Generalisations of Gödel’s $L$

Desirable features of Gödel’s $L$:

a. Definable wellordering (strong form of AC)
b. GCH
c. Jensen’s $\Diamond$, $\Box$ and Morass

The theory $\text{ZFC} + V = L$ is mathematically strong

Problem. For many interesting $\varphi$ in set theory:

$\text{ZFC} + \varphi$ proves $\text{Con}(\text{ZFC})$

But $\text{ZFC} + V = L$ does NOT prove $\text{Con}(\text{ZFC})$

$\text{ZFC} + V = L$ is consistency weak
Large cardinal axioms (LC’s): Inaccessible, measurable, strong, Woodin, superstrong, …

Empirical fact: ZFC+LC’s is consistency strong: For any $\phi$

$$\text{Con}(\text{ZFC} + \text{LC}) \rightarrow \text{Con}(\text{ZFC} + \phi)$$

for some large cardinal axiom LC. In fact:

$$\text{Con}(\text{ZFC} + \text{LC} + \epsilon) \rightarrow \text{Con}(\text{ZFC} + \phi) \rightarrow \text{Con}(\text{ZFC} + \text{LC}),$$

for some large cardinal axiom LC and small $\epsilon$

Q1. Can we combine the mathematical power of $V = L$ with the consistency power of LC’s?

Q2. Are large cardinals needed solely for the analysis of consistency strength, or do they follow from basic logical principles?
Q1: Large cardinals and $L$-like models

*Inner model program:* Show that any model with large cardinals has an $L$-like inner model with large cardinals.

Contributors: Gödel, Silver, Dodd, Jensen, Mitchell, Steel, Neeman and others.

Example 1: Inaccessible cardinals

Easy: If $\kappa$ is inaccessible, then $L \models \kappa$ inaccessible.

Example 2: Measurable cardinals

Scott: $L \models$ There is no measurable cardinal

What inner model shall we use?
Relativised $L$: $\mathcal{L}^E_\alpha = (L^E_\alpha, \in, E_\alpha), \alpha \in \text{Ord}$

$\mathcal{L}^E_0 = (\emptyset, \emptyset, \emptyset)$

$\mathcal{L}^E_{\alpha+1} = (\text{Def}(\mathcal{L}^E_\alpha), \in, E_{\alpha+1})$ (in fact $E_{\alpha+1} = \emptyset$)

$\mathcal{L}^E_\lambda = (\bigcup_{\alpha<\lambda} L^E_\alpha, \in, E_\lambda),$

Desired inner model is $L[\langle E_\alpha \mid \alpha \in \text{Ord} \rangle] = L[E]$. But what is $E$?

First: What is a measurable cardinal?

$\exists \text{ measurable iff } \exists \ j : V \to M$

$[ \ j \text{ is an elementary embedding from } (V, \in) \text{ to } (M, \in) \text{ for some inner model } M, \ j \text{ is not the identity} ]$

Idea: Approximate the class embedding $j : V \to M$ by set embeddings $E_\lambda$. 
Theorem 1. Suppose that there is a measurable cardinal. Then there exists \( E = (E_\alpha \mid \alpha \in \text{Ord}) \) such that:

1. For limit \( \lambda \), \( E_\lambda \) is either empty or an embedding \( E_\lambda : L_\lambda^E \rightarrow L_\lambda^E \) for some \( \alpha < \lambda \).
2. \( L[E] \models \) There is a measurable cardinal.
3. \( E \) is definable over \( L[E] \).
4. **Condensation:** With mild restrictions, \( M \prec L_\alpha^E \) implies \( M \) is isomorphic to some \( L_\alpha^E \).
5. \( L[E] \models \lozenge, \Box \) and (gap 1) Morass

\[ 3 \rightarrow \text{definable wellordering} \]
\[ 4 \rightarrow \text{GCH} \]

Theorem 1 has been generalised after great effort to stronger large cardinal properties.

Why is the Inner Model Program so difficult?
Condensation: $M \prec \mathcal{L}^E_\alpha = (L^E_\alpha, \in, E_\alpha)$ implies $M$ is isomorphic to some $\mathcal{L}^E_\bar{\alpha} = (L^E_\bar{\alpha}, \in, E_\bar{\alpha})$.

With Gödel’s methods, $M$ is isomorphic to some $\mathcal{L}^F_\alpha = (L^F_\alpha, \in, F_\alpha)$

Goal: $\mathcal{L}^F_\alpha = \mathcal{L}^E_\alpha$

Only known technique: Comparison method

Let $\bar{M}$, $\bar{N}$ denote $\mathcal{L}^F_\alpha$, $\mathcal{L}^E_\alpha$. Construct chains of embeddings

$\bar{M} = \bar{M}_0 \rightarrow \bar{M}_1 \rightarrow \bar{M}_2 \rightarrow \cdots \rightarrow \bar{M}_\lambda$
$\bar{N} = \bar{N}_0 \rightarrow \bar{N}_1 \rightarrow \bar{N}_2 \rightarrow \cdots \rightarrow \bar{N}_\lambda$

until $M_\lambda = N_\lambda$. Then conclude that $\bar{M} = \bar{N}$.

Where do the embeddings come from?
\[ \tilde{M} = (L_{\alpha}^F, \in, F_{\alpha}), \text{ where } F = \langle F_\beta \mid \beta < \bar{\alpha} \rangle \]

Choose \( \beta \leq \bar{\alpha} \). Then
\[ F_\beta : L_{\beta}^F \to L_\beta^F. \]

Extend \( F_\beta \) to
\[ F_\beta^* : L_{\alpha}^F \to L_{\alpha^*}^F. \]

Now adjoin the predicate \( F_{\alpha} \) to get
\[ F_\beta^* : \tilde{M} = (L_{\alpha}^F, \in, F_{\alpha}) \to (L_{\alpha^*}^{F^*}, \in, F_{\alpha^*}^{F^*}) = \tilde{M}^*. \]

\[ F_\beta^* : \tilde{M} \to \tilde{M}^* \] is the ultrapower embedding of \( \tilde{M} \) via \( F_\beta \).

Thus the chains
\[ \tilde{M} = \tilde{M}_0 \to \tilde{M}_1 \to \tilde{M}_2 \to \cdots \to \tilde{M}_{\lambda} \]
\[ \tilde{N} = \tilde{N}_0 \to \tilde{N}_1 \to \tilde{N}_2 \to \cdots \to \tilde{N}_{\lambda} \]

are obtained by taking iterated ultrapowers
**Key question:** Is $\tilde{M}$ iterable, i.e., are the models $\tilde{M} = \tilde{M}_0 \rightarrow \tilde{M}_1 \rightarrow \tilde{M}_2 \rightarrow \cdots \rightarrow \tilde{M}_\lambda$ well-founded?

If so, comparison works and Condensation can be proved!

**Iterability problem.** Show that there are iterable structures $M = (L^E_\alpha, \in, E_\alpha)$ which contain large cardinals.

Still open; solved only up to a Woodin limit of Woodin cardinals.
**Outer model program.** Show that any model with large cardinals has an $L$-like outer model with large cardinals.

Are there any (proper) outer models?

Treat $V$ as a *countable* transitive model of GB (Gödel-Bernays class theory)

**Outer model** of $V = a$ countable transitive model of GB which contains all the sets and classes of $V$

By forcing, $V$ has many outer models

The *inner model program* has reached Woodin limits of Woodin cardinals.

But the *outer model program* has gone *all the way!*
Theorem 2. Suppose that there is a superstrong cardinal. Then there exists an outer model \( L[A] \) of \( V \) (obtained by forcing) such that:

1. \( A \) is a class of ordinals.
2. \( L[A] \models \) There is a superstrong cardinal.
3. \( A \) is definable over \( L[A] \).
4. **Condensation:** With mild restrictions, \( M \prec (L_\alpha[A], \in, A \cap \alpha) \) implies \( M \) is isomorphic to some \( (L_\beta[A], \in, A \cap \beta) \).
5. \( L[A] \models \diamondsuit, \Box \) and (gap 1) Morass

3 \( \rightarrow \) definable wellordering

4 \( \rightarrow \) GCH

What is a superstrong cardinal?
Suppose $j : V \to M$.

Critical point of $j = \text{least ordinal } \kappa$ such that $j(\kappa) \neq \kappa$.

$j$ is $\alpha$-strong iff $V_\alpha \subseteq M$

Superstrong = $j(\kappa)$-strong
Hyperstrong = $j(\kappa) + 1$-strong
$n$-superstrong = $j^n(\kappa)$-strong
$\omega$-superstrong = $j^\omega(\kappa)$-strong

$j^\omega(\kappa) + 1$-strong is inconsistent!

$\omega$-superstrong is at the edge of inconsistency

$\kappa$ is $n$-superstrong iff $j$ is $n$-superstrong
(similarly for hyperstrong, $\omega$-superstrong)
Hyperstrong → □ fails

Theorem 3. With □ omitted, Theorem 2 holds for ω-superstrong

Conclusion:
\textit{L-like} is consistent with superstrong
\textit{L-like without □} is consistent with all large cardinals
Q2: The inner model hypothesis

Inner model hypothesis. If a sentence $\varphi$ holds in an inner model of some outer model of $V$ (i.e., in some model compatible with $V$), then it already holds in some inner model of $V$.

The IMH implies that there are no large cardinals in $V$:

Theorem 4. The IMH implies that for some real $R$, there is no transitive set model of ZFC containing $R$. In particular, there are no inaccessible cardinals and the Singular Cardinal Hypothesis is true.
The IMH implies however that there are large cardinals in inner models:

Theorem 5. The IMH implies the existence of an inner model with measurable cardinals of arbitrarily large Mitchell order.

The IMH is consistent relative to large cardinals:

Theorem 6. The consistency of the IMH follows from the consistency of a Woodin cardinal with an inaccessible cardinal above it.
The strong inner model hypothesis

Fact: The IMH with arbitrary ordinal parameters or with arbitrary real parameters is inconsistent.

The parameter $p$ is \textit{(globally) absolute} iff there is a parameter-free formula which has $p$ as its unique solution in all outer models of $V$ with the same cardinals as $V$ up to $\text{hcard}\,(p)$, the cardinality of the transitive closure of $p$.

\textit{Strong inner model hypothesis}. Suppose that $p$ is absolute, $V^*$ is an outer model of $V$ with the same cardinals $\leq \text{hcard}\,(p)$ as $V$ and $\varphi$ is a sentence with parameter $p$ which holds in an inner model of $V^*$. Then $\varphi$ holds in an inner model of $V$.
The SIMH solves the continuum problem:

Theorem 7. Assume the SIMH. Then CH is false. In fact, $2^{\aleph_0}$ cannot be absolute and therefore cannot be $\aleph_\alpha$ for any ordinal $\alpha$ which is countable in $L$.

The SIMH implies that there are very large cardinals in inner models:

Theorem 8. The SIMH implies the existence of an inner model with a strong cardinal.

Is the SIMH consistent relative to large cardinals?
Gödel

Referring to maximum principles in set theory, Gödel said:

"I believe that the basic problems of abstract set theory, such as Cantor’s continuum problem, will be solved satisfactorily only with the help of axioms of this kind."

I think that Gödel would have liked the Inner Model Hypothesis!

But will the IMH be adopted by the set theory community?

Time will tell...