Projective Singletons

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Assuming large cardinals, the structure of $\Pi^1_n$-singletons for odd $n$ is well understood: using the prewellordering property of the pointclass $\Pi^1_n$, it follows that they are prewellordered by $\Delta^1_n$-reducibility. But for even $n$, $\Pi^1_n$ the analysis of $\Pi^1_n$-singletons requires new techniques.

For $n = 2$ one has:

Theorem 1. ([2]) There is a $\Pi^1_2$-singleton $R$ such that $0 <_L R <_L 0^\#$. Moreover, there are two such $\Pi^1_2$-singletons of incomparable $L$-degree.

The proof of this result makes essential use of $L$-coding ([1], [4]). To generalize this result to level 4, it is necessary to understand the correct analogues of $L$-reducibility and $0^\#$ for the pointclass $\Pi^1_4$, and to develop an appropriate generalized coding method.

The correct analogue of $0^\#$ at level 4 is $M^\#_2$, the least mouse with two Woodin cardinals and a sharp above. The correct analogue of $L$ at level 4 is $M_2$, the iterable extender model with 2 Woodin cardinals that results after iterating the sharp at the top of $M^\#_2$ to infinity. And the correct analogue of $L$-reducibility is $M_2$-reducibility: $R <_4 S$ iff $R$ belongs to $M^S_2$, the canonical iterable extender model with 2 Woodin cardinals containing $S$.

Next we make a few comments about Woodin cardinals. There are two equivalent definitions of this concept. An inaccessible cardinal $\delta$ is Woodin iff either of the following equivalent properties holds:

1. For any $f : \delta \rightarrow \delta$ there is $\kappa < \delta$ closed under $f$ and an embedding $\pi : V \rightarrow M$ with critical point $\kappa$ which is $f$-strong, i.e., such that $V_{\pi(f)(\kappa)}$ is
contained in $M$.

2. For any $A \subseteq \delta$, there is $\kappa < \delta$ which is $A$-strong up to $\delta$: for any $\alpha < \delta$ there is $\pi : V \to M$ with critical point $\kappa$ such that $A \cap \alpha = \pi(A) \cap \alpha$.

A set $S$ of extenders (on $V$) is a weak Woodin witness for $\delta$ iff $\delta$ is Woodin via the first definition above, using ultrapower embeddings given by extenders in $S$. $S$ is a strong Woodin witness for $\delta$ iff $\delta$ is Woodin via the second definition, using ultrapower embeddings given by extenders in $S$.

We shall also need the following technical fact, which follows directly from the results of [5].

**Theorem 2.** Assume that there are two Woodin cardinals with a measurable above them. Suppose that $x$ is a real in $W$, an inner model which is $(\omega_1 + 1)$-iterable with respect to a countable set of extenders strongly witnessing the Woodinness of two cardinals. Then $W$ contains all the reals of $M^2_x$.

We are now ready to discuss the analogue of Theorem 1 for $\Pi^1_4$. First we need a class forcing technique in the context of Woodin cardinals.

**Theorem 3.** Suppose that $W = L^E$ is a good extender model (like $M_2$, with sufficient Condensation and $\square$, and with the properties that the class of extenders on the $E$-sequence is sufficient to witness Woodinness and is closed under cutbacks to cardinals of $W$). Then there is a real $x$ which is class-generic but not set-generic over $W$ such that every Woodin cardinal of $W$ remains Woodin in $W[x]$. Moreover Woodinness in $W$ has a $W$-definable weak witness, consisting of extenders on the $E$-sequence which have cardinal (of $W$) true length and lift to $W[x]$.

By combining the techniques used in the proofs of Theorems 1 and 3, we can produce a real $x$ with the following properties:

1. $x$ is class generic over $M_2$, $x$ belongs to $L[M^2_2]$ but not to $M_2$ and $M_2$, $M_2[x]$ have the same cofinalities.
2. For some formula $\varphi$, $x$ is the unique real such that $(M_2 \upharpoonright \alpha)[x] \models \varphi(x)$ for each ordinal $\alpha$.
3. The Woodinness of the two Woodin cardinals of $M_2$ can be weakly witnessed by a set of extenders $T \in M_2$, each element of which is on the $E$-sequence of $M_2$, has $M_2$-cardinal true length and lifts to $M_2[x]$. 
It can be shown that property 2 is $\Pi^1_4$ in any real which codes $M_2 \upharpoonright \delta_1$, where $\delta_1$ is the smaller Woodin cardinal of $M_2$. It follows that $R$ is a $\Pi^1_4$ singleton relative to any such real.

Thus we have obtained an analogue of Theorem 1 at level 4 provided we can show that $M^*_2$ does not belong to $M_2^\sharp$, the canonical iterable extender model with 2 Woodin cardinals containing $x$. It is clear that the real $M^*_2$ does not belong to $M_2[x]$, as the latter has the same cardinals as $M_2$. But this does not suffice, as $M_2[x]$ and $M^*_2$ might be different models. (In the $\Pi^1_2$ singleton result, there is no distinction between $L^x$ and $L[x]$.)

By Theorem 2 it suffices to show that the model $M_2[x]$ is $(\omega_1 + 1)$-iterable with respect to the extenders in a countable strong witness $T^\ast$ to Woodinness in $M_2[x]$, for then $M_2[x]$ contains all reals of $M_2^\sharp$ and therefore the real $M^*_2$ cannot belong to $M_2^\sharp$. We take $T^\ast$ to be the set of all liftings to $M_2[x]$ of extenders on the $M_2$-sequence which have $M_2$-cardinal true length. By Property 3 above, this is a weak witness to Woodinness in $M_2[x]$; $T^\ast$ is in fact a strong witness: Suppose not, and choose $\delta$ to be Woodin in $M_2$ and $A \subseteq \delta$ such that no $\kappa < \delta$ is $A$-strong up to $\delta$ via extenders in $T^\ast$. For each $\kappa < \delta$ let $f(\kappa)$ be several $M_2$-cardinals past the supremum of those $\alpha < \delta$ such that $\kappa$ is $A$-strong up to $\alpha$ via extenders in $T^\ast$. As $T^\ast$ is a weak witness to the Woodinness of $\delta$ in $M_2[x]$, we can choose $\kappa < \delta$ closed under $f$ and an extender $E \in T^\ast$ such that $E$ witnesses $f$-strength. In $\text{Ult}(M_2[x], E)$, $E(f)(\kappa)$ is several $M_2$-cardinals past the supremum of those $\alpha < \delta$ such that $\kappa$ is $E(A)$-strong up to $\alpha$ via extenders in $E(T^\ast)$. Let $F$ be obtained from $E$ by cutting its length back by one $M_2$-cardinal. Then $F$ belongs to $\text{Ult}(M_2[x], E)$ and witnesses that $\kappa$ is $E(A)$-strong up to its length. But $F$ is the lifting of a cutback to an $M_2$-cardinal of an extender on the extender sequence of $M_2$ and therefore of $\text{Ult}(M_2, E)$, by coherence. Since such cutbacks are also on the extender sequence of $\text{Ult}(M_2, E)$, $F$ belongs to $E(T^\ast)$, a contradiction.

Finally we argue that $M_2[x]$ is iterable with respect to the extenders in $T^\ast$. It suffices to show that any iteration $\langle N_i[x], E^*_i \mid i < \lambda \rangle$ of $M_2[x]$ via liftings of extenders from $M_2$ of $M_2$-cardinal length is a lifting to $M_2[x]$ of an iteration $\langle N_i, E_i \mid i < \lambda \rangle$ of $M_2$. This is clear by induction, provided the lifted extender $E^*_i$ is applied to the model $N_j[x]$, when $E_i$ is applied to $N_j$, and the resulting ultrapower embedding of $N_j[x]$ via $E^*_i$ lifts the corresponding
ultrapower embedding of $N_j$ via $E_i$. By definition, $j$ is least so that the critical point of $E_i$ is less than the true length of $E_j$. So for the first of these properties it suffices to show that $E_i$ and $E_i^*$, the lifting of $E_i$, have the same critical point and true length. The fact that $E_i^*$ is a lifting of $E_i$ implies that $E_i$ and $E_i^*$ have the same critical point and that the true length of $E_i^*$ is at most that of $E_i$. But since $x$ preserves cardinals over $N_i$ and the true length of $E_i$ is a cardinal of $N_i$, it follows that the true length of $E_i^*$ cannot be smaller than the true length of $E_i$. So $E_i^*$ is indeed applied to $N_j[x]$, as desired. The second property follows from the special nature of the forcing $P$ that produces $x$. Indeed, using density-reduction for $P$, any lifting $E^*$ of an extender $E$ from a model $N$ to $N[x]$, where $x$ is $P$-generic over $N$, gives rise to an ultrapower embedding of $N[x]$ which lifts the ultrapower embedding given by $E$.

In summary we have:

*Theorem 4.* There is a real $R$ such that $0 <_4 R <_4 M_2^R$ and $R$ is a $\Pi^1_4$ singleton an any real coding $M_2 \upharpoonright \delta_1$, where $\delta_1$ is the smaller Woodin cardinal of $M_2$. Moreover there are two such reals which are $<_4$-incomparable.

The natural analogue of Theorem 3 also holds at higher even levels, where $\delta_1$ is the least Woodin cardinal of $M_{2n}$. At level $\omega$ however, one needs a code for the entire $M_\omega$, and not just $M_\omega \upharpoonright \delta_1$.

*References*

3. Friedman, Sy D., Genericity and large cardinals, to appear.