Post's Problem without Admissibility

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INTRODUCTION

This paper is a contribution to \( \beta \)-recursion theory; i.e., recursion theory on arbitrary limit ordinals. The basic definitions, motivation, and results were set forth in [4]. In brief, the setting for \( \beta \)-recursion theory is \( S_\beta \), the \( \beta \)th level of Jensen's \( S \)-hierarchy for \( L \) (see [1, p. 82]) and \( \beta \)-recursively enumerable (\( \beta \)-r.e.) sets are those subsets of \( S_\beta \) which are \( \Sigma_1 \)-definable over \( \langle S_\beta, \epsilon \rangle \). \( A \subseteq S_\beta \) is \( \beta \)-recursive if both \( A \) and \( S_\beta - A \) are \( \beta \)-r.e. and is \( \beta \)-finite if \( A \in S_\beta \). As in \( \alpha \)-recursion theory, there is a \( \beta \)-r.e. enumeration \( \{W_\epsilon\}_{\epsilon \in \beta} \) of the \( \beta \)-r.e. sets and using this we define: \( A \) is weakly \( \beta \)-recursive in \( B \) (\( A \in \text{ws} B \)) if for some \( \epsilon \)

\[ x \in A \iff \exists K_1 \exists K_2 [(0, x, K_1, K_2) \in W_\epsilon \text{ and } K_1 \subseteq B \text{ and } K_2 \subseteq S_\beta - B], \]

\[ x \in S_\beta - A \iff \exists K_1 \exists K_2 [(1, x, K_1, K_2) \in W_\epsilon \text{ and } K_1 \subseteq B \text{ and } K_2 \subseteq S_\beta - B], \]

where \( K_1, K_2 \) vary over \( \beta \)-finite sets. \( A \) is \( \beta \)-recursive in \( B \) (\( A \leq_\beta B \)) if \( \{Z \in S_\beta \mid Z \subseteq A\} \) and \( \{Z \in S_\beta \mid Z \subseteq S_\beta - A\} \) are both weakly \( \beta \)-recursive in \( B \). \( \leq_\beta \) is transitive and the \( \beta \)-degree of \( A = \{B \mid A \leq_\beta B, B \leq_\beta A\} \).

The program of \( \beta \)-recursion theory is to generalize theorems of ordinary recursion theory to arbitrary limit ordinals. This paper focuses on the \( \beta \)-degrees of \( \beta \)-r.e. sets and provides a partial solution to:

**Post's Problem.** Show that there are \( \leq_{\omega_\beta} \)-incomparable \( \beta \)-r.e. sets.

Post's problem was solved in ordinary recursion theory by Friedberg [2] and Muchnik [10] independently, and generalized to \( \alpha \)-recursion theory (i.e., recursion theory on admissible ordinals) by Sacks and Simpson [14]. (Purely "syntactic" techniques suffice to solve the following weaker formulation of Post's problem for every inadmissible \( \beta \): Find a \( \beta \)-r.e. set of \( \beta \)-degree strictly between 0 and 0\(^*\), the largest \( \beta \)-degree of a \( \beta \)-r.e. set. See Theorem 3.4 of [4].)

Sacks and Simpson used ideas from the structure of \( L \) devised by Gödel to prove the Generalized Continuum Hypothesis in \( L \) [6]. We extend the application of techniques from the fine structure of \( L \) to recursion theory by making use of Jensen's \( \diamond \)-principle in our solution to Post's problem. An effectivized version of Fodor's theorem is also developed and used in our proof. For background material on \( \diamond \) and stationary sets, see [1].
In Section 1, Fodor’s theorem, ◊, and other facts about stationary sets are reviewed. Section 2 uses these ideas to solve Post’s problem in the special case when $\beta^* = \aleph_1^\kappa$. Finally, in Section 3 the material of the first section is effectivized and applied to the case where $\beta^*$ is only regular with respect to functions $\Sigma_1$ over $S_\beta$. Thus Post’s problem is solved if $\beta^*$ is a successor $\beta$-cardinal. Also, the generalization of the construction to $\Sigma_\alpha$ predicates is discussed in Section 4.

1. Review of Stationary Sets and the ◊-Principle

We begin by recalling some basic definitions and facts from combinatorial set theory.

**Definition.** Let $\kappa > \omega$ be a regular cardinal. $C \subseteq \kappa$ is closed if for all $\gamma < \kappa$, $C \cap \gamma$ unbounded in $\gamma \rightarrow \gamma \in C$. $S \subseteq \kappa$ is stationary if $S \cap C \neq \emptyset$ for every closed unbounded $C \subseteq \kappa$.

**Proposition 1.** Suppose $\{C_\alpha\}_{\alpha < \gamma}$ is a collection of closed unbounded sets and $\gamma < \kappa$. Then $C = \bigcap_{\alpha < \gamma} C_\alpha$ is closed and unbounded.

**Proof.** Let $\beta_{0,0}$ be arbitrary. Inductively, define

$\beta_{n,0} = \text{some member of } C_0 \text{ greater than } \bigcup_{\alpha < \gamma} \beta_{n-1,\alpha}$

$\beta_{n,\delta} = \text{some member of } C_\delta \text{ greater than } \bigcup_{\alpha < \delta} \beta_{n,\alpha} \text{ if } \delta > 0$.

Then $\beta = \bigcup_{n,\delta} \beta_{n,\delta} = \bigcup_{n} \beta_{n,\alpha}$ for all $\alpha$. Thus $C$ is unbounded. But $C \cap \delta$ unbounded in $\delta$

$\rightarrow C_\alpha \cap \delta$ unbounded in $\delta$, $\forall \alpha$

$\rightarrow \delta \in C_\alpha$ for all $\alpha$

$\rightarrow \delta \in C$.  

**Corollary 2.** If $C \subseteq \kappa$ is closed and unbounded, then $C$ is stationary.

**Proof.** Just apply Proposition 1 when $\gamma = 2$.

**Fodor’s Theorem.** Suppose $S \subseteq \kappa$ is stationary, $f : S \rightarrow \kappa$, and $f(\alpha) < \alpha$ for every $\alpha \in S$ (that is, $f$ is regressive on $S$). Then for some $\gamma$, $\{\alpha \in S \mid f(\alpha) < \gamma\}$ is a stationary subset of $S$.

**Proof.** Suppose not. Let $\gamma_0 < \kappa$ be arbitrary. Choose a closed unbounded $C_0$ so that $\alpha \in C_0 \cap S \Rightarrow f(\alpha) \geq \gamma_0$. Let $\gamma_0 < \gamma_1 \in C_0$. Choose a closed unbounded $C_1 \subseteq C_0$ so that $\alpha \in C_1 \cap S \Rightarrow f(\alpha) \geq \gamma_1$ (by Proposition 1). Choose
γ₀ < γ₁ < γ₂ ∈ C₁. Continuing in this way define γ₀ < γ₁ < ... < C₀, C₁, ...
so that for all β < κ

(a) γₙ ∈ Cₙ for all β′ < β,
(b) β limit → γₙ = ⋃ₙ<β γₙ′, Cₙ = ⋂ₙ<β Cₙ′,
(c) α ∈ Cₙ ∩ S → f(α) ≥ γₙ.

This is possible as β limit → γₙ ∩ Cₙ′ unbounded in γₙ for β′ < β → γₙ ∈ Cₙ′. Then γₙ ∈ S for some limit β, as S is stationary and {yₙ | β limit} is closed, unbounded. But then γₙ ∈ Cₙ and so f(γₙ) ≥ γ₉ by (c), contradicting the hypothesis that f is regressive on S. 1

Jensen's ◇-principle (see [7]) is a strong axiom true in Gödel’s L which can be used to diagonalize over subsets of κ in only κ-many steps. Jensen used it to construct a Souslin Tree in L.

Let κ be a regular cardinal. ◇ says: There is a sequence <Sα | α < κ> such that

(i) Sα ⊆ α for each α,
(ii) If X ⊆ κ, then {α | Sα = X ∩ α} is stationary.

Proposition 3. (a) ◇κ⁺ implies that 2κ = κ⁺.
(b) ◇κ implies that there is a collection {Xα | α < κ} of stationary subsets of κ such that α₁ ≠ α₂ implies Xα₁ ∩ Xα₂ = ∅.

Proof. (a) Define f: 2κ → κ⁺ by f(Y) = least α > κ such that Sα = Y. Then f is well defined and 1-1.
(b) For each α < κ, let Xα = {β > α | Sβ = {α}}. Then each Xα is stationary and the Xα's are pairwise disjoint. 1

Theorem 4 (Jensen). ◇κ is true in L. Moreover, there is a ◇κ-sequence which is Σ₁-definable over <Lκ, ∈>.

Proof. We define Sα by induction on α. S₀ = ∅, Sα+1 = ∅ for all α. If Sα has been defined for all α < λ, λ limit, we define Sα as follows: Let <X, C> be the least (in the canonical well-ordering of L) pair so that

(i) X, C ⊆ λ,
(ii) C is closed, unbounded in λ,
(iii) α ∈ C → Sα ≠ X ∩ α.

Define Sλ = X if such a pair exists, and Sα = ∅ otherwise. Clearly, <Sα | α < κ> is Σ₁-definable over <Lκ, ∈>.
We claim that \( \langle S_\alpha \mid \alpha < \kappa \rangle \) satisfies \( \diamond \). If not, let \( \langle X, C \rangle \) be the least (in the canonical well-ordering of \( L \)) pair so that

(i) \( X, C \subseteq \kappa \),
(ii) \( C \) is closed, unbounded in \( \kappa \),
(iii) \( \alpha \in C \rightarrow S_\alpha \neq X \cap \alpha \).

Now, \( \langle X, C \rangle \in L_{\kappa^+} \). Let \( M < L_{\kappa^+} \) so that

(a) \( M \) has \( (L-) \) cardinality \( \kappa \).
(b) \( \kappa, \langle X, C \rangle \in M, K \subseteq M \) is an ordinal.

By Gödel Condensation, there is a unique \( \pi: \langle M, \in \rangle \cong \langle L_\alpha, \in \rangle, \alpha < \kappa \). Then let \( \pi(\kappa) = \lambda \). It is easily seen that \( \pi \mid \lambda = \text{id} \mid \lambda \), so \( \pi(\langle X, C \rangle) = \langle X \cap \lambda, C \cap \lambda \rangle \). Moreover, \( \pi(S_\alpha) = S_\alpha \) for \( \alpha < \lambda \). Thus, since \( \pi \) is an isomorphism,

\( \langle X \cap \lambda, C \cap \lambda \rangle \) is the least (in the canonical well-ordering of \( L \)) pair so that

(i) \( X \cap \lambda, C \cap \lambda \subseteq \lambda \),
(ii) \( C \cap \lambda \) is closed, unbounded in \( \lambda \),
(iii) \( \alpha \in C \cap \lambda \rightarrow S_\alpha \neq X \cap \lambda \cap \alpha = X \cap \alpha \).

But then by definition, \( S_\lambda = X \cap \lambda \). But since \( C \) is closed and \( C \cap \lambda \) is unbounded in \( \lambda, \lambda \in C \). This contradicts \( \alpha \in C \rightarrow S_\alpha \neq X \cap \alpha \).

Let \( E \) be a stationary subset of \( \kappa \). We relativize \( \diamond \) to \( E \). \( \diamond(E) \) says: There is a sequence \( \langle S_\alpha \mid \alpha \in E \rangle \) so that

(i) \( S_\alpha \subseteq \alpha \) for \( \alpha \in E \),
(ii) If \( X \subseteq \kappa \), then \( \{ \alpha \in E \mid S_\alpha = X \cap \alpha \} \) is a stationary subset of \( E \).

**Theorem 5 (Jensen).** \( \diamond(E) \) is true in \( L \). Moreover, there is a \( \diamond(E) \)-sequence which is \( \Sigma_1 \)-definable over \( \langle L_\kappa, \in, E \rangle \).

**Proof.** Define \( S_\alpha \) for \( \alpha \in E \) by induction on \( \alpha \). \( S_0 = \emptyset \). \( S_{\alpha+1} = \emptyset \) for every \( \alpha \). If \( S_\alpha \) has been defined for \( \alpha < \lambda, \lambda \) limit, let \( \langle X, C \rangle \) be the least pair such that

(i) \( X, C \subseteq \lambda \),
(ii) \( C \) is closed, unbounded in \( \lambda \),
(iii) \( \alpha \in C \cap E \rightarrow S_\alpha \neq X \cap \alpha \).

If such a pair exists, let \( S_\lambda = X \); \( S_\lambda = \emptyset \) otherwise. Clearly \( \langle S_\alpha \mid \alpha \in E \rangle \) is \( \Sigma_1 \)-definable over \( \langle L_\kappa, \in, E \rangle \).
Suppose \( \langle S_\alpha \mid \alpha \in E \rangle \) does not satisfy \( \diamondsuit_\kappa(E) \). Let \( \langle X, C \rangle \) be the least pair so that

(i) \( X, C \subseteq \kappa \),

(ii) \( C \) is closed, unbounded in \( \kappa \),

(iii) \( \alpha \in C \cap E \rightarrow S_\alpha \neq X \cap \alpha \).

We define a \( \kappa \)-sequence \( M_0 < M_1 < \cdots < M_\lambda < \cdots \) of elementary sub-models of \( L_{\kappa^+} \) as follows:

\[ M_0 = \text{an elementary submodel of } L_{\kappa^+} \text{ such that } E, X, C, \kappa \in M_0 \text{ and } M_0 \cap \kappa = \text{an ordinal less than } \kappa, \]

\[ M_{\alpha+1} = \text{an elementary submodel of } L_{\kappa^+} \text{ such that } M_\alpha \cup \{ M_\alpha \} \subseteq M_{\alpha+1} \text{ and } M_{\alpha+1} \cap \kappa = \text{an ordinal less than } \kappa, \]

\[ M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha \text{ for limit } \lambda. \]

Let \( \beta_\alpha = M_\alpha \cap \kappa \). Then \( \{ \beta_\alpha \mid \alpha < \kappa \} \) is a closed unbounded subset of \( \kappa \), so there is a \( \lambda = \beta_\alpha \in E \), since \( E \) is stationary. Let \( \pi: M_\lambda \prec L_\nu \). Then \( \pi(E) = E \cap \lambda \), \( \pi(\langle X, C \rangle) = \langle X \cap \lambda, C \cap \lambda \rangle \) since \( \pi(\kappa) = \beta_\alpha = \lambda \). So \( \langle X \cap \lambda, C \cap \lambda \rangle \) is the least pair so that

(i) \( X \cap \lambda, C \cap \lambda \subseteq \lambda \),

(ii) \( C \cap \lambda \) is closed, unbounded in \( \lambda \),

(iii) \( \alpha \in C \cap \lambda \cap E \cap \lambda \rightarrow S_\alpha \neq X \cap \lambda \cap \alpha \).

But then \( S_\lambda = X \cap \lambda \) by definition and \( \lambda \in C \). As before this contradicts the choice of \( \langle X, C \rangle \). \( \blacksquare \)

2. Post's Problem When \( \beta^* = \aleph_1^L \)

The theory of stationary sets and \( \diamondsuit \) can be useful in a priority construction. In this section we prove

**Theorem 6.** Suppose \( \beta \) is a limit ordinal and \( \beta > \beta^* = \aleph_1^L \). Then there exist \( \beta \)-r.e. sets \( A, B \subseteq \aleph_1^L \) such that \( A \leq_{w^A} B, B \leq_{w^B} A \).\(^1\)

**Proof.** As in any construction to solve Post's problem, we wish to satisfy the requirements

\[ R^A_\beta: \aleph_1^L \setminus B \neq W^A_\epsilon, \quad R^B_\beta: \aleph_1^L \setminus A \neq W^B_\epsilon, \quad \epsilon \in S_\beta. \]

\(^1\) The argument that we present will also handle the case \( \beta = \aleph_1^L \), or \( \beta^* = \) any successor \( L \)-cardinal.
Here, $W_e^A = \text{eth set } \beta\text{-r.e. in } A$. If $\{W_e\}_{e \in \beta}$ is a uniformly $\beta$-r.e. listing of the $\beta$-r.e. sets, then

$$W_e^A = \{x \mid \exists x_1 \exists x_2[x_1, x_2 \in S_\beta, \langle x, x_1, x_2 \rangle \in W_e \text{ and } x_1 \subseteq A, x_2 \subseteq \beta - A]\}.$$  

In this construction, there will be added requirements to ensure that $A, B$ are simple and weakly-$\text{t.r.e.}$

**Definitions.** $A \subseteq \beta^*$ is **simple** if $A$ is $\beta$-r.e., $\beta^* - A$ is unbounded in $\beta^*$, and $A$ intersects any $\beta$-r.e. $B$ which is unbounded in $\beta^*$.

$A \subseteq \beta$ is **tamely-$\text{t.r.e.}$** ($\text{t.r.e.}$) if $\{z \in S_\beta \mid z \subseteq A\}$ is $\beta$-r.e. $A$ is **weakly-$\text{t.r.e.}$** if $\{z \in S_\beta \mid z - A \text{ is bounded in } \beta^*\}$ is $\beta$-r.e.

The requirements for simplicity are

**$S_e^A$:** $W_e$ unbounded in $S_1^{\beta L} \rightarrow A \cap W_e \neq \emptyset$,

**$S_e^B$:** $W_e$ unbounded in $S_1^{\beta L} \rightarrow B \cap W_e \neq \emptyset$, $e \in S_\beta$.

Of course, we must also guarantee that $\beta^* - A$ is unbounded in $\beta^*$.

$\text{t.r.e.-ness}$ is automatic when $\beta$ is $\Sigma_1$-admissible. However, we cannot hope to build $\text{t.r.e.}$ sets when $\Sigma_1 \text{eff} \beta = \omega$ (see [4, p. 36]). As a result, our attempts at $R_e^A$ will not only keep ordinals out of $A$, but put ordinals into $A$. Weakly-$\text{t.r.e.-ness}$ is essential in that it allows the positive part of these attempts to be countable. The requirements for weakly-$\text{t.r.e.-ness}$ are

**$T_e^A$:** $e - A$ bounded in $S_1^{\beta L} \rightarrow \exists \text{ stage } \sigma[e - A^\sigma \text{ is bounded in } S_1^{\beta L}]$,

**$T_e^B$:** $e - B$ bounded in $S_1^{\beta L} \rightarrow \exists \text{ stage } \sigma[e - B^\sigma \text{ is bounded in } S_1^{\beta L}]$  

for $e \in S_\beta$. $A^\sigma$ = the part of $A$ enumerated by stage $\sigma$ of the construction.

**Remark about stages.** Unless $\omega^\beta = \beta$ (for example, $\beta$ primitive-recursively closed), one cannot naturally identify members of $S_\beta$ with ordinals less than $\beta$. However, there is always a canonical $\beta$-recursive well-ordering $\prec \beta$ of $S_\beta$ with the property that $\text{pr}_\beta(x) = \{z \mid z \prec \beta x\}$ is a $\beta$-recursive function from $S_\beta$ into $S_\beta$ (see [1, p. 84]). Accordingly, we can identify stages $\sigma$ with members of $S_\beta$, viewed as positions in the canonical well-ordering $\prec \beta$.

Let $f: S_\beta \rightarrow S_1^{\beta L}$ be $\beta$-recursive and $1\text{-}1$. $f$ can be used to arrange the above requirements into a list of order type $S_1^{\beta L}$, assigning each requirement a "priority" less than $S_1^{\beta L}$. For example, we can assign:

$$f'(R_e^A) = f(\langle 0, e \rangle), \quad f'(R_e^P) = f(\langle 1, e \rangle),$$

$$f'(S_e^A) = f(\langle 2, e \rangle), \quad f'(S_e^P) = f(\langle 3, e \rangle),$$

$$f'(T_e^A) = f(\langle 4, e \rangle), \quad f'(T_e^P) = f(\langle 5, e \rangle).$$
This assigns a single priority to each requirement. However, this is not good enough for our purposes. We would like to allow each requirement to have a stationary set of different priorities.

By Theorem 4, let \( \langle S_\alpha \mid \alpha < R_1 \rangle \) be a \( \beta \)-finite \( \diamondsuit_{R_1} \)-sequence. As in Proposition 3(b), \( \{ \alpha \mid S_\alpha = \{ y \} \} \) is stationary for each \( y < R_1 \). Define \( g : R_1 \rightarrow \{ \text{Requirements} \} \) by \( g(\delta) = R \) if and only if \( S_\delta = \{ f'(R) \} \); \( g(\delta) \) undefined if no such \( R \) exists. Then each requirement \( R \) has the stationary set of priorities \( g^{-1}(\{ R \}) \). Note that \( f' \circ g \) is partial \( \beta \)-recursive.

Thus each requirement \( R \) can be viewed as the whole stationary collection of auxiliary requirements \( R_\delta \), \( g(\delta) = R \). The reason for arranging this is that unlike priority arguments in the admissible case, we will not be able to guarantee that every auxiliary requirement \( R_\delta \) can eventually be satisfied. Instead, we can argue (rather easily) that a closed unbounded collection of auxiliary requirements \( R_\delta \) will be satisfied and in fact will never be injured. Then since \( g^{-1}(\{ R \}) \) is stationary, there will be an auxiliary requirement \( R_\delta \), \( g(S) = R \) which will be satisfied and hence so will \( R \).

We now describe how an attempt is made at an auxiliary requirement \( R_\delta \). All attempts will be of the form \( (K, H) \) where \( K, H \in L_{R_1} \); if \( (K, H) \) is an \( A \)-attempt then the members of \( K \) will be put into \( A \) by the attempt, and the attempt tries to preserve \( A \cap H = \emptyset \). Similarly for \( B \)-attempts. In addition, if \( (K, H) \) is an attempt at \( R_\delta \), then \( K \cup H \) will contain no ordinals \( < \delta \). Finally (much in contrast to the admissible case):

1. At most one \( A \)-attempt and at most one \( B \)-attempt will be made at each \( R_\delta \).
2. If \( (K, H) \) is an attempt at \( R_\delta \) at stage \( \sigma \), then no attempt can be made after stage \( \sigma \) at any \( R_{\delta'} \), \( \delta < \delta' \leq \sup(K \cup H) \).

A consequence of (2) is that if \( \delta < \delta' \), then no attempt at \( R_{\delta'} \) can ever injure an attempt at \( R_\delta \). (The \( A \)-attempt \( (K_1, H_1) \) injures the \( A \)-attempt \( (K_2, H_2) \) if \( K_1 \cap H_2 \neq \emptyset \). Similarly for \( B \)-attempts.)

Having laid the groundwork, we are now ready to examine the separate cases \( g(\delta) = R_\alpha^A, S_\alpha^A, T_\alpha^A \). The cases \( R_\alpha^B, S_\alpha^B, T_\alpha^B \) are handled similarly.

If \( g(\delta) = S_\alpha^A \), then an \( A \)-attempt is made at \( R_\delta \) at stage \( \sigma \) if \( g(\delta) \) is defined by stage \( \sigma \) and

1. No attempt has already been made at \( R_\delta \).
2. No attempt \( (K, H) \) at some \( R_{\delta'} \), \( \delta' < \delta \) has been made such that \( \sup(K \cup H) \geq \delta \);
3. \( A^\sigma \cap W_\delta^\sigma = \emptyset \), but \( \exists \gamma (\gamma \in W_\delta^\sigma \wedge \gamma > \delta) \).

Then the \( A \)-attempt at \( R_\delta \) at stage \( \sigma \) is \( (\{ \gamma_0 \}, \emptyset) \), where \( \gamma_0 = \mu \gamma (\gamma \in W_\delta^\sigma \wedge \gamma > \delta) \). No \( B \)-attempts are ever made at \( R_\delta \).
If $g(\delta) = T_s^A$, then an $A$-attempt is made at $R_s$ at stage $\sigma$ if $g(\delta)$ is defined by stage $\sigma$ and

(a) As before,
(b) as before,
(c) $\sigma \subseteq A^\sigma \cup (\delta + 1)$.

Then the $A$-attempt at $R_s$ at stage $\sigma$ is $(\varnothing, \{\gamma_0\})$ where $\gamma_0 = \mu(\gamma \in \epsilon - A^\sigma \wedge \gamma > \delta)$. No $B$-attempts are ever made at $R_s$.

The case $g(\delta) = R_s^A$ requires an extended discussion. We would like to find an argument $x > \delta$ and a pair $(z_1, z_2)$ of $\beta$-finite sets such that $A^\sigma \cap z_2 = \varnothing$ and $(x, z_1, z_2) \in W_s^\sigma$. If we can make the $A$-attempt $(z_1, z_2)$ and the $B$-attempt $\langle\{x\}, \varnothing\rangle$, then (assuming $z_2 \cap A = \varnothing$ is preserved):

$$x \in B \quad \text{and} \quad x \in W_s^A.$$ 

Thus $R_s^A$ will be satisfied. Requirements $\{S_s^A, T_s^A\}_{s \in \beta}$ guarantee that $A$ will be both simple and weakly-t.r.e., so it suffices to look for $(z_1, z_2)$ as above where $z_1 \subseteq A^\sigma \cup z_1'$ and $z_1', z_2$ are disjoint and countable. Then we would like to make the $A$-attempt $(z_1, z_2)$.

Unfortunately, it may happen that $z_1' \cup z_2$ contains ordinals less than $\delta$ (and hence $(z_1', z_2)$ cannot qualify as an attempt at $R_s$). We can, however, make the $A$-attempt $(z_1 - \delta, z_2 - \delta)$. What is needed is some way to "guess" $A \cap \delta$.

This guessing procedure is provided by $\diamond_{R_s^A}(E)$. Let $\langle S_\delta \mid \delta \in E \rangle$ be a $\beta$-finite $\diamond_{R_s^A}(E)$-sequence, where $E = \{\delta \mid g(\delta) = R_s^A\}$, provided by Theorem 5.

We use $S_\delta$ as our guess at $A \cap \delta$. Since $\{\delta \in E \mid A \cap \delta = S_\delta\}$ is a stationary subset of $E$, this guess will be correct for stationary-many $\delta$'s.

We are now ready to describe how attempts are made at $R_s$ when $g(\delta) = R_s^A$. Attempts are made at stage $\sigma$ if $g(\delta)$ is defined by stage $\sigma$ and

(a) As before,
(b) as before,
(c) there is an $x > \delta$ and a pair $(z_1, z_2)$ such that $(x, z_1, z_2) \in W_s^\sigma$, $A^\sigma \cap z_2 = \varnothing$, $z_1 \cap (\sigma \subseteq S_\delta$, $z_2 \cap S_\delta = \varnothing$, $z_1 - A^\sigma$, $z_2$ are disjoint and countable.

Then the $A$-attempts $\langle z_1 - A^\sigma - \delta, z_2 - \delta \rangle$ and the $B$-attempt $\langle\{x\}, \varnothing\rangle$ are made at $R_s$ at stage $\sigma$.

Having described how attempts are made at the auxiliary requirements $R_s$, one may describe the construction as follows: Let $L_s : S_\delta \rightarrow \mathcal{K}_s^L$ be an enumeration

\[A^\sigma \cap \delta = A \cap \delta.\]
of $\mathfrak{N}_1^L$ (in the sense of [4, p. 193]) such that $L^{-1}(\{\delta\})$ is unbounded in $<_\beta$, for all $\delta < \mathfrak{N}_1^L$. For example,

$$L(z) = \delta, \quad z = \langle x, \delta \rangle \quad \text{for some } x,$$

$$\quad - 0, \quad \text{otherwise.}$$

Then at stage $\sigma$, attempts are made at $R_{L(\sigma)}$ subject to the conditions described above. If the $A$-attempt $(K, H)$ is made at stage $\sigma$, then members of $K$ are put into $A$ and $A^{\sigma+1} = A^\sigma \cup K$. Similarly for $B$.

We are now ready to verify that the requirements $S^A, T^A, R^A$ will be satisfied (the argument for $S^B, T^B, R^B$ is similar).

**Claim 1.** For $\delta < \mathfrak{N}_1^L$, let

$$h(\delta) = \max\{\sup(K \cup H) \mid (K, H) \text{ is an attempt at } R_\delta\}$$

$$= \delta \quad \text{if no attempts are made at } R_\delta.$$

Then $h: \mathfrak{N}_1^L \to \mathfrak{N}_1^L$ and $C = \{\delta \mid h[\delta] \subset \delta\}$ is a closed unbounded set.

**Proof.** Since $\{(K, H) \mid (K, H) \text{ is an attempt at } R_\delta\}$ is a set of cardinality $\leq 2$, clearly $h: \mathfrak{N}_1^L \to \mathfrak{N}_1^L$. $C$ is certainly closed: if $\gamma_0 < \gamma_1 < \cdots$ and $\gamma = \bigcup \gamma_n$, then

$$h[\gamma_n] \subset \gamma_n \forall n \Rightarrow h[\gamma] = \bigcup_n h[\gamma_n] \subset \bigcup_n \gamma_n = \gamma.$$

Now define $h'(\delta) = \mu[\gamma \mid h[\gamma] \geq h(\delta)]$. Clearly $h'(\delta) \leq h(\delta)$ and also, $\{\delta \mid h'(\delta) < \gamma\}$ is bounded by $\sup h[\gamma] < \mathfrak{N}_1^L$. So by Fodor's theorem, $\{\delta \mid h'(\delta) < \gamma\}$ is not stationary, so certainly $\{\delta \mid h'(\delta) = \delta\}$ is unbounded. But this latter set is $C$. 

Let $A = \bigcup_{\sigma \in S} A^\sigma$, $B = \bigcup_{\sigma \in S} B^\sigma$.

**Claim 2.** $\mathfrak{N}_1^L - A, \mathfrak{N}_1^L - B$ are unbounded in $\mathfrak{N}_1^L$.

**Proof.** We just consider $\mathfrak{N}_1^L - A$. Let $S = \{\delta \mid g(\delta) = S^A\}$. Then $S$ is stationary (by the definition of $g$) and hence $S' = S \cap C$ is unbounded.

**Subclaim.** If $(K, H)$ is an $A$-attempt, then $K \cap S' = \emptyset$.

**Proof.** Let $\delta' \in S'$. Let $(K, H)$ be an attempt at $R_\delta$. If $\delta > \delta'$, then $K \cap \delta = \emptyset \rightarrow \delta' \notin K$. If $\delta < \delta'$, then since $\delta' \in C$, $\sup(K \cup H) < \delta'$ and so again $\delta' \notin K$. If $\delta = \delta'$, then $\delta' \notin K$ by construction (see definition of attempts at $R_\delta$ when $g(\delta) = S^A$).

But of course $A = \bigcup \{K \mid K \text{ is an } A\text{-attempt}\}$. Thus $A \cap S' = \emptyset$. 


CLAIM 3. The requirements $S_e^A$, $S_e^B$ are satisfied for all $e \in S_\beta$.

Proof. We just consider $S_e^A$. Suppose $W_e$ is unbounded in $\mathfrak{R}_1^L$. Let $S = \{\delta | g(\delta) = S_e^A\}$. As $S$ is stationary, let $\delta \in S \cap C$.

Let $\sigma$ be a stage such that $L(\sigma) = \delta$, $g(\delta)$ is defined by stage $\sigma$ and $W_{e^\sigma} = (\delta + 1) \not\in \varnothing$. $\sigma$ exists since $W_e$ is unbounded in $\mathfrak{R}_1^L$ and $L^{-1}(\{\delta\})$ is unbounded in $<\beta$. If an $A$-attempt is made at $R_\delta$ at stage $\sigma$, then some member of $W_{e^\sigma}$ is put into $A$ and so $S_e^A$ is satisfied. If not, then either

(a) An attempt has already been made at $R_\delta$, or

(b) an attempt $(K, H)$ has been made at some $R_{\delta'}$, $\delta' < \delta$ such that $\sup(K \cup H) \geq \delta$.

Condition (b) contradicts the assumption $\delta \in C$. Condition (a) implies $A^\sigma \cap W_{e^\sigma} \neq \varnothing$, so again $S_e^A$ is satisfied.  

CLAIM 4. The requirements $T_e^A$, $T_e^B$ are satisfied for all $e \in S_\beta$.

Proof. We just consider $T_e^A$. Suppose $e - A$ is bounded in $\mathfrak{R}_1^L$ (where $e \subseteq \mathfrak{R}_1^L$). Let $S = \{\delta | g(\delta) = T_e^A\}$. As $S$ is stationary, $S \cap C$ is unbounded in $\mathfrak{R}_1^L$.

Let $\delta \in S \cap C$, and let $\sigma$ be a stage such that $g(\delta)$ is defined by stage $\sigma$ and $L(\sigma) = \delta$. If $e \subseteq A^\sigma \cup (\delta + 1)$, then $e - A^\sigma$ is bounded and $T_e^A$ is satisfied. Otherwise, as $\delta \in C$, either an $A$-attempt at $R_\delta$ has already been made or one will be made at stage $\sigma$. Let $(\varnothing, \{\gamma_0\})$ be an $A$-attempt at $R_\delta$ made at some stage $\sigma' \leq \sigma$.

SUBCLAIM. If $(K, H)$ is an $A$-attempt made after stage $\sigma'$, then $\gamma_0 \notin K$.

Proof. Let $(K, H)$ be an $A$-attempt at $R_{\delta'}$. If $\delta' < \delta$, then $\sup(K \cup H) < \delta < \gamma_0$ since $\delta \in C$. If $\delta' > \delta$, then since $\sup(\varnothing \cup \{\gamma_0\}) = \gamma_0$, we must have $\gamma_0 < \delta'$ (otherwise no $A$-attempt could be made at $R_{\delta'}$). But then $K \cap \delta' = \varnothing \rightarrow \gamma_0 \notin K$. No $A$-attempts at $R_\delta$ can be made after stage $\sigma'$ since one has already been made.

But then $\gamma_0 \notin A = \cup \{K | (K, H) \text{ is an } A\text{-attempt}\}$. So we have shown that either

(a) $\exists \sigma[e - A^\sigma \text{ is bounded}]$, or

(b) $\forall \delta \in S \cap C[\exists \gamma_0 > \delta \text{ s.t. } \gamma_0 \in e - A]$

Condition (b) contradicts the assumption that $e - A$ is bounded.  

CLAIM 5. Each $R_e^A$, $R_e^B$ is satisfied, $e \in S_\beta$.

Proof. We just consider $R_e^A$. Let $E = \{\delta | g(\delta) = R_e^A\}$, a stationary subset of $\mathfrak{R}_1^L$. 


Let \( \langle S_\delta | \delta \in E \rangle \) be the \( \diamond \)-sequence chosen earlier (and used in our attempts at \( R_\delta, \delta \in E \)). Then \( E' = \{ \delta | \delta \in E \text{ and } S_\delta = A \cap \delta \} \) is a stationary subset of \( E \).
Now pick any \( \delta \in E' \cap C \).

Suppose \( \kappa_1 \rightarrow B_w \rightarrow W_\delta \). We work now toward a contradiction. By Claim 2, choose \( x \in \kappa_1 \rightarrow B \) such that \( x > \delta \). Then there is a pair \( (z_1, z_2) \) such that

1. \( z_1 \subseteq A, z_2 \subseteq \kappa_1 \rightarrow A \),
2. \( \langle x, z_1, z_2 \rangle \in W_\delta \).

By Claim 3, \( z_2 \) is bounded in \( \kappa_1 \rightarrow \) (i.e., is countable). By Claim 4, there is a stage \( \sigma \) such that \( L(\sigma) = \delta, \langle x, z_1, z_2 \rangle \in W_\sigma \) and \( z_1 \rightarrow A_\sigma \) is countable. Also, since \( S_\delta = A \cap \delta \), we have \( z_1 \cap \delta \supseteq S_\delta \) and \( z_2 \cap S_\delta = \emptyset \). Thus since \( \delta \in C \), either a pair of attempts is made at \( R_\delta \) at stage \( \sigma \), or a pair of attempts at \( R_\delta \) was already made at some earlier stage.

**Subclaim.** Let the \( A \)-attempt \( \langle z_1 \rightarrow A_\sigma \rightarrow \delta, z_2 \rightarrow \delta \rangle \) and the \( B \)-attempt \( \langle [x], \emptyset \rangle \) be made at \( R_\delta \) at stage \( \sigma \). Then \( z_2 \cap A = \emptyset \).

**Proof.** Otherwise some \( A \)-attempt \( (K, H) \) made at some \( R_\delta \), at some later stage has \( K \cap z_2 \neq \emptyset \). If \( \delta' < \delta \), this can't happen since \( \delta \in C \supset \sup(K \cup H) < \delta \) and \( z_2 \cap \delta \subseteq (\delta - S_\delta) = \delta - A \). If \( \delta' = \delta \), then no attempt can be made at \( R_\delta \) at any stage after stage \( \sigma \). If \( \delta' > \delta \), then \( \delta' > \sup((z_1 \rightarrow A_\sigma) \cup z_2) \) since otherwise no attempt can be made after stage \( \sigma \) at \( R_\delta \). But then \( K \cap \delta' = \emptyset \Rightarrow K \cap z_2 = \emptyset \).

The pair of attempts in the subclaim guarantees that \( z_1 \subseteq A, z_2 \subseteq \kappa_1 \rightarrow A, \langle x, z_1, z_2 \rangle \in W_\delta \) and \( x \in B \). Thus \( W_\delta \rightarrow A \neq \kappa_1 \rightarrow B \).

### 3 The General Case

In this section we treat the case: \( \beta^* \) is \( \beta \)-recursively regular. This means that every \( \beta \)-recursive function \( f : \gamma \rightarrow \beta^*, \gamma < \beta^* \) has range bounded in \( \beta^* \).

By Theorem 3.13 of \([4]\), we may assume that \( \Sigma_1 \) is \( \beta \)-regular; this assumption is actually not needed for our proof when \( \beta^* \) is a successor \( \beta \)-cardinal (or \( \beta^* = \beta \) and there exists a largest \( \beta \)-cardinal).

**Proposition 7.** If \( \beta^* \) is a successor \( \beta \)-cardinal, then \( \beta^* \) is \( \beta \)-recursively regular.

**Proof.** See \([4, \text{Proposition 1.18}]\).
should be replaced by “intersects every $\prod^b_1$-definable closed unbounded subset of $\beta^*$.” However, with only the assumption of $\beta$-recursive regularity of $\beta^*$, it may be that there exist $\prod^b_1$-definable unbounded subsets of $\beta^*$ of order type $\omega$; in this case, sets having the above property analogous to stationary must be final segments of $\beta^*$. So we discard the concept of “stationary” and instead choose to effectivize the following weaker versions of Fodor’s theorem and $\Diamond$

**Weak Fodor’s Theorem.** Suppose $\gamma < \kappa$ and $f: \kappa \times \gamma \rightarrow \kappa$. Then \( \{ \delta < \kappa \mid f[\delta \times \gamma] \subseteq \delta \} \) is a closed unbounded subset of $\kappa$.

**Weak $\Diamond_\kappa$-Principle.** There exists a sequence $\langle S_\delta \mid \delta < \kappa \rangle$ such that

(i) $S_\delta \subseteq 2^\kappa$, $S_\delta < \kappa$,

(ii) If $X \subseteq \kappa$, then $\{ \delta \mid X \cap \delta \in S_\delta \}$ is a closed unbounded subset of $\kappa$.

*Proof of Weak Fodor.* Let

$$g(\delta) = \mu^\delta' < \delta[f[\delta' \times \gamma] \subseteq \delta]$$

$$= \delta, \quad \text{otherwise.}$$

Clearly $g(\delta) \lessdot \delta$ for all $\delta < \kappa$, and also $g^{-1}[\delta]$ is bounded for all $\delta$ since $\kappa$ is regular and $f$ is a function. Applying Fodor’s theorem to $g$, we get that $\{ \delta \mid g(\delta) = \delta \}$ is certainly unbounded. But $g(\delta) = \delta \leftrightarrow f[\delta \times \gamma] \subseteq \delta$, and since $\{ \delta < \kappa \mid f[\delta \times \gamma] \subseteq \delta \}$ is certainly closed, we are done. $\blacksquare$

The proof of Weak $\Diamond_\kappa$ will be deferred until the proof of its effective version.

We proceed to describe our effective versions of Weak Fodor and Weak $\Diamond$. In our application of these effectivized principles to some particular $\beta$-recursive $f$ (in Weak Fodor) and $\beta$-r.e. $X$ (in Weak $\Diamond$), it will be important to know that

$$\{ \delta \mid f[\delta \times \gamma] \subseteq \delta \} \cap \{ \delta \mid X \cap \delta \in S_\delta \}$$

is closed, unbounded. As these sets might have ordertype $\omega$, we cannot simply argue that the intersection of two closed unbounded sets is closed unbounded ($\beta^*$ is not regular enough). Instead our effective versions will specify the above sets exactly, given defining parameters for $f$ as a $\beta$-recursive function and $X$ as a $\beta$-r.e. set.

**Definition.** Let $\beta^*$ be $\beta$-recursively regular and $p \in S_\delta$. Then $\delta < \beta^*$ is $p$-stable if $h_F[(\delta \cup \{p\}) \times \omega] \cap \beta^* \subseteq \delta$, where $h_F$ is a parameter-free $\Sigma_1$ Skolem function for $S_\delta$. (For a summary of the basic facts concerning Skolem functions, see [4, Chap. 1].)
Since \( h_1[(\delta \cup \{p\}) \times \omega] \) is a \( \Sigma_1 \)-elementary substructure of \( S_\beta \), we see that \( \delta < \beta^* \) is \( p \)-stable if and only if there is an \( H < \Sigma_1 S_\beta \) such that \( H \cap \beta^* = \delta \).

Let \( C_p = \{ \delta < \beta^* \mid \delta \) is \( p \)-stable\}\).

**Lemma 8.** For each \( p \in S_\beta \), \( C_p \) is a closed unbounded subset of \( \beta^* \).

**Proof.** Case 1. \( \beta^* \) is a successor \( \beta \)-cardinal. It is enough to show that \( C_p \) is unbounded. Let \( \gamma_0 = \) the largest \( \beta \)-cardinal less than \( \beta^* \). Let \( \gamma > \gamma_0 \), \( \gamma < \beta^* \) be arbitrary. Let \( H = h_1[(\gamma \cup \{p\}) \times \omega] \). As \( h_1 \) is \( \Sigma_1 \), \( H \cap \beta^* \) must be bounded in \( \beta^* \).

**Claim.** \( H \cap \beta^* = an \ ordinal. \)

**Proof.** This is because \( \delta \in H \cap \beta^* \rightarrow \exists f \in H[\delta \rightarrow \gamma_0] \). Then \( f^{\gamma_0} \subseteq H \) since \( \gamma_0 \subseteq \beta^* \).

Let \( \delta = H \cap \beta^* \). Then \( \delta \gg \gamma, \delta \in C_p \).

Case 2. \( \beta^* \) is a limit \( \beta \)-cardinal. In this case, we use the assumption \( \Sigma_1 \text{eff} \beta < \beta^* \). Let \( \gamma_0 = \Sigma_1 \text{eff} \beta < \beta^* \) and let \( f: \gamma_0 \rightarrow \beta \) be \( \beta \)-recursive, unbounded, and order preserving. As \( h_1 \) is \( \Sigma_1 \), let \( h_1^{\gamma_0} \) be the part of graph(h_1) enumerated by stage \( \sigma \) (in some fixed \( \beta \)-recursive enumeration of graph(h_1)). Let \( \delta_0 < \beta^* \) be arbitrary. Define, inductively,

\[
\delta_1 = \sup h_1^{(\delta_0)}[(\delta_0 \cup \{p\}) \times \omega] \cap \beta^*,
\]

\[
\delta_{\gamma+1} = \sup h_1^{(\gamma)}[(\gamma \cup \{p\}) \times \omega] \cap \beta^*,
\]

\[
\delta_\lambda = \sup \{ \delta_{\gamma} \mid \gamma < \lambda \} \quad \text{for limit } \lambda
\]

for \( \gamma, \lambda < \gamma_0 \). Then \( \delta_{\gamma_0} = \sup \{ \delta_{\gamma} \mid \gamma < \gamma_0 \} < \beta^* \) and \( \beta^* \cap h_1[(\delta_0 \cup \{p\}) \times \omega] \subseteq \delta_0 \). Then \( \delta_0 \leq \delta_{\gamma_0} \in C_p \).

Case 3. \( \beta^* = \beta \). Then \( C_p = \{ \gamma < \beta \mid p \in S_\gamma \) and \( \gamma \) is \( \beta \)-stable\} which is unbounded in \( \beta \).

**Effectivized Fodor.** Suppose \( \gamma < \beta^*, A \subseteq \beta^* \times \gamma \) and the partial function \( f: A \rightarrow \beta^* \) is \( \Sigma_1 \) over \( S_\delta \) with parameter \( p \). Then \( \delta > \gamma, \delta \in C_p \) implies \( f[\delta \times \gamma] \subseteq \delta \).

**Effectivized \( \diamond_{\beta^*} \).** There exists a \( \beta^* \)-recursive sequence \( \langle S_\delta \mid \delta < \beta^* \rangle \) such that

(i) \( S_\delta \subseteq 2^\delta, \beta \)-card. \( S_\delta < \beta^* \),

(ii) if \( X \subseteq \beta^* \) is \( \Sigma_1 \) over \( S_\delta \) with parameter \( p \), then \( \delta \in C_{<\omega, \beta^*} \) implies \( X \cap \delta \in S_\delta \).

**Proof of Effectivized Fodor.** Let \( \delta > \gamma, \delta \in C_p \) and \( H = h_1[(\delta \cup \{p\}) \times \omega] \).

Then \( \delta \times \gamma \subseteq H \) and so \( f[\delta \times \gamma] \subseteq H \) since \( p \in H \). But \( H \cap \beta^* = \delta \).
Proof of Effectivized $\diamond_{\beta^*}$. Case 1. There is a largest $\beta$-cardinal less than $\beta^*$. Then let this $\beta$-cardinal be $\gamma_0$. If $\delta \leq \gamma_0$, set $S_\delta = \emptyset$. Otherwise, if $\gamma_0 < \delta < \beta^*$, let $S_\delta = 2^\delta \cap J_\delta$, where $\delta = \mu^\gamma \geq \delta [J_\delta \Vdash "\delta\text{ is not a cardinal}''].$

If $\delta \in C_{<\beta^*}$, $X \subseteq \beta^*$ is $\Sigma_1$ over $S_\delta$ with parameter $p$, then let $H = h_\delta[\delta \cup \{p, \beta^*)\}] \times \omega$. Thus $\gamma_0 \in H$ and $\delta = H \cap \beta^*$. So $\gamma_0 < \delta$.

Let $c : \langle H_\delta, \epsilon \rangle \cong \langle J_\delta, \epsilon \rangle$ be the transitive collapse. Now $J_\delta \Vdash "\delta\text{ is a cardinal}'" since $\delta = c(\beta^*)$. But $X$ is $\Sigma_1$-definable with parameter $p$ over $S_\delta$, and hence $X \cap H$ is $\Sigma_1$-definable with parameter $p$ over $H$. Since $X \cap H = X \cap \delta$ and $c \upharpoonright \delta = \text{id} \upharpoonright \delta$, $X \cap \delta$ is definable over $J_\delta$. But then $X \cap \delta \in J_{\delta+1} \subseteq J_\delta$. So $X \cap \delta \in S_\delta$.

Case 2. $\beta^*$ is the limit of smaller $\beta$-cardinals. Then if $\delta < \beta^*$, let $S_\delta = 2^\delta \cap J_\delta$. Since $X$ is $\beta$-r.e., $\delta < \beta^*$ implies $X \cap \delta \in J_{\beta^*}$, we are done.

We are now ready to prove:

**Theorem 9.** If $\beta^*$ is $\beta$-recursively regular, then there exist $\beta$-r.e. $A, B \subseteq \beta^*$ such that $A \equiv_{\omega_1} B, B \equiv_{\omega_1} A$.

**Proof.** As before, we have the requirements

\[
\begin{align*}
R^A_e: & \quad \beta^* - B \neq W^A_e, & R^B_e: & \quad \beta^* - A \neq W^B_e, \\
S^A_e: & \quad W^A_e \text{ unbounded in } \beta^* \rightarrow A \cap W^A_e \neq \emptyset, & S^B_e: & \quad W^B_e \text{ unbounded in } \beta^* \rightarrow B \cap W^B_e \neq \emptyset, \\
T^A_e: & \quad (e - A) \text{ bounded in } \beta^* \rightarrow \exists \sigma[e - A^\sigma \text{ bounded in } \beta^*], & T^B_e: & \quad (e - B) \text{ bounded in } \beta^* \rightarrow \exists \sigma[e - B^\sigma \text{ bounded in } \beta^*]
\end{align*}
\]

for $e \in S_\delta$. Previously, we considered auxiliary requirements $\{R_\delta\}_{\delta < \beta^*}$ so that each $R_\delta$ constituted an attempt at one of the above requirements, each of them being attempted by stationary-many $R_\delta$'s. In this construction, each $\delta < \beta^*$ will be responsible for a whole (size $<\beta^*$) collection of requirements, each requirement being attempted by a final segment of $\delta$'s.

More precisely, fix a 1-1 $\beta$-recursive $f : S_\delta \rightarrow \beta^*$ and let $p'$ be the parameter which defines $f$. $\langle \langle p', \beta^* \rangle \rangle$ will be the parameter for the construction if $\beta^* < \beta$.) Then $\delta$ is responsible for $R^A_e, R^B_e, S^A_e, S^B_e, T^A_e, T^B_e$ where $f(e) < \delta$. As before, "guesses" will have to be made at $A \cap \delta, B \cap \delta$ for the sake of requirements $R^A_e, R^B_e$; as $\diamond_{\beta^*}$ provides us with a collection of such guesses, the requirements $R^A_e, R^B_e$ will in fact be treated by $\delta$ as collections of requirements, one for each guess given by $\diamond_{\beta^*}$.

**The auxiliary requirements.** Fix a $\beta^*$-recursive $\diamond_{\beta^*}$ sequence $\langle S_\delta \mid \delta < \beta^* \rangle$. Then the collection $\mathcal{B}_\delta$ of auxiliary requirements at level $\delta$ consists of:
(a) \( \{ S_y^A \mid \gamma < \delta \} \cup \{ S_y^B \mid \gamma < \delta \} \),
(b) \( \{ T_y^A \mid \gamma < \delta \} \cup \{ T_y^B \mid \gamma < \delta \} \),
(c) \( \{(R_y^A, g) \mid \gamma < \delta, g \in S_\delta \} \cup \{(R_y^B, g) \mid \gamma < \delta, g \in S_\delta \} \). \)

\( S_y^A (T_y^A) \) is intended to represent \( S_y^A (T_y^A) \) where \( f(e) = \gamma \). Similarly for \( S_y^B \), \( T_y^B \). \( (R_y^A, g) \) represents \( R_y^A \) with the guess \( g \) for \( A \cap \delta \), where \( f(e) = \gamma \). Similarly for \( (R_y^B, g) \). Note that there may be \( \gamma < \delta \) which are not in range \( f \), in which case auxiliary requirements with subscript \( \gamma \) will never be active.

Now for each \( \delta < \beta^* \), \( R_\delta \) has \( \beta^- \) cardinality less than \( \beta^* \), so let \( \{ R_\xi^\delta \}_{\xi < \delta < \beta^*} \) be a fixed well-ordering of \( R_\delta \).

As before, each \( \delta \) will make certain \( A^- \) attempts and \( B^- \) attempts of the form \( (K, H) \) where \( K, H \) are bounded in \( \beta^* \) and \( K \cap \delta = \emptyset = H \cap \delta \). \( \delta \) will make at most one \( A^- \) attempt and one \( B^- \) attempt at each \( R_\delta^\delta \). Also, if \( \delta \) makes the attempt \( (K, H) \) at stage \( \sigma \), then no \( \delta' \) such that \( \delta < \delta' \leq \sup(K \cup H) \) can ever make an attempt at any later stage.

Two \( A^- \) attempts \( (K_1, H_1) \) and \( (K_2, H_2) \) are compatible if \( K_1 \cap H_2 = \emptyset = K_2 \cap H_1 \). Similarly for \( B^- \) attempts. At each stage \( \sigma \), some \( \delta < \beta^* \) will be examined, and then \( A^- \) and \( B^- \) attempts will be made successively at \( R_1^\delta, R_2^\delta, R_3^\delta, \ldots \); an attempt at \( R_\xi^\delta \) at this stage must be compatible with attempts at \( R_\xi^\delta \), \( \xi' < \xi \) at this stage and all attempts at members of \( R_\delta \) at earlier stages. Thus no two attempts at members of \( R_\delta \) will ever conflict, and each member of \( R_\delta \) has an opportunity to act at each stage where \( \delta \) is being examined.

We now describe exactly how attempts are made and how the construction proceeds. Let \( L: S_\delta \rightarrow \beta^* \) be a fixed enumeration of \( \beta^* \) such that for each \( \delta < \beta^*, L^{-1}(\{\delta\}) \) is unbounded in \( <_\beta \) and \( L \) is \( \Sigma_1 \)-definable over \( S_\delta \) with parameter \( \beta^* \). (For example,

\[
L(x) = \delta \quad \text{if} \quad x = \langle \delta, \gamma \rangle, \quad \delta < \beta^*
\]

\[
= 0, \quad \text{otherwise}
\]

is an example of such an \( L \).

Stage \( \sigma \). We consider \( L(\sigma) = \delta \). If some attempt \( (K, H) \) has been made at some \( \delta' < \delta \) where \( \sup(K \cup H) \geq \delta \), go to the next stage. Otherwise, begin by making the \( A^- \) attempt and \( B^- \) attempt \( (\emptyset, \{\delta\}) \). (This is to ensure \( \beta^* - A, \beta^* - B \) unbounded in \( \beta^* \).) Then successively make \( A^- \) and \( B^- \) attempts at \( R_1^\delta, R_2^\delta, R_3^\delta, \ldots \); an attempt at \( R_\xi^\delta \) at this stage must be compatible with attempts at \( R_\xi^\delta \), \( \xi' < \xi \) at this stage and all attempts at members of \( R_\delta \) at earlier stages. We describe how to act on \( R_\xi^\delta \). If an attempt has already been made at \( R_\xi^\delta \) at some earlier stage \( \sigma' \), \( L(\sigma') = \delta \), then go on to \( R_{\xi+1}^\delta \). Now assume \( R_\xi^\delta \) is of one of the forms \( S_y^A, T_y^A, (R_y^A, g) \). The cases \( S_y^B, T_y^B, (R_y^B, g) \) are treated similarly. If \( \gamma \) has not yet appeared in Range \( f \) (in some fixed enumeration of the \( \beta^- \) r.e. set range \( f \)), go on to \( R_{\xi+1}^\delta \). Otherwise, let \( f(e) = \gamma \).
$R^A_\epsilon = S^A_\epsilon$. If there is an $A$-attempt ($\gamma', \emptyset$) compatible with earlier $A$-attempts at members of $R_\delta$ such that $\gamma' > \delta$ and $\gamma' \in W_\epsilon^\alpha$, then make the $A$-attempt ($\gamma_0, \emptyset$) where $\gamma_0$ is the least such $\gamma'$. Otherwise go to $R^B_\epsilon$.

Make no $B$-attempts.

$R^A_\epsilon = T^A_\epsilon$. If there is an $A$-attempt ($\emptyset, (\gamma')$) compatible with earlier $A$-attempts at members of $R_\delta$ such that $\gamma' > \delta$ and $\gamma' \notin W_\epsilon^\alpha$, then make the $A$-attempt ($\emptyset, (\gamma_0)$) where $\gamma_0$ is the least such $\gamma'$. Otherwise go to $R^B_\epsilon$.

Make no $B$-attempts.

$R^A_\epsilon = (R^A_\epsilon, g)$. Suppose that there are $x > \delta$ and a pair $(x_1, x_2)$ such that $(x, x_1, x_2) \in W_\epsilon^\alpha$, $x_1 \cap \delta \subseteq g$, $x_2 \cap g = \emptyset$, $x_2 \cap A_\delta = \emptyset$ and $x_2 - A_\delta$, $x_2$ are disjoint and bounded in $\beta^*$. Then if such an $x > \delta$ and $(x_1, x_2)$ can be found so that the $A$-attempt $(x_1 - A_\delta - \delta, x_2 - \delta)$ and the $B$-attempt $(x, \emptyset)$ are compatible with earlier attempts at members of $R_\delta$, make the least such pair of attempts. Otherwise go to $R^B_\epsilon$.

This ends the description of the construction. Note that the only parameters needed in the construction are $\delta'$ and $\beta^*$. Let $p'$ be a parameter which defines a $\beta$-recursive $g: \Sigma_1^{cf} \beta \rightarrow \beta$, range $g$ unbounded. Let $p = (p', \beta, p^*)$.

**Claim 1.** Suppose $\delta \in C_p$; let $(K, H)$ be an attempt at some member of $R_\delta$, $\delta' < \delta$. Then $\sup(K \cup H) < \delta$.

**Proof.** Case 1. There is a largest $\beta$-cardinal less than $\beta^*$, $\gamma$. Then each $R_\delta'$ is well-ordered in length $\gamma$. Define the partial $\Sigma_1$ function $f$ by $f(\delta', \gamma') = \max\{\sup(K \cup H) : (K, H) \text{ is an attempt at } R^A_\delta\}$. Then by Effectivized Fodor, $\delta \in C_p$, $\delta > \gamma \rightarrow f[\delta \times \gamma] \subseteq \delta$. But it is easily seen that $\delta \in C_p \rightarrow \delta > \gamma$ and so we are done.

Case 2. Otherwise. In this case we use the assumption $\Sigma_1^{cf} \beta < \beta^*$. Let $\gamma = \Sigma_1^{cf} \beta$ and define $f$ by $f(\delta', \gamma') = \sup\{\sup(K \cup H) : (K, H) \text{ is an attempt at some member of } R_\delta \}

Then by Effectivized Fodor, $\delta \in C_p \rightarrow f[\delta \times \gamma] \subseteq \delta$. So we are done.

**Claim 2.** If $\delta \in C_p$, then $A \cap \delta$, $B \cap \delta \subseteq S_\delta$.

**Proof.** This is immediate from Effectivized $\diamondsuit_{\beta^*}$.

Thus we see that any $A$-attempt $(K, H)$ made at a member of $R_\delta$, $\delta \in C_p$ is permanent; i.e., $K \subseteq A$ and $A \cap H = \emptyset$. (Similarly, for $B$-attempts.) Moreover, by Claim 2, a correct "guess" was made at $A \cap \delta$ and at $B \cap \delta$. These facts make it easy to check that the desired requirements have been met.
CLAIM 3. $\beta^* - A, \beta^* - B$ are unbounded in $\beta^*$.

Proof. The first attempts made by $\delta \in C_\beta$ are the $A$-attempts and $B$-attempts ($\emptyset, \{\delta\}$). These attempts are permanent so $\delta \notin A \cup B$. $C_\beta$ is unbounded by Lemma 8. □

CLAIM 4. The requirements $S_\epsilon^A, S_\epsilon^B$ are satisfied for all $\epsilon \in S_\beta$.

Proof. Let $f(\epsilon) = \gamma$. We just consider $S_\epsilon^A$. Choose $\delta \in C_\beta$, $\delta > \gamma$. If $W_\epsilon$ is unbounded in $\beta^*$, there must be a stage $\sigma$ such that

(i) $L(\sigma) = \delta$,

(ii) $f(\epsilon)$ is defined by stage $\sigma$,

(iii) there is an $\epsilon \in W_\epsilon$, $\epsilon > \delta'$, where $\delta' =$ least member of $C_\beta$ greater than $\delta$.

Then by Claim 1, the $A$-attempt ($\{x\}, \emptyset$) must be compatible with earlier $A$-attempts at members of $S_\delta$. So either an attempt must be made at $S_\epsilon^A$ at stage $\sigma$ or one must have already been made. But then $A \cap W_\epsilon \neq \emptyset$. □

CLAIM 5. The requirements $T_\epsilon^A, T_\epsilon^B$ are satisfied for all $\epsilon \in S_\beta$.

Proof. Let $f(\epsilon) = \gamma$. We just consider $T_\epsilon^A$. Suppose that for no stage $\sigma$, do we have $\epsilon - A^\sigma$ bounded. We show that $\epsilon - A$ is unbounded. Let $\gamma' > \gamma$ be arbitrary and $\delta \in C_\beta$, $\delta > \gamma'$. Let $\sigma$ be any stage such that $L(\sigma) = \delta$ and $f(\epsilon)$ is defined by stage $\sigma$. Since $\epsilon - A^\sigma$ is unbounded, there must be an $A$-attempt ($\emptyset, \{\gamma_0\}$, $\gamma_0 \in A - A^\sigma$, which is compatible with all earlier $A$-attempts at members of $S_\delta$. Then either such an $A$-attempt was made at stage $\sigma$ or an earlier attempt was made at $T_\gamma^A$. But this attempt is permanent since $\delta \in C_\beta$. So we have shown that $\exists x > \gamma'[x \in \epsilon - A]$. Since $\gamma'$ was arbitrary, $\epsilon - A$ is unbounded. □

CLAIM 6. Each $R_\epsilon^A, R_\epsilon^B$ is satisfied, $\epsilon \in S_\beta$.

Proof. We just consider $R_\epsilon^A$. Let $f(\epsilon) = \gamma$ and $\gamma < \delta \in C_\beta$. Let $\delta' =$ least member of $C_\beta$ greater than $\delta$. Since $\beta^* - B$ is unbounded choose $x \in \beta^* - B$, $x > \delta'$. Suppose $\beta^* - B = W_\epsilon^A$. Then there must be a pair $(x_1, x_2)$ such that $(x, x_1, x_2) \in W_\epsilon, x_1 \subseteq A, x_2 \subseteq \beta^* - A$. Since $A$ is simple (Claim 4) and weakly-t.r.e. (Claim 5), there is a stage $\sigma$ such that

(i) $L(\sigma) = \delta$, $f(\epsilon)$ is defined by stage $\sigma$,

(ii) $x_1 - A^\sigma$, $x_2$ are bounded in $\beta^*$.

Since $\delta \in C_\beta$, there is a $g \in S_\delta$, $g = A \cap \delta$. Then of course $z_1 \cap \delta = g$, $z_2 \cap g = \emptyset$. Then the $A$-attempt ($x_1 - A^\sigma \delta, z_2 \delta$) and the $B$-attempt ($\{x\}, \emptyset$) must be compatible with all earlier attempts since all of these attempts
are permanent. But then attempts at \((R,^4, g)\) must have been made at stage \(\sigma\) or some earlier stage. Since this attempt must be permanent, it guarantees \(B \neq \mathcal{W}_n^4\). 

4. Generalizing to \(\Sigma_n, n > 1\)

Jensen's master codes can be used to extend the above results to \(\Sigma_n\) sets when the \(\Sigma_n\)-projectum of \(\beta, \rho_n^\beta\), is regular with respect to the functions \(\Sigma_n\) over \(S_\beta\).

Recall [7]

**Theorem (Jensen).** For each \(n > 0\) there is a subset \(A_n^\beta\) of \(\rho_n^\beta\) which is \(\Sigma_n\) over \(S_\beta\) such that

\[
\Sigma_{n+m}^\beta \cap 2^{\rho_n^\beta} = \Sigma_m^\rho_n^\beta \cdot A_n^\beta
\]

for all \(m > 0\).

Thus, as far as subsets of \(\rho_n^\beta\) are concerned, \(\Sigma_{n+m}\)-definability over \(S_\beta\) reduces to \(\Sigma_m\)-definability over the amenable structure \(\langle S_\rho^\beta, A_n^\beta \rangle\).

Now for \(A, B \subseteq S_\beta\), define \(A \leq_{u\beta} B\) if there are \(\Sigma_n\) predicates \(W_1, W_2\) such that

\[
x \in A \iff \exists x_1 \exists x_2 [\langle x, x_1, x_2 \rangle \in W_1 \land x_1 \subseteq B \land x_2 \subseteq S_\beta - B],
\]

\[
x \notin A \iff \exists x_1 \exists x_2 [\langle x, x_1, x_2 \rangle \in W_2 \land x_1 \subseteq B \land x_2 \subseteq S_\beta - B].
\]

Then \(\leq_{u\beta} = \leq^1_{u\beta}\).

**Theorem 10.** Suppose \(\rho_n^\beta\) is regular with respect to functions \(\Sigma_n\) over \(S_\beta\). Then there exist sets \(A, B \subseteq S_\beta\) which are \(\Sigma_n\) over \(S_\beta\) and such that \(A \leq_{u\beta} B, B \leq_{u\beta} A\).

**Definition.** If \(\mathcal{U} = \langle S_\beta, \epsilon, A \rangle\) is an amenable structure, then if \(B, C \subseteq S_\beta\) we define: \(B \leq_{\mathcal{U}} C\) if there are predicates \(W_1, W_2, \Sigma\) over \(\mathcal{U}\) such that

\[
x \in B \iff \exists x_1 \exists x_2 [\langle x, x_1, x_2 \rangle \in W_1 \land x_1 \subseteq C \land x_2 \subseteq S_\beta - C],
\]

\[
x \notin B \iff \exists x_1 \exists x_2 [\langle x, x_1, x_2 \rangle \in W_2 \land x_1 \subseteq C \land x_2 \subseteq S_\beta - C].
\]

**Proof of Theorem 10.** Let \(\mathcal{U} = \langle S_{\rho_n^\beta-1}, \epsilon, A_{n-1}^\beta \rangle\) (we can assume \(n \geq 2\)). A straightforward relativization of Theorem 9 yields: There are \(A, B \subseteq \rho_1^\mathcal{U} = \rho_n^\beta\) which are \(\Sigma_1\) over \(\mathcal{U}\) (hence \(\Sigma_n\) over \(S_\beta\)) such that \(A \leq_{\mathcal{U}} B, B \leq_{\mathcal{U}} A\). If \(\rho_n^\beta \leq_{\mathcal{U}} \rho_n^\beta-1\), then every \(\beta\)-finite subset of \(\rho_n^\beta\) belongs to \(S_{\rho_n^\beta-1}\), so \(\leq_{\mathcal{U}} = \leq_{u\beta}^n\) for subsets of \(\rho_n^\beta\). Thus \(A \leq_{u\beta}^n B, B \leq_{u\beta}^n A\). Otherwise, \(\mathcal{U}\) is an admissible
structure (since then $\rho_{n-1}^\beta = \rho_n^\beta$ which is $\Sigma_n^\beta$-regular by assumption). Then do the usual construction [16] of incomparable $\Sigma_1$ sets $A, B$ over the admissible structure $\mathcal{M}$, but in addition guaranteeing

$$R \in \Sigma_1, \quad R \text{ unbounded in } \rho_{n-1}^\beta \rightarrow A \cap R \neq \emptyset, \quad B \cap R \neq \emptyset.$$ 

Then all $\beta$-finite subsets of $S_{\rho_{n-1}^\beta} - A, S_{\rho_{n-1}^\beta} - B$ belong to $S_{\rho_{n-1}^\beta}$. Using this, it can be seen that $A \preceq_{\omega_1^{\beta}} B, B \preceq_{\omega_1^{\beta}} A$.

Last, it is quite pertinent to ask for a stronger incomparability in Theorem 10: If $A, B \subseteq S_\beta$, say that $A$ is $\Delta_n$ in $B$ if $A$ is $\Delta_n$-definable over $\langle S_\beta[B], \epsilon, B \rangle$. Then do there exist $\Sigma_n$ sets $A, B$ such that neither is $\Delta_n$ in the other? Shore [15] handles the case where $\beta$ is $\Sigma_n$-admissible. See [17] for progress on the case where $\rho_n^\beta$ is regular with respect to functions $\Sigma_n$ over $S_\beta$.

5. Recent Developments

We have now produced ordinals for which Post's problem has a negative solution: Let $\beta = \kappa_{\omega_1}^L + \omega$. If $\alpha = \kappa_{\omega_1}^L$ then there are no $\alpha$-degrees strictly between $0'$ and $0''$. These results will appear in a series of papers entitled "Some Negative Solutions to Post's Problem," the first of which is listed as [18].

However, the complete situation regarding Post's problem is not fully understood. We end with some questions.

1. For which $\beta$ can Post's problem be solved positively? See [18] for a conjecture on this.
2. For which $\beta$ are there incomparable $\beta$-r.e. degrees?
3. For which $\beta$ are the $\beta$-r.e. degrees dense?

References

15. R. Shore, \(\Sigma_n\) sets which are \(\Delta_n\) incomparable, *J. Symbolic Logic* 39 (1974), 295–304.