsets of $\alpha$ such that $Z$ is a stationary subset of $X \cap \alpha$, $C$ is club in $\alpha$ and $D_\beta$ does not equal $Z \cap \beta$ for $\beta$ in $Y \cap C$ (if there is no such pair then set $D_\alpha = \emptyset$). Then take $D_\alpha$ to be $Z$. For future use also define $C_\alpha$ to be $C$.

If the above sequence of $D_\alpha$’s does not work then let $(Z, C)$ be the $L$-least pair of subsets of $\kappa$ such that $Z$ is a stationary subset of $X$, $C$ is club in $\kappa$ and $D_\beta$ does not equal $Z \cap \beta$ for $\beta$ in $Y \cap C$. Let $M$ be an elementary submodel of some large $H(\theta)$ which contains $X$ as an element such that $\alpha = M \cap \kappa$ is an element of $Y$ and $Z \cap \alpha$ is stationary in $\alpha$. Such an $M$ exists by Lemma 19. Let $\bar{M}$ be the transitive collapse of $M$. Then $(Z \cap \alpha, C \cap \alpha)$ is the $L$-least pair $(z, c)$ in $\bar{M}$ such that $z$ is a subset of $X \cap \alpha$ which is stationary in $\bar{M}$, $c$ is club in $\alpha$ and $D_\beta$ does not equal $z \cap \beta$ for $\beta$ in $Y \cap c$. As $Z \cap \alpha$ really is stationary in $\alpha$ (and not just stationary with respect to clubs in $\bar{M}$), the pair $(Z \cap \alpha, C \cap \alpha)$ is in fact equal to the pair $(D_\alpha, C_\alpha)$ defined above. But this contradicts the choice of the pair $(Z, C)$, as $\alpha$ belongs to $Y \cap C$ and $D_\alpha$ does equal $Z \cap \alpha$. □

Finally, to get a consistent Borel reducibility from $E^{\omega_1}_\omega$ to $E^{\omega_2}_\omega$, we start with a weak compact $\kappa$ in $L$ and force with an $\omega$-closed Lévy collapse to make $\kappa$ into $\omega_2$. We show that in the extension $L[G]$, $S^{\omega_2}_\omega$ $\Diamond$ reflects to $Y =$ the $L$-regular cardinals less than $\kappa$. Note that $Y$ is stationary in $L[G]$ as it is stationary in $L$ and any club in $\kappa$ in $L[G]$ contains a club in $\kappa$ in $L$, by the $\kappa$-cc of the forcing. As $Y$ is a stationary subset of $S^{\omega_2}_\omega$ in $L[G]$ it follows that $S^{\omega_2}_\omega$ also $\Diamond$ reflects to $S^{\omega_2}_\omega$ in $L[G]$.

In $L[G]$ we define the sequence $\langle D_\mu \mid \mu \in Y \rangle$ as follows. Let $f^G : \kappa \times \omega_1 \to \kappa$ be the generic function added by $G$; i.e., $f^G$ maps $\{\mu\} \times \omega_1$ onto $\mu$ for each $\mu < \kappa$ and is forced with countable conditions. For each $L$-regular $\mu \in [\omega_2^L, \kappa)$ let $\langle \sigma^\mu_i \mid i < (\mu^+)^L \rangle$ enumerate the nice $P(< \mu)$-names for subsets of $\mu$. Then we take $D_\mu$ to be the interpretation of the $P(< \mu)$-name $\sigma^\mu_{f^G(\mu^+, \tau, 0)}$ via the $P(< \mu)$-generic $G \cap P(< \mu)$ (provided this is stationary in $\mu$; $D_\mu = \mu$ otherwise).

We claim that the sequence $\langle D_m u \mid \mu \in Y \rangle$ works. Indeed, suppose that the condition $p$ forces $\bar{S}$ to be a nice $P$-name for a stationary subset of $S^{\omega_2}_\omega$ and $C$ is a club in $\kappa$; we show that $p$ has an extension $q$ forcing $\bar{S} \cap \mu = D_\mu$ for some $\mu$ in $Y \cap C$. This suffices, as any club subset of $\kappa = \omega_2$ in $L[G]$. 

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contains a club subset in \( L \). Apply \( \Pi_1^1 \) reflection to get \( \mu \in Y \cap C \) so that \( p \) belongs to \( P(<\mu) \) and \( p \) forces that \( S \cap \mu \) is stationary in \( \mu \). Now extend \( p \) to \( q \) by choosing \( q((\mu^+)^L,0) \) to be \( i \) where \( \hat{S} \cap \mu \) is the \( P(<\mu) \)-name \( \sigma_i^\mu \). \( \Box \)

\[ 11.-12. \text{Vorlesungen} \]

\( \text{Clinton's Question} \)

I'll give a partial answer to a question of Clinton Conley: What is the relationship between \( E_0 \) and \( E_\omega^\omega \)?

**Theorem 21** It is consistent that under Borel reducibility, \( E_{\omega_1}^{\omega_2} \) and \( E_\omega^{\omega_2} \) are incomparable and neither reduces to \( E_0 \).

**Theorem 22** In \( L \), \( E_0 \) reduces to both \( E_{\omega_1}^{\omega_2} \) and \( E_\omega^{\omega_2} \).

**Proof of Theorem 21.** First recall the basic technical lemma that we used to get the incomparability of \( E_{\omega_1}^{\omega_2} \) and \( E_\omega^{\omega_2} \):

**Lemma 23** Suppose that \( X \subseteq S_{\omega_2}^{\omega_2} \) is stationary but has nonstationary intersection with \( \alpha \) for all \( \alpha \in S_{\omega_2}^{\omega_1} \). Suppose that \( (T,l) \) is a code for a Borel reduction from \( E_X \) to \( E_{\omega_1}^{\omega_2} \). Then \( (T,l) \) is not such a code in some generic extension by an \( \omega_2 \)-closed forcing of size \( \omega_2 \).

This was proved as follows: First add an \( \omega_2 \)-Cohen set \( G \). If \( (T,l) \) does not code a Borel reduction from \( E_X \) to \( E_{\omega_1}^{\omega_2} \) in \( V[G] \) then we are done. Otherwise let \( f \) be the Borel reduction coded by \( (T,l) \) in \( V[G] \) and note that \( G \cap X \) is stationary. As \( f \) is a reduction it follows that \( (f(G \cap X) \Delta f(\emptyset)) \cap S_{\omega_1}^{\omega_2} = Z \) is stationary. Now force to kill the stationarity of \( G \cap X \). Using the hypothesis that \( X \) is a “nonreflecting” stationary set, this forcing preserves the stationarity of \( Z \) and therefore “kills” the reduction \( (T,l) \). Finally, the two-step iteration \( \omega_2 \)-Cohen \( \ast \) (Kill stationarity of \( G \cap X \)) is \( \omega_2 \)-closed.

The existence of \( X \) as in the Lemma follows from \( \square_{\omega_1} \) and therefore holds in \( L \). By an \( \omega_2 \)-iteration one can use the Lemma to kill all Borel codes for reductions of \( E_X \) to \( E_{\omega_1}^{\omega_2} \). As \( E_X \) is Borel reducible to \( E_{\omega_1}^{\omega_2} \), it follows that in the resulting model, \( E_{\omega_2}^{\omega_2} \) is not Borel-reducible to \( E_{\omega_1}^{\omega_2} \).

There is a similar lemma for killing reductions from \( E_{\omega_1}^{\omega_2} \) to \( E_\omega^{\omega_2} \), using the fact that the entire \( S_{\omega_1}^{\omega_2} \) trivially has “nonstationary” intersection with each \( \alpha \in S_{\omega}^{\omega_2} \).
Now the point is that the same technique shows how to kill reductions from \(E^{\omega_2}_\omega\) or \(E^{\omega_2}_{\omega_1}\) to \(E_0\).

**Lemma 24** Suppose that \((T, l)\) is a code for a Borel reduction from either \(E^{\omega_2}_\omega\) or \(E^{\omega_2}_{\omega_1}\) to \(E_0\). Then in an \(\omega_2\)-closed forcing extension, \((T, l)\) is no longer such a code.

**Proof.** Suppose that \((T, l)\) codes a Borel reduction from \(E^{\omega_2}_\omega\) to \(E_0\) and add an \(\omega_2\)-Cohen set \(G\). If this kills the code \((T, l)\) then we are done. Otherwise, let \(f\) be the reduction coded by \((T, l)\) in \(V[G]\). As \(G \cap S^{\omega_2}_\omega = Z\) is stationary and \(f\) is a reduction, we have that \(f(G) \triangle f(\emptyset)\) is unbounded. Now force to kill the stationarity of \(Z\). As before, the two-step iteration \(\omega_2\)-Cohen \(*\) (Kill the stationarity of \(Z\)) is equivalent to an \(\omega_2\)-closed forcing. As \(f(G) \triangle f(\emptyset)\) trivially remains unbounded, \((T, l)\) no longer codes a Borel reduction in the extension. □

**Proof of Theorem 22.** Let \((D_\alpha \mid \alpha < \omega_2)\) be the canonical \(\Box\) sequence in \(L\) at \(\omega_2\). Then for each \(X \subseteq \omega_2\), both of the following sets are stationary:

- \(\{ \alpha \in S^{\omega_2}_\omega \mid D_\alpha = X \cap \alpha \}\)
- \(\{ \alpha \in S^{\omega_2}_{\omega_1} \mid D_\alpha = X \cap \alpha \}\)

Now to reduce \(E_0\) to \(E^{\omega_2}_\lambda\) (\(\lambda = \omega\) or \(\omega_1\)) send \(X\) to \(f(X) = \{ \alpha \in S^{\omega_2}_\lambda \mid D_\alpha \\setminus \bar{\alpha} = (X \cap \alpha) \setminus \bar{\alpha}\) for some \(\bar{\alpha} < \alpha\}\). If \(X \triangle Y\) is bounded by \(\alpha\) it follows that \(f(X), f(Y)\) agree on all ordinals in \(S^{\omega_2}_\lambda\) greater than \(\alpha\). If \(X \triangle Y\) is unbounded it follows that \(f(X), f(Y)\) disagree on all cofinality \(\lambda\) limit points \(\alpha\) of \(X \triangle Y\) where \(D_\alpha = X \cap \alpha\). So we get the desired reductions. □

**Remark.** It follows by work of Shelah that assuming GCH, \(E_0\) is Borel reducible to \(E^{\kappa}_\lambda\) except for the case \(\kappa = \mu^+, \lambda = \text{the cofinality of } \mu\). What happens in this case remains open.

**Another application of “code-killing”**

Another special case of our general lemma for killing codes for Borel reductions is the following.

**Lemma 25** Suppose that \(S, T\) are disjoint stationary subsets of \(S^\kappa_\lambda\) and \((T, l)\) is a code for a Borel reduction from \(E_S\) to \(E_T\). Then in a forcing extension by a \(\kappa\)-closed forcing of size \(\kappa\), \((T, l)\) is no longer such a code.
Again, the key point in the proof is that \( S \cap \alpha \) is “nonstationary” for all \( \alpha \in T \), simply because the ordinals in \( S \) and \( T \) all have the same cofinality.

Now we use this to prove:

**Theorem 26** Consistently: There is an injective, order-preserving embedding from \((\mathcal{P}(\kappa), \subseteq)\) into the partial order of \( \Sigma_1^1 \) equivalence relations under Borel reducibility.

It won’t be proved here, but one can in fact get the embedding to have range included in the \( \Delta_1^1 \) equivalence relations. The case of Borel equivalence relations is open:

**Question.** Are there Borel equivalence relations which are incomparable under Borel reducibility (\( \kappa \) uncountable)? Is this at least consistent?

**Proof of Theorem 26.** Let \( (S_i \mid i < \kappa) \) be a partition of \( S_\kappa^\kappa \) into \( \kappa \)-many pairwise disjoint stationary sets. For any \( A \subseteq \kappa \) let \( E(A) \) be the equivalence relation \( E_{X(A)} \) where \( X(A) = \bigcup_{i \in A} S_i \). If \( A \) is a subset of \( B \) then \( E(A) \) is Borel reducible to \( E(B) \) because \( X(A) \) is a subset of \( X(B) \). Now suppose that \( A \) is not a subset of \( B \) and choose \( \alpha \in A \setminus B \). Of course \( E(\{\alpha\}) \) is Borel reducible to \( E(A) \) and \( E(B) \) is Borel reducible to \( E(\kappa \setminus \{\alpha\}) \). So to get the desired conclusion that \( E(A) \) is not Borel reducible to \( E(B) \) it would suffice to know that that \( E(\{\alpha\}) \) is not Borel reducible to \( E(\kappa \setminus \{\alpha\}) \).

By Lemma 25 there is for each \( \alpha \) a \( \kappa \)-closed forcing of size \( \kappa \) which kills a code for a Borel reduction of \( E(\{\alpha\}) \) to \( E(\kappa \setminus \{\alpha\}) \). By iteration, we can kill all such codes for all \( \alpha \) and thereby obtain the desired embedding. \( \Box \)

**Shelah Classification and Higher Descriptive Set Theory**

Let \( T \) be countable, complete and first-order. Then \( T \) is **classifiable** iff there is a “structure theory” for its models. Example: Algebraically closed fields (transcendence degree).

\( T \) is **undeclassifiable** otherwise. Example: Dense linear orderings.

**Shelah’s Characterisation (Main Gap):** \( T \) is classifiable iff \( T \) is superstable without the OTOP and without the DOP.
A classifiable $T$ is *deep* iff it has the maximum number of models in all uncountable powers. Example: Acyclic undirected graphs, every node has infinitely many neighbours.

Another way of classifying theories uses the higher descriptive set theory that we have developed. For simplicity assume GCH and $\kappa = \lambda^+$ where $\lambda$ is uncountable and regular. $\text{Isom}_T^\kappa$ is the isomorphism relation on the models of $T$ of size $\kappa$.

**Theorem 27 (SDF-Hyttinen-Kulikov)**

(a) $T$ is classifiable and shallow iff $\text{Isom}_T^\kappa$ is Borel.

(b) $T$ is classifiable iff for all regular $\mu < \kappa$, $E_{S_\mu}$ is not Borel reducible to $\text{Isom}_T^\kappa$.

(c) In $L$, $T$ is classifiable iff $\text{Isom}_T^\kappa$ is $\Delta^1_1$.

The proof uses Ehrenfeucht-Fraissé games:

**The Game $\text{EF}_t^\kappa(A,B)$**

$A, B$ are structures of size $\kappa$, $t$ is a tree. Player $I$ chooses size $< \kappa$ subsets of $A \cup B$ and nodes along an initial segment of a branch through $t$; player $II$ builds a partial isomorphism between $A$ and $B$ which includes the sets that player $I$ has chosen. Player $II$ wins iff he survives until a cofinal branch is reached.

The tree $t$ captures $\text{Isom}_T^\kappa$ iff for all size $\kappa$ models $A, B$ of $T$, $A \simeq B$ iff Player $II$ has a winning strategy in $\text{EF}_t^\kappa(A,B)$.

Now there are 4 cases:

**Case 1: $T$ is classifiable and shallow.**

Then Shelah’s work shows that some well-founded tree captures $\text{Isom}_T^\kappa$. We use this to show that $\text{Isom}_T^\kappa$ is Borel.

**Case 2: $T$ is classifiable and deep.**

Then Shelah’s work shows that no fixed well-founded tree captures $\text{Isom}_T^\kappa$. We use this to show that $\text{Isom}_T^\kappa$ is not Borel.
Shelah’s work also shows that $L_{\infty\kappa}$ equivalent models of $T$ of size $\kappa$ are isomorphic. This means that the tree $t = \omega$ (with a single infinite branch) captures $\text{Isom}_T^\kappa$. As the games $\text{EF}^\kappa(\mathcal{A}, \mathcal{B})$ are determined, this shows that $\text{Isom}_T^\kappa$ is $\Delta^1_1$.

We must also show: $E_{S\kappa}^\kappa$ (equality modulo the $\mu$-nonstationary ideal) is not Borel reducible to $\text{Isom}_T^\kappa$ for any regular $\mu < \kappa$. This is because (in this case) $\text{Isom}_T^\kappa$ is absolutely $\Delta^1_1$, whereas $\mu$-stationarity is not.

Now we look at the unclassifiable cases. Recall: Classifiable means superstable without DOP and without OTOP.

**Case 3: $T$ is unstable, superstable with DOP or superstable with OTOP.**

Work of Hyttinen-Shelah and Hyttinen-Tuuri shows that in this case no tree of size $\kappa$ without branches of length $\kappa$ captures $\text{Isom}_T^\kappa$. This can be used to show $\text{Isom}_T^\kappa$ is not $\Delta^1_1$.

But $E_{S\kappa}^\kappa \leq_B \text{Isom}_T^\kappa$ is harder. Following Shelah, there is a Borel map $S \mapsto \mathcal{A}(S)$ from subsets of $\kappa$ to Ehrenfeucht-Mostowski models of $T$ built on linear orders so that $\mathcal{A}(S_0) \simeq \mathcal{A}(S_1)$ iff $S_0 = S_1$ modulo the $\lambda$-nonstationary ideal.

**Case 4: $T$ is stable but not superstable.**

This is the hardest case and requires some new model theory. Hyttinen replaces Ehrenfeucht-Mostowski models built on linear orders with primary models built on trees of height $\omega + 1$ to show $E_{S\kappa} \leq_B \text{Isom}_T^\kappa$. (We don’t know if $E_{S\kappa} \leq_B \text{Isom}_T^\kappa$ or if $\text{Isom}_T^\kappa$ could be $\Delta^1_1$ in this case.)

Now we have all we need to prove the Theorem mentioned earlier:

(a) $T$ is classifiable and shallow iff $\text{Isom}_T^\kappa$ is Borel.

We showed that if $T$ is classifiable and shallow then $\text{Isom}_T^\kappa$ is Borel and if it is classifiable and deep it is not. If $T$ is not classifiable then some $E_{S\mu}^\kappa$ Borel reduces to $\text{Isom}_T^\kappa$, so the latter cannot be Borel.

(b) $T$ is classifiable iff for all regular $\mu < \kappa$, $E_{S\mu}^\kappa$ is not Borel reducible to $\text{Isom}_T^\kappa$. 

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We showed that if $T$ is not classifiable then $E_{\kappa_\mu}$ is Borel reducible to $\text{Isom}_T$ where $\mu$ is either $\lambda$ or $\omega$. We also showed that if $T$ is classifiable and deep then no $E_{\kappa_\mu}$ is Borel reducible to $\text{Isom}_T$, by an absoluteness argument. When $T$ is classifiable and shallow there is no such reduction as $\text{Isom}_T$ is Borel.

(c) In $L$, $T$ is classifiable iff $\text{Isom}_T^\kappa$ is $\Delta^1_1$.

We showed that if $T$ is classifiable then $\text{Isom}_T^\kappa$ is $\Delta^1_1$, in ZFC. If $T$ is not classifiable then $E_{\kappa_\mu}$ Borel reduces to $\text{Isom}_T^\kappa$ for some $\mu$, and in $L$, $E_{\kappa_\mu}$ is not $\Delta^1_1$.

13.-14. Vorlesungen

Definability of Isomorphism relations

We assume GCH and fix $\kappa$ to be the successor of a regular and at least $\omega_2$. Then the definability of $\text{Isom}_T^\kappa$ for a countable, complete first-order theory $T$ is closely related to the stability-theoretic properties of $T$. To establish this we first have to understand the definability of $\text{Isom}_T^\kappa$ in terms of Ehrenfeucht-Fraïssé games. In what follows we refer to the relation $\text{Isom}_T^\kappa$ simply as “isomorphism for $T$”. Recall that the tree $t$ captures isomorphism for $T$ iff for any models $A, B$ of $T$ of size $\kappa$, $A$ and $B$ are isomorphic iff $II$ has a winning strategy in the game $EF^\kappa_t(A, B)$.

**Theorem 28** Isomorphism for $T$ is Borel iff some well-founded, rooted tree $t$ of size $\kappa$ captures isomorphism for $T$.

**Theorem 29** (a) If isomorphism for $T$ is $\Delta^1_1$ then some rooted tree $t$ of size $\kappa$ without branches of length $\kappa$ captures isomorphism for $T$.

(b) Let $t^*_{\omega}$ be the ill-founded tree consisting of a single branch of ordertype $\omega$. Then $t^*_{\omega}$ captures isomorphism for $T$ iff any two $L_{\infty, \kappa}$ equivalent models of $T$ of size $\kappa$ are isomorphic, and in this case isomorphism for $T$ is $\Delta^1_1$.

**Proof of Theorem 28.** First suppose that some well-founded, rooted tree $t$ of size $\kappa$ captures isomorphism for $T$; we show that isomorphism for $T$ is Borel. This means that for some well-founded, rooted tree $u$ of size $\kappa$ and labelling $h$ of the terminal nodes of $u$ by open subsets of $\kappa^\kappa \times \kappa^\kappa$, $A_\eta, A_\xi$ are isomorphic iff player $II$ has a winning strategy in the game $G(u, h, (\eta, \xi))$, where $\eta, \xi \in \kappa^\kappa$ are codes for the models $A_\eta, A_\xi$ of $T$ with universe $\kappa$. 

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We take $u$ to be the tree of all sequences $((p_0, A_0), f_0, \cdots, (p_n, A_n), f_n)$ which are valid positions in a game $EF^\kappa_t(A, B)$ where $A, B$ have universe $\kappa$.

I.e., we require:

1. $p_i \in t$ and $A_i \subseteq \kappa$, card$(A_i) < \kappa$.
2. $f_i$ is a partial function of size $< \kappa$ from $\kappa$ to $\kappa$ whose domain and range includes $A_i$.
3. The $p_i$'s form an initial segment of a branch through $t$, the $A_i$'s and $f_i$'s are increasing.

Then for the labelling $h$ we take the value of $h$ on the terminal node $((p_0, A_0), f_0, \cdots, (p_n, A_n), f_n)$ of $u$ to be the set of pairs $(\eta, \xi)$ such that $f_n$ is a partial isomorphism from $A_\eta$ to $A_\xi$. This is an open set because it uses only boundedly much information about the structures $A_\eta, A_\xi$.

It is easy to convert a winning strategy for II in the game $EF^\kappa_t(A_\eta, A_\xi)$ into a winning strategy for II in the game $G(u, h, (\eta, \xi))$ and vice-versa. By hypothesis isomorphism for $T$ is captured by the tree $t$ and therefore we conclude that isomorphism for $T$ is Borel.

For the converse we need a theorem of Lopez-Escobar. We say that $B \subseteq \kappa^\kappa$ is closed under permutations iff whenever $A_\eta$ is isomorphic $A_\xi$, $\eta$ belongs to $B$ iff $\xi$ belongs to $B$.

**Theorem 30** A subset $B$ of $\kappa^\kappa$ is Borel and closed under permutations iff for some sentence $\varphi$ of $L_{\kappa+\kappa}$, $B = \{\eta \mid A_\eta \models \varphi\}$.

Given this, argue as follows. A basic connection between Ehrenfeucht-Fraïssé games and infinitary languages is the following.

**Lemma 31** Let $A, B$ be models of size $\kappa$. Then $A$ and $B$ are $L_{\kappa+\kappa}$ equivalent iff II has a winning strategy in $EF^\kappa_t(A, B)$ for each wellfounded tree $t$ of size $\kappa$. More exactly, $A, B$ satisfy the same sentences of quantifier-rank $< \alpha$ iff II has a winning strategy in the game $EF^\kappa_{t_\alpha}(A, B)$ where $t_\alpha$ is the tree of finite descending sequences through the ordinal $\alpha$.

Now suppose that isomorphism for $T$ is Borel. Then by the (2-dimensional version of the) Lopez-Escobar Theorem there is a sentence $\varphi$ of $L_{\kappa+\kappa}$ such that $A_\eta, A_\xi$ are isomorphic iff the model $(A_\eta, A_\xi)$ satisfies $\varphi$. If we let $\alpha$ be greater
than the quantifier rank of \( \varphi \) then we claim that \( t_\alpha \) captures isomorphism for models of \( T \) (of size \( \kappa \)): Suppose that \( II \) has a winning strategy in \( \text{EF}^*_\kappa(A,B) \). Then clearly \( II \) also has a winning strategy in \( \text{EF}^*_\kappa((A,B),(A,A)) \). It follows by the choice of \( \alpha \) that \((A,B)\) and \((A,A)\) are isomorphic and therefore \( A \) is isomorphic to \( B \).

**Proof of Theorem 30.** The direction from right to left is immediate, as one shows by induction on the rank of \( \varphi \) that for any parameter choice \( \bar{a} = (a_i \mid i < \alpha) \) from \( \kappa \), the set of \( \eta \) such that \( A_\eta \models \varphi(\bar{a}) \) is Borel.

Now we turn to the harder direction, using “Vaught transforms”. Note that the set \( S_\kappa \) of permutations of \( \kappa \) is Borel. For any \( u \in \kappa^{<\kappa} \) we let \( \bar{u} \) denote the set of permutations \( p \in \kappa^\kappa \) such that \( u \subseteq p \). Then for any \( A \subseteq 2^\kappa \) and \( u \in \kappa^{<\kappa} \) define

\[
A^u = \{ \eta \mid \{ p \in \bar{u} \mid \eta \circ p \in A \} \text{ is comeager in } \bar{u} \}.
\]

We let \( Z \) denote the set of Borel \( A \subseteq 2^\kappa \) such that \( A^u \) is \( L_{\kappa^{+\kappa}} \)-definable for all \( u \in \kappa^{<\kappa} \). We show that all Borel sets belong to \( Z \). The desired result then follows by setting \( u = \emptyset \), as \( A^\emptyset = A \) for sets \( A \) closed under permutations.

It is straightforward to show that basic open sets belong to \( Z \). For intersections of size \( \kappa \) use the formula

\[
\bigcap_{i<\kappa}(A_i^u) = (\bigcap_{i<\kappa} A_i)^u.
\]

For complements we use:

\[(*) \quad \eta \in (\sim A)^u \text{ iff for all } v \supseteq u, \eta \notin A^v.\]

**Proof of (*)&:** Here we use the property of Baire for Borel sets. Suppose \( \eta \notin (\sim A)^u \). Then \( B = \{ p \in \bar{u} \mid \eta \circ p \in A \} \) is not meager in \( \bar{u} \) \( \rightarrow \) There is a \( \bar{v} \subseteq \bar{u} \) such that \( B \) is comeager in \( \bar{v} \) \( \rightarrow \) There exists \( v \supseteq u, \eta \in A^v \).

Conversely, if \( \eta \in (\sim A)^u \) then

\[
\{ p \in \bar{u} \mid \eta \circ p \in A \} \text{ is meager } \rightarrow \\
\text{For all } v \supseteq u, \{ p \in \bar{v} \mid \eta \circ p \in A \} \text{ is meager } \rightarrow \\
\text{For all } v \supseteq u, \eta \notin A^v.
\]
This completes the proof of Theorem 30 and therefore also of Theorem 28.

Proof of Theorem 32 (b) Note that the games $EF_{\kappa}^\omega(A, B)$ are closed games for player II and therefore determined. If $t^*_\omega$ captures isomorphism for $T$ and $A, B$ are not isomorphic then I must have a winning strategy in $EF_{\kappa}^\omega(A, B)$ and therefore for some $\alpha$, I will have a winning strategy in the game $EF_{t^\alpha}^\kappa(A, B)$; it follows that $A, B$ cannot be $L_{\kappa+\kappa}$ equivalent. And again, as the games $EF_{t^\alpha}^\kappa(A, B)$ are determined, it follows that isomorphism for $T$ is $\Delta^1_1$.

15.-16. Vorlesungen

Theorem 32 (a) If isomorphism for $T$ is $\Delta^1_1$ then some tree $t$ of size $\kappa$ without branches of length $\kappa$ captures isomorphism for $T$.

(b) Let $t^*_\omega$ be the ill-founded tree consisting of a single branch of ordertype $\omega$. Then $t^*_\omega$ captures isomorphism for $T$ iff any two $L_{\kappa\omega}$ equivalent models of $T$ of size $\kappa$ are isomorphic, and in this case isomorphism for $T$ is $\Delta^1_1$.

We proved (b); we now turn to (a). First we need a lemma.

Lemma 33 Let $Z$ be $\Sigma^1_1$ and write

$$\xi \in Z \text{ iff } t(\xi) \text{ has a } \kappa\text{-branch}$$

where $t$ is a tree on $\kappa^{<\kappa} \times \kappa^{<\kappa}$ and $t(\xi) = \{ p \in \kappa^{<\kappa} \mid (\xi \upharpoonright |p|, p) \in t \}$. If $Z$ is $\Delta^1_1$ then there is a tree $t'$ of size $\kappa$ with no $\kappa$-branch such that

$$\xi \in Z \iff t(\xi) \not\leq t'$$

(where for trees $u$ and $v$, $u \leq v$ means that $u$ embeds into $v$, i.e., there is a strictly order-preserving map from $u$ to $v$).

Proof. If $Z$ is $\Delta^1_1$ then choose $t$ to be a tree of size $\kappa$ such that

$$\xi \notin Z \iff \tilde{t}(\xi) \text{ has a } \kappa\text{-branch}$$

Now form the tree $t'$ consisting of triples $(p, q, r)$ where $p, q, r$ have the same length, $(p, q)$ belongs to $t$ and $(p, r)$ belongs to $\tilde{t}$. Clearly $t'$ has no $\kappa$-branch, else the first components of that branch would yield an element of both $Z$. 

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and its complement. If \( \xi \) does not belong to \( Z \) then choose a branch \( \eta \) through \( t(\xi) \) and embed \( t(\xi) \) into \( t' \) by sending \( q \) to \( (\xi \upharpoonright |\!|q\!|, q, \eta \upharpoonright |\!|q\!|) \). \( \Box \) (Lemma 33)

**Proof of Theorem 32 (a).** Suppose that isomorphism is \( \Delta^1_1 \). We want to produce a tree \( u \) of size \( \kappa \) with no \( \kappa \)-branch that captures isomorphism for \( T \).

Let \( t \) be the tree of partial isomorphisms from \( A_\eta \) to \( A_\xi \). (Recall that \( A_\xi \) is the model coded by \( \xi \).) As isomorphism is \( \Delta^1_1 \) we can apply Lemma 33 to get a tree \( u \) of size \( \kappa \) without \( \kappa \)-branches such that \( A_\eta \simeq A_\xi \) iff \( t(\eta, \xi) \not\in u \). We show that \( u \) captures isomorphism for \( T \). Suppose not and choose \( \xi_1, \xi_2 \) such that \( A_{\xi_1} \) and \( A_{\xi_2} \) are not isomorphic yet \( II \) wins the game \( EF^u_{\kappa}(A_{\xi_1}, A_{\xi_2}) \).

**Claim 1.** Let \( v \) be a tree and let \( \sigma v \) denote the tree of downward-closed linear subsets of \( v \). Then there is no embedding of \( \sigma v \) into \( v \).

**Proof.** Suppose that \( g \) were such an embedding and define

\[
\begin{align*}
x_0 &= g(\emptyset) \\
x_\alpha &= g(\{ y \in v \mid y \leq_v x_\beta \text{ for some } \beta < \alpha \})
\end{align*}
\]

Then the \( x_\alpha \)'s yield a branch of length the ordinals through \( v \), contradiction. \( \Box \)

**Claim 2.** There is an embedding of \( \sigma u \) into \( t(\xi_1, \xi_2) \).

It follows from the Claims that \( t(\xi_1, \xi_2) \) does not embed into \( u \), contradicting the fact that \( A_{\xi_1} \) is not isomorphic to \( A_{\xi_2} \) and the choice of \( u \).

**Proof of Claim 2.** Let \( \tau \) be a winning strategy for \( II \) in \( EF^u_{\kappa}(A_{\xi_1}, A_{\xi_2}) \). We define an embedding \( g \) of \( \sigma u \) into \( t(\xi_1, \xi_2) \) inductively. If \( s \) belongs to \( \sigma u \) and \( g \) is defined for \( s' <_{\sigma u} s \), then play \( EF^u_{\kappa} \) up to \( \sup(s) \in u \). If \( s \) does not contain its sup then put \( g(s) = \cup_{s' <_{\sigma u} s} g(s') \). Otherwise apply \( II \)'s strategy to produce a partial isomorphism \( \nu_\alpha \) of length \( \alpha \) extending \( \cup_{s' <_{\sigma u} s} g(s') \) and set \( g(s) = \nu_\alpha \). \( \Box \)

Theorem 32 (b) is used to show that isomorphism is \( \Delta^1_1 \) for classifiable theories. Theorem 32 (a) shows that it is not \( \Delta^1_1 \) for most unclassifiable theories; the exceptional case is that of “strictly stable” theories, i.e., theories which are stable but not superstable.
It is open whether isomorphism is provably not $\Delta^1_1$ for strictly stable theories. However, for all unclassifiable theories there is some $\lambda < \kappa$ such that $E^\kappa_\lambda$ is Borel reducible to isomorphism, so we get a provable characterisation of classifiability via the next result.

**Theorem 34** If $T$ is classifiable then $E^\kappa_\lambda$ is not Borel reducible to isomorphism for $\lambda < \kappa$.

What we use about classifiability in the following proof is the following characterisation due to Shelah: $T$ is classifiable iff the tree $t^*_\kappa$ captures isomorphism for $T$, i.e., models $A, B$ of $T$ of size $\kappa$ are isomorphic iff $I$ wins the game $EF^\kappa_{t^*_\kappa}(A, B)$. Note that the latter is equivalent to $I$ not having a winning strategy in $EF^\kappa_{t^*_\kappa}(A, B)$, as this game is closed for $II$.

**Proof.** Suppose that $r : 2^\kappa \rightarrow 2^\kappa$ were a Borel reduction of $E^\kappa_\lambda$ to $\text{Isom}^\kappa_T$. So for each $\eta \in 2^\kappa$, $r(\eta)$ is a model of $T$ and $\eta E^\kappa_\lambda \xi$ iff $A_{r(\eta)}$ is isomorphic to $A_{r(\xi)}$. Let $M$ be a transitive, $\kappa$-closed model of ZF$^-\kappa$ containing a code for $r$ and consider $P = \kappa$-Cohen forcing over $M$. Choose a condition $p \in P$ which either forces the models $A_{r(\vec{0})}$ and $A_{r(\dot{g})}$ to be isomorphic or forces them to not be isomorphic, where $\vec{0}$ is the constant sequence with value 0 and $\dot{g}$ denotes the $\kappa$-Cohen generic. Note that since $T$ is classifiable, $II$ has a winning strategy in $EF^\kappa_{t^*_\kappa}(A_{r(\vec{0})}, A_{r(\dot{g})})$ when they are isomorphic, and otherwise $I$ does.

But we can choose $\gamma_0, \gamma_1$ below $p$ to be $\kappa$-Cohen generic over $M$ such that, identifying them with subsets of $\kappa$, $\gamma_0$ is nonstationary and $\gamma_1$ contains a club in the real world. If $p$ forces $A_{r(\vec{0})}$ and $A_{r(\dot{g})}$ to be isomorphic then we get a contradiction using $\gamma_1$, as then $A_{r(\vec{0})}$ and $A_{r(\gamma_1)}$ are isomorphic in $M[\gamma_1]$ and therefore in the real world, whereas $\vec{0}$ and $\gamma_1$ are clearly not $E^\kappa_\lambda$ equivalent. If $p$ forces $A_{r(\vec{0})}$ and $A_{r(\dot{g})}$ to not be isomorphic then we get a contradiction using $\gamma_0$, as then $A_{r(\vec{0})}$ and $A_{r(\gamma_0)}$ are not isomorphic in $M[\gamma_1]$, hence $I$ wins $EF^\kappa_{t^*_\kappa}(A_{r(\vec{0})}, A_{r(\gamma_0)})$ in $M[\gamma_0]$, hence $I$ wins $EF^\kappa_{t^*_\kappa}(A_{r(\vec{0})}, A_{r(\gamma_0)})$ in the real world (as $M$ is closed under $< \kappa$ sequences) and hence $A_{r(\vec{0})}$ and $A_{r(\gamma_0)}$ are not isomorphic in the real world, contradicting the fact that $\vec{0}$ and $\gamma_0$ are $E^\kappa_\lambda$ equivalent. $\square$