The $\Pi_2^1$-Singleton Conjecture: An Introduction
Sy D. Friedman
MIT

Solovay conjectured that there is a $\Pi_2^1$-Singleton $R$ such that $0 <_L R <_L 0^\#$. My purpose here is to provide a setting for this conjecture and to give an idea of its proof. This article is intended for the non-specialist; I will do my best to explain all the basic notions.

I will begin with the following result of Cohen:

1) There can be a nonconstructible real.

By this, I mean that $\exists R \subseteq \omega (R \notin L)$ is consistent with ZFC. It is natural to ask for something stronger:

2) Can there be a definable nonconstructible real?

This was answered affirmatively by McAloon, who showed using Cohen's methods that $R = \{ n | 2^n = \aleph_{n+1} \}$ can be nonconstructible. For our present purposes a more definitive result is due to Silver and Solovay:

3) If there is a measurable cardinal then $0^\# = Thy(L, \aleph_1, \aleph_2, \ldots)$ is a definable nonconstructible real.

By this we mean that the set of Gödel numbers of formulas $\phi(x_1 \ldots x_n)$ in the language of set theory such that $(L, \in) \models \phi(\aleph_1, \ldots, \aleph_n)$ is the definable, nonconstructible real $0^\#$. When we write $\aleph_1, \aleph_2, \ldots$ we mean the first $\omega$-many uncountable cardinals of $V = \text{the real world}$, not of $L = \text{the constructible universe}$. In fact if there is a measurable cardinal then $\aleph_1, \aleph_2, \ldots$ are all countable ordinals.

It is not difficult to generate other examples of definable, nonconstructible reals:

4) $\omega - 0^\#$ is definable, nonconstructible.

5) $0^{\#\#} = Thy(L[0^\#], \aleph_1, \aleph_2, \ldots)$ is definable, nonconstructible.

We would like to eliminate examples as in 4), by modding out by a suitable equivalence relation. Define $R \leq_L S$ iff $R \in L[S], R <_L S$ iff $(R \leq_L S, S \notin L R), R =_L S$ iff $(R \leq_L S, S \leq_L R)$. Then $=_L$ is an equivalence relation.

Solovay's conjecture addresses the following question: Are $0^\#, 0^{\#\#}, 0^{\#\#\#}, \ldots$ the only "canonical" nonconstructible reals, up to $=_L$? Specifically: Is $0^\#$ the $\leq_L$-least "canonical" nonconstructible real? We must be careful with the word "canonical". If we simply take this to mean "definable" we have a counterexample:

6) There is a real $R \leq_L 0^\#$ which is Cohen generic over $L$. Now $0^\#$ is not $=_L$-equivalent
to any real Cohen generic over $L$. Thus $R_\# = (L[0#]-$least real Cohen generic over $L$)
is a definable, nonconstructible real and $0# \notin L[R_\#]$. But $R_\#$ is not really "canonical", because to define $R_\#$ we need to refer to $0#$ and $0#$ is not constructible from $R_\#$. Thus we are looking for reals defined in a more
absolute way, as follows:

Definition $R$ is a \textit{Solovay Singleton} if for some formula $\phi(x)$ with ordinal parameters, $R$ is the unique real $R$ such that $L[R] \models \phi(R)$.

Now $0#$ is a Solovay Singleton of a special type:

Definition \ A \Pi^1_1$-formula is a formula of the form $\forall S \exists T \psi$ where $\psi$ is arithmetical and $S, T$ range over reals. $R$ is a \Pi^1_2$-Singleton if $R$ is the unique solution to a \Pi^1_2$ formula.

Levy–Shoenfield absoluteness implies that:

7) $R$ is a $\Pi^1_2$-Singleton $\iff R$ is a Solovay Singleton via a formula $\phi(x)$ that is $\Pi_1$ in the Levy hierarchy and has no ordinal parameters.

Thus a $\Pi^1_2$-Singleton is the simplest type of Solovay Singleton that could be noncon-
structible.

8) $0#$ is a $\Pi^1_2$-Singleton.

\textit{Solovay's $\Pi^1_2$-Singleton Conjecture} There is a $\Pi^1_2$-Singleton $R$, $0 < L R < L 0#$.

Results

Theorem 1 There is a $\Pi^1_2$-Singleton $R$, $0 < L R < L 0#$.

A question related to the $\Pi^1_2$-Singleton Conjecture is due to Kechris. A set $X$ is
$\Pi^1_1$ if $X = \{R|\phi(R)\}$ where $\phi$ is a $\Pi^1_2$ formula.

Question Does every nonempty countable $\Pi^1_2$ set contain a $\Pi^1_2$-singleton?

Theorem 2 There is a nonempty countable $\Pi^1_2$ set containing no
$\Pi^1_2$-singleton.

On the other hand we have:

Theorem (Harrington–Kechris) If $X$ is a nonempty countable $\Pi^1_2$ set then $X$ contains
a Solovay Singleton, defined using parameters $\aleph_1, \ldots, \aleph_n$ for some $n$.

Theorem 2' For each $n$ there is a nonempty countable $\Pi^1_2$ set $X_n$ such that no
element of $X_n$ is a Solovay Singleton defined using parameters $\aleph_1, \ldots, \aleph_n$. Thus the
Harrington–Kechris result is best possible.

On the Proof of Theorem 1

$R$ arises as the generic real for an $L$-definable forcing $\mathcal{P}$. This is a \textit{class forcing}: $\mathcal{P}$ is
not a set. $\mathcal{P}$ is built out of three types of forcings:

1) Jensen Coding: This enables $R$ to code a class of information.
2) Backwards Easton Forcing: This is used to add certain CUB subsets to L-inaccessible cardinals. Such a forcing is an iteration $\mathcal{P}_\alpha \ast \mathcal{P}_1 \ast \mathcal{P}_2 \ast \cdots \ast \mathcal{P}_\alpha \ast \cdots$ through the ordinals with the restriction that the support of any condition is bounded in any L-inaccessible.

3) Set Forcing: This helps to make $R$ a $\Pi^1_2$-Singleton, via a trick of Solovay.

The existence of $0^#$ entails the existence of a canonical CUB class of indiscernibles $I$ for $(L, \epsilon)$ such that $L = \Sigma_1$ Skolem hull $(I)$ in $(L, \epsilon)$. Our desired $\mathcal{P}$-generic $G \subseteq \mathcal{P}$ arises in a very natural way from $I : G = \{p|p(i_1, \ldots, i_n) \leq p \text{ for some } i_1, \ldots, i_n \in I\}$ where $(i_1, \ldots, i_n) \rightarrow p(i_1, \ldots, i_n)$ is a $\Sigma_1(L, \epsilon)$ procedure.

Our goal is to show that there is only one $\mathcal{P}$-generic. This is where the Backwards Easton forcing comes in. There is a method of adding CUB sets (in a Backwards Easton fashion) for the purpose of "killing" guesses $(i_1 \ldots i_n)$ at an $n$-tuple of indiscernibles. No correct guess $(i_1 \ldots i_n) \in I^n$ can be killed. Our forcing is set up so that the generic $G$ will kill any guess $(i_1 \ldots i_n)$ which via the $\Sigma_1(L, \epsilon)$ procedure above produces information $p(i_1 \ldots i_n)$ contradicting $G$. So there can only be our one generic $G$, as another generic $H$ would have to kill the correct guesses $(i_1 \ldots i_n) \in I^n$ which produce $p(i_1 \ldots i_n) \notin H$.

The other key ingredient in the proof is the Recursion Theorem. To create the forcing $\mathcal{P}$ we need a $\Sigma_1$ index for the procedure $(i_1 \ldots i_n) \rightarrow p(i_1 \ldots i_n)$ in order to know which guesses $(i_1 \ldots i_n)$ to kill. But this procedure in turn cannot be defined explicitly without knowing the forcing $\mathcal{P}$! This circularity is dealt with by using the Recursion Theorem to obtain the desired index.

A similar technique can be used to establish Theorem 2.
References


Harrington–Kechris [77]  “$\Pi^1_2$-Singletons and $0^\#$, Fundamenta Mathematica.