Descriptive Set Theory, Sommersemester 2003

1. Vorlesung

My last course dealt with the techniques of Pure Set Theory:

Constructible Universe $L$
Set-forcing over $L$
Class-forcing over $L$
Inner models $K$ for large cardinals
Set-forcing and Class-forcing over $K$

In the present course I consider applications of these techniques. There are two kinds of applications:

Consistency results
Con($\text{ZFC} + \text{a large cardinal}$) $\rightarrow$ Con(something interesting).
Some examples:
Con(GCH)
Con(Suslin's Hypothesis)
Con(Inaccessible) $\rightarrow$ Con(All projective sets of reals are measurable)
Con(Hypermeasurable) $\rightarrow$ Con(Failure of the singular cardinal hypothesis)
Con(Woodin) $\rightarrow$ Con(The nonstationary ideal on $\omega_1$ is saturated)

Theorems
ZFC $+$ a large cardinal $\rightarrow$ something interesting.
Some examples in descriptive set theory:
$\Sigma^1_1$ sets of reals are measurable
Measurable cardinal $\rightarrow$ $\Sigma^1_2$ sets of reals are measurable
Infinitely many Woodin cardinals $\rightarrow$ All sets of reals in $L(\mathcal{R})$ are measurable

There is a good reason why the latter examples are taken from descriptive set theory: Large cardinals appear to give a complete understanding of the behaviour of sets of reals in $L(\mathcal{R})$. But if we go beyond that, we are faced with CH, which remains undecidable even with the addition of large cardinal hypotheses.

The aim of this course is to study applications of the second type, within descriptive set theory. After establishing as much as possible in ZFC alone,
we shall introduce large cardinals in order to complete the picture. Here is an outline:

**ZFC Results**
The Borel and Projective Hierarchies
Borel= $\Delta^1_1$
The Suslin Property and Regularity for Analytic Sets
Determinacy and the Wadge Property for Borel Sets
$\Pi^1_1$ Uniformisation

**The Constructible Universe**
Failure of $\Delta^2_2$ Regularity
Failure of $\Pi^1_2$ Determinacy and Wadge
Projective Uniformisation

**Forcing Extensions of L**
Solovay’s Model: Projective Regularity
Failure of $\Pi^1_2$ Uniformisation

**Sharps**
Regularity for $\Sigma^1_2$ Sets
$\Pi^1_1$ Determinacy and Wadge
$\Pi^1_2$ Uniformisation

**Woodin Cardinals**
Projective Regularity
Projective Uniformisation
Projective Determinacy and Wadge
Projective Equivalence Relations
Projective Basis Theorems

**Part 1: ZFC Results**

A *Polish space* is a topological space that is homeomorphic to a complete, separable metric space.

Lemma 1. Let $X$ be a Polish space. Then there is a continuous function from Baire Space $\mathcal{N} = \{ f \mid f \text{ is a function from } N \text{ to } N \}$ onto $X$. 

2
Proof. By induction on the length of $s \in \text{Seq} =$ the set of finite sequences of elements of $N$, we define $C_s$ such that $C_0 = X$ and for non-empty $s$:

i. $C_s$ is a closed ball of diameter $\leq 1/n$, where $n =$ length $s$.

ii. $C_s \subseteq \bigcup_{k=0}^{\infty} C_{s+k}$ (all $s \in \text{Seq}$).

iii. $s \subseteq t \rightarrow \text{center}(C_t) \in C_s$.

For each $a \in \mathcal{N}$ let $f(a)$ be the unique point in $\bigcap \{C_s \mid s \subseteq a\}$. Then $f$ is continuous and has range $X$. $\Box$

**Borel Sets**

Let $X$ be a Polish space. $A \subseteq X$ is **Borel** if it belongs to the smallest $\sigma$-algebra of subsets of $X$ containing all closed sets. The Borel hierarchy is defined as follows: For each $\alpha < \omega_1$ define $\Sigma^0_\alpha$, $\Pi^0_\alpha$ as follows:

$\Sigma^0_1 = \text{Open Sets}$

$\Pi^0_1 = \text{Closed Sets}$

$\Sigma^0_\alpha = \text{All sets } A = \bigcup_{n=0}^{\infty} A_n$, where each $A_n$ belongs to $\Pi^0_\beta$ for some $\beta < \alpha$

$\Pi^0_\alpha = \text{Complements of sets in } \Sigma^0_\alpha$

It is clear that the above sets are all Borel. As every open set is the countable union of closed sets, we have $\Sigma^0_0 \subseteq \Sigma^0_2$ and then by induction:

$\alpha < \beta \rightarrow \Sigma^0_\alpha \cup \Pi^0_\alpha \subseteq \Sigma^0_\beta \cap \Pi^0_\beta$.

Thus $\bigcup \Sigma^0_\alpha = \bigcup \Pi^0_\alpha$ is the collection of Borel sets. For each $\alpha$, $\Sigma^0_\alpha$, $\Pi^0_\alpha$ are closed under finite unions, finite intersections and inverse images by continuous functions.

We show that in the case where $X$ is the Baire space, the Borel hierarchy is strict: $\Sigma^0_\alpha \not\subseteq \Pi^0_\alpha$ and therefore $\Sigma^0_\alpha \neq \Sigma^0_{\alpha+1}$ for each $\alpha$. The proof generalises to arbitrary uncountable Polish spaces.

**Lemma 2.** (Universal $\Sigma^0_\alpha$ Sets) For each $\alpha \geq 1$ there exists a set $U \subseteq N^2$ such that $U$ is $\Sigma^0_\alpha$ and for every $\Sigma^0_{\alpha}$ set $A \subseteq N$ there is $a \in N$ such that $A = \{x \mid (x,a) \in U\}$.

Proof. By induction on $\alpha$. For $\alpha = 1$ let $G_1, G_2, \ldots$ be a list of all basic open sets and $G_0 = \emptyset$. We define $U = \{(x,y) \mid x \in G_{y(n)} \text{ for some } n\}$. $U$
is open and if $G$ is an arbitrary open set, then $G = \{x \mid (x, a) \in U\}$ where $G = \bigcup_n G_{\alpha(n)}$.

Suppose now that $U_\beta$ is a universal $\Sigma_\beta^0$ set for each $\beta < \alpha$ and we shall construct a universal $\Sigma_\alpha^0$ set $U$. Choose $\alpha_0 \leq \alpha_1 \leq \cdots$ less than $\alpha$ either with supremum $\alpha$ or maximum the ordinal predecessor to $\alpha$. Choose a continuous mapping of $\mathcal{N}$ onto $\mathcal{N}^\omega$ and for each $a \in \mathcal{N}$ let $(a)_n$ be the $n$-th coordinate of the image of $a$. We define $U = \{(x, y) \mid (x, (y)_n) \notin U_{\alpha_n} \text{ for some } n\}$. Then $U$ is $\Sigma_\alpha^0$. If $A$ is $\Sigma_\alpha^0$ then $A$ is the union of sets $A_n$ where $A_n$ is $\Pi_{\alpha_n}^0$. For each $n$ let $a_n$ be such that $A_n = \{x \mid (x, a_n) \notin U_{\alpha_n}\}$ and let $a$ be such that $(a)_n = a_n$ for each $n$; then $A = \{x \mid (x, a) \in U\}$. □

Corollary 3. For each $\alpha \geq 1$ there is a set $A \subseteq \mathcal{N}$ that is $\Sigma_\alpha^0$ but not $\Pi_\alpha^0$.

Proof. Let $U$ be a universal $\Sigma_\alpha^0$ set and consider $A = \{x \mid (x, x) \in U\}$. □

2. Vorlesung

Analytic Sets

The continuous image of a Borel set need not be Borel. Let $X$ be a Polish space.

Definition. $A \subseteq X$ is analytic iff there is a continuous function $f : \mathcal{N} \to X$ with range $A$. The projection of a set $S \subseteq X \times Y$ is the set $P = \{x \mid (x, y) \in S$ for some $y \in Y\}$.

Lemma 4. The following are equivalent:
(a) $A$ is the continuous image of a Borel set in some Polish space.
(b) $A$ is analytic.
(c) $A$ is the projection of a closed set in $X \times \mathcal{N}$.
(d) $A$ is the projection of a Borel set in $X \times Y$ for some Polish space $Y$.

Proof. We will show that every Borel set is analytic. Then (a)$\rightarrow$(b) follows. (b)$\rightarrow$(c) holds since if $A$ is the range of $f : \mathcal{N} \to X$ then $A$ is the projection of the closed set $\{(f(x), x) \mid x \in \mathcal{N}\} \subseteq X \times \mathcal{N}$. (c)$\rightarrow$(d)$\rightarrow$(a) is trivial.

Note that every closed set in a Polish space forms a Polish space and therefore is analytic by Lemma 1. So it suffices to show that each Borel subset of a Polish space $X$ is the projection of a closed set in $X \times \mathcal{N}$. We show that the family $P$ of all projections of closed sets in $X \times \mathcal{N}$ is closed
under countable unions and intersections. As $P$ clearly contains all closed sets and each open set is the countable union of closed sets, it follows that $P$ contains all Borel sets, as desired.

Let $A_n$ be the projection of the closed set $F_n \subseteq X \times \mathcal{N}$ for each $n$. We shall show that $\bigcup_n A_n$, $\bigcap_n A_n$ are projections of closed sets. As before, choose a continuous mapping of $\mathcal{N}$ onto $\mathcal{N}^\omega$ and for each $a \in \mathcal{N}$ let $(a)_n$ be the $n$-th coordinate of the image of $a$.

\[
\begin{align*}
x \in \bigcup_n A_n &\iff \\
\exists n \exists a (x,a) \in F_n &\iff \\
\exists a \exists b (x,a) \in F_{b(0)} &\iff \\
\exists c (x,(c)_0) \in F_{(c)_1(0)}. &\iff \\
x \in \bigcap_n A_n &\iff \\
\forall n \exists a (x,a) \in F_n &\iff \\
\exists c \forall n (x,(c)_n) \in F_n &\iff \\
\exists c (x,c) \in \bigcap_n \{(x,c) \mid (x,(c)_n) \in F_n\}. &\iff \\
\end{align*}
\]

Hence $\bigcup_n A_N$ is the projection of the closed set $\{(x,c) \mid (x,(c)_0) \in F_{(c)_1(0)}\}$ and $\bigcap_n A_N$ is the projection of an intersection of closed sets. $\Box$

Lemma 5. The collection of analytic sets is closed under countable unions and intersections, as well as continuous images and preimages.

Proof. Closure under countable unions and intersections was established in the proof of the previous lemma. Closure under continuous images is clear by definition. Suppose that $x \in A \iff f(x) \in B$, where $f$ is continuous and $B$ is analytic. Write $y \in B \iff \exists z C(y,z)$, where $C$ is Borel. Then $x \in A \iff \exists z C(f(x),z)$, so $A$ is the projection of the Borel set $\{(x,z) \mid C(f(x),z)\}$ and therefore is analytic. $\Box$

The Projective Hierarchy

The collection of analytic sets is not closed under complementation. For $n \geq 1$ we define the $\Sigma^1_n$, $\Pi^1_n$ and $\Delta^1_n$ subsets of a Polish space $X$ as follows:

- $\Sigma^1_1 = \text{Analytic}$
- $\Pi^1_1 = \text{Coanalytic} = \text{Complements of Analytic sets}$
- $\Sigma^1_{n+1} = \text{Projections of} \ \Pi^1_n \ \text{subsets of} \ X \times \mathcal{N}$
\[ \Pi_{q+1}^1 = \text{Complements of } \Sigma_{n+1}^1 \text{ sets} \]
\[ \Delta_n^1 = \Sigma_n^1 \cap \Pi_n^1. \]

A set is \textit{projective} iff it is \( \Sigma_n^1 \) or \( \Pi_n^1 \) for some \( n \). It is obvious that \( \Delta_n^1 \subseteq \Sigma_n^1 \subseteq \Delta_{n+1}^1 \), \( \Delta_n^1 \subseteq \Pi_n^1 \subseteq \Delta_{n+1}^1 \); we shall show that \( \Sigma_n^1 \neq \Pi_n^1 \) and therefore these inclusions are proper.

Lemma 6. (Universal \( \Sigma_n^1 \) Sets) For each \( n \geq 1 \) there is a set \( U \subseteq \mathcal{N}^2 \) such that \( U \) is \( \Sigma_n^1 \) and for every \( \Sigma_n^1 \) \( A \subseteq \mathcal{N} \) there is some \( v \in \mathcal{N} \) such that \( A = \{ x \mid (x, v) \in U \} \).

Proof. Let \( h \) be a homeomorphism of \( \mathcal{N}^2 \) with \( \mathcal{N} \). For notational convenience, define \( \Sigma_0^1 \) to be \( \Sigma_0^0 \). We prove the Lemma by induction on \( n \geq 0 \). By Lemma 2 there does exists a universal \( \Sigma_0^1 \) set. Inductively, assume that \( V \) is a universal \( \Sigma_{n-1}^1 \) set and define \( U = \{ (x, y) \mid (h(x, a), y) \notin V \text{ for some } a \in \mathcal{N} \} \). Then \( U \) is \( \Sigma_n^1 \). If \( A \subseteq \mathcal{N} \) is \( \Sigma_n^1 \) then there is a \( \Pi_{n-1}^1 \) set \( B \) such that \( A = \{ x \mid (x, a) \in B \text{ for some } a \in \mathcal{N} \} \). The set \( C = \mathcal{N} - h[B] \) is \( \Sigma_{n-1}^1 \) and since \( V \) is universal there exists a \( v \) such that \( C = \{ u \mid (u, v) \in V \} \); then

\[
\begin{align*}
x \in A \leftrightarrow \\
(x, a) \in B \text{ for some } a \leftrightarrow \\
h(x, a) \notin C \text{ for some } a \leftrightarrow \\
(h(x, a), v) \notin V \text{ for some } a \leftrightarrow \\
(x, v) \in U.
\end{align*}
\]

So \( U \) is a universal \( \Sigma_n^1 \) set. \( \Box \)

Corollary 7. For each \( n \geq 1 \) there is a \( \Sigma_n^1 \) set which is not \( \Pi_n^1 \).

Every Borel set is \( \Delta_1^1 \). This follows from the fact that the latter is closed under countable unions and intersections, and contains all open and closed sets. Conversely:

Theorem 8 (Suslin’s Theorem). Every \( \Delta_1^1 \) set is Borel.

Proof. We say that a set \( D \) \textit{separates} two disjoint sets \( A, B \) iff \( A \) is contained in \( D \) and \( B \) is disjoint from \( D \). We shall show that any two disjoint analytic sets can be separated by a Borel set, which clearly implies the Theorem.
First note that if $A = \bigcup_{n} A_n$, $B = \bigcup_{n} B_n$ and the pair $A_m, B_n$ can be separated by a Borel set $D_{m,n}$ for each $m, n$, then the pair $A, B$ can be separated by a Borel set, namely by $D = \bigcup_{m} \bigcap_{n} D_{m,n}$.

Now let $A, B$ be analytic and choose continuous functions $f, g$ such that $A = f[\mathcal{N}], B = g[\mathcal{N}]$. For each $s \in \text{Seq}$ let $A_s = f[N_s], B_s = g[N_s]$, where $N_s$ is the basic open set $\{ f \mid s \subseteq f \}$ in Baire space. For each $a, b \in \mathcal{N}$, \{f(a)\} = \bigcap_n A_{a|n}$ and \{g(b)\} = \bigcap_n B_{b|n}. It follows that for any $a, b \in \mathcal{N}$, there exists $n$ such that $A_{a|n}$ and $B_{b|n}$ can be separated by an open set, as if $U_a, U_b$ are disjoint open sets separating $f(a)$ from $g(b)$, the sets $A_{a|n}, B_{b|n}$ will be contained in $U_a, U_b$, respectively, for large enough $n$.

Now suppose that $A, B$ cannot be separated by a Borel set. Then for some $m_0, n_0, A_{(m_0)}, B_{(n_0)}$ cannot be separated by a Borel set. Then for some $m_1, n_1$, the sets $A_{(m_0, m_1)}, B_{(n_0, n_1)}$ cannot be separated by a Borel set, etc. Let $a = \langle m_0, m_1, \ldots \rangle$ and $b = \langle n_0, n_1, \ldots \rangle$. Then $A_{a|n}, B_{b|n}$ cannot be separated by a Borel set for any $n$, in contradiction to the previous paragraph. $\Box$
Suslin Sets and Regularity Properties

For any set $S$ let $\text{Seq}(S)$ denote the collection of finite sequences of elements of $S$. A tree $T$ on $S$ is a subset of $\text{Seq}(S)$ closed under initial segments. A branch through $T$ is a function $a : \omega \to S$ such that $a \upharpoonright n \in T$ for all $n$. $T$ is well-founded iff $T$ has no branch.

Let $\kappa$ be an infinite cardinal. A tree on $\omega \times \kappa$ is a set $T$ of pairs $(s, h) \in \text{Seq}(\omega) \times \text{Seq}(\kappa)$ such that length $(s) = \text{length } (h)$ and for each $n \leq \text{length } (s)$, $(s \upharpoonright n, h \upharpoonright n) \in T$. For each $x \in \mathcal{N}$, $T(x) = \{h \mid (x \upharpoonright n, h) \in T, \text{ where } n = \text{length } (h)\}$. Then $T(x)$ is a tree on $\kappa$ for each $x \in \mathcal{N}$. The projection of $T$ is defined by

$$p[T] = \{x \in \mathcal{N} \mid T(x) \text{ has a branch}\}.$$ 

A set of reals $A$ is $\kappa$-Suslin iff it is the projection of a tree on $\omega \times \kappa$.

Proposition 9. Analytic subsets of $\mathcal{N}$ are $\omega$-Suslin.

Proof. If $A$ is analytic, then $A$ is the projection of a closed set $C$ on $\mathcal{N} \times \mathcal{N}$. Now consider the following tree on $\omega \times \omega$:

$$T = \{(a \upharpoonright n, b \upharpoonright n) \mid (a, b) \in C\}.$$ 

A branch through $T$ is essentially a pair $(a, b) \in C$, and conversely, every pair $(a, b) \in T$ is a branch through $T$. As $A$ is the projection of $C$, it is also the projection of the tree $T$. $\Box$

3. Vorlesung

A set $C$ is perfect iff it is nonempty, closed and has no isolated points. $\text{ZFC}^-\Sigma_1$ is the theory obtained from $\text{ZFC}$ by restricting the Replacement Axiom to $\Sigma_{100}$ formulas.

Theorem 10. Suppose that $A$ is the projection of a tree $T$ on $\omega \times \kappa$ which belongs to the transitive $\text{ZFC}^-\Sigma_1$ model $M$. If $A$ has an element not belonging to $M$ then $A$ contains a perfect subset.

Corollary 11. An uncountable analytic set has a perfect subset.
Proof of Corollary 11. If $A$ is analytic then choose a tree $T$ on $\omega \times \omega$ such that $A = p[T]$. Choose a countable $\text{ZFC}^-$ model $M$ such that $T \in M$. By the Theorem, $A$ is either a subset of $M$, and therefore countable, or contains a perfect subset. \(\Box\)

Proof of Theorem 10: Define $T_0 = T$ and inductively:

$$T_{\alpha+1} = \{(s, h) \in T_\alpha \mid \text{There exist } (s_0, h_0), (s_1, h_1) \in T_\alpha \text{ extending } (s, h) \text{ such that } s_0, s_1 \text{ are incompatible}\}.$$

For limit $\lambda$, $T_\lambda = \bigcap \{T_\alpha \mid \alpha < \lambda\}$. As $T$ belongs to $M$ and $M$ is a model of $\text{ZFC}^-$, the sequence $\langle T_\beta \mid \beta \in \text{Ord}(M) \rangle$ is definable in $M$ and for some least ordinal $\alpha \in M$, $T_\alpha = T_{\alpha+1}$.

Let $x$ belong to $A$ and choose $f$ so that $(x, f)$ is a branch through $T$. If $(x, f)$ is not a branch through $T_\alpha$ then there is some least ordinal $\beta$ such that $(x, f)$ is a branch through $T_\beta$ but not through $T_{\beta+1}$. So there is some $(s, h) \subseteq (x, f)$ in $T_\beta - T_{\beta+1}$ and therefore $x$ is the union of $\{s' \mid (s', h') \text{ extends } (s, h) \text{ and belongs to } T_\beta\}$. It follows that $x$ belongs to $M$, since $T_\beta$ does.

As $A$ has an element not belonging to $M$, it must be that $T_\alpha$ has a branch and therefore is nonempty. If $(s, h)$ belongs to $T_\alpha$ then we can choose $(s_0, h_0), (s_1, h_1)$ extending $(s, h)$ in $T_\alpha$ with $s_0, s_1$ incompatible. Then we can choose extensions $(s_{00}, h_{00}), (s_{01}, h_{01})$ of $(s_0, h_0)$ in $T_\alpha$ such that $s_{00}, s_{01}$ are incompatible, and similarly for $(s_1, h_1)$. Continuing in this way we can build a subtree of $T_\alpha$ whose projection is a perfect subset of $A$. \(\Box\)

A null set is a set of reals of Lebesgue measure 0. A meager set is the countable union of nowhere dense sets. A set of reals is measurable iff it differs by a null set from a Borel set (equivalently, from a countable union of closed sets or from a countable intersection of open sets). It has the Baire property iff it differs by a meager set from a Borel set (equivalently, from an open set).

Theorem 12. Suppose that $A$ is the projection of a tree $T$ on $\omega \times \kappa$ which belongs to the transitive $\text{ZFC}^-$ model $M$. Suppose that $M$ has only countably many reals. Then $A$ is measurable and has the property of Baire.

Corollary 13. Analytic sets are measurable and have the Baire property.
Proof of Corollary 13. If $A$ is analytic then $A$ is the projection of a tree $T$ on $\omega \times \omega$. There is a countable ZFC$^-$ model that contains $T$. So by the Theorem, $A$ is measurable and has the Baire property. \( \square \)

Before proving Theorem 12, we must introduce Borel codes and absoluteness. Let $I_1, I_2, \ldots$ be a recursive enumeration of the basic open sets of $N$. Let $c$ belong to $N$. We define $u(c) \in N$ by $u(c)(n) = c(n + 1)$ for all $n$. Let $\Gamma$ be a recursive bijection from $N \times N$ onto $N$. For each $i \in N$ we define $v_i(c)$ by $v_i(c)(n) = c(\Gamma(i, n) + 1)$ for all $n$. For each positive countable ordinal $\alpha$ we define coding sets $\Sigma_\alpha, \Pi_\alpha$ as follows:

- $c \in \Sigma_1$ iff $c(0) > 1$
- $c \in \Pi_\alpha$ iff either $c \in \Sigma_\beta \cup \Pi_\beta$ for some $\beta < \alpha$ or $c(0) = 0$ and $u(c) \in \Sigma_\alpha$

For $\alpha > 1$: $c \in \Sigma_\alpha$ iff either $c \in \Sigma_\beta \cup \Pi_\beta$ for some $\beta < \alpha$ or $c(0) = 1$ and $v_i(c) \in \bigcup_{\beta < \alpha} (\Sigma_\beta \cup \Pi_\beta)$ for all $i$.

If $c \in \Sigma_\alpha$ we call $c$ a $\Sigma_\alpha^0$-code, similarly for $\Pi_\alpha^0$-codes. The union of all $\Sigma_\alpha$ is the set $BC$ of Borel codes. The Borel code $c$ codes the Borel set $A_c$ defined as follows:

- If $c \in \Sigma_1$ then $A_c = \bigcup \{I_n \mid c(n) = 1\}$
- If $c \in \Pi_\alpha$ and $c(0) = 0$ then $A_c = \sim A_{u(c)}$
- If $c \in \Sigma_\alpha$ and $c(0) = 1$ then $A_c = \bigcup I_i A_{v_i(c)}$.

It is clear that for every $\alpha > 0$, if $c \in \Sigma_\alpha$ then $A_c \in \Sigma_\alpha^0$, similarly for $\Pi_\alpha$. Conversely, every $\Sigma_\alpha^0, \Pi_\alpha^0$ set $B$ is coded by some $c \in \Sigma_\alpha, \Pi_\alpha^0$, respectively. Thus $\{A_c \mid c \in BC\}$ is the collection of all Borel sets.

We introduce the hierarchy of $\Sigma_\alpha$ and $\Pi_\alpha$ formulas. A $\Sigma_1$ formula with parameter $p \in N$ is a formula of the form

$$\varphi(y_1, \ldots, y_n) \leftrightarrow \exists z \psi(y_1, \ldots, y_n, z, p),$$

where $\psi$ is arithmetical, i.e., a formula in the language of second-order arithmetic with only number quantifiers. The subsets of $N^n$ which are definable by $\Sigma_1$ formulas with parameters are exactly the analytic subsets of $N^n$. A $\Pi_1$ formula with parameter $p$ is the negation of a $\Sigma_1$ formula with parameter $p$. Inductively: A $\Sigma_{k+1}$ formula with parameter $p$ is a formula of the form $\varphi(y_1, \ldots, y_n) \leftrightarrow \exists z \psi(y_1, \ldots, y_n, z)$, where $\psi$ is a $\Pi_k$ formula with parameter.
$p$ and a $\Pi_{k+1}^1$ formula with parameter $p$ is the negation of such a formula.
A subset of $\mathcal{N}^n$ is $\Sigma^1_n(p)$, $\Pi^1_n(p)$ iff it is definable by a $\Sigma^1_n$, $\Pi^1_n$ formula with
parameter $p$ and is $\Delta^1_n(p)$ iff it is both $\Sigma^1_n(p)$ and $\Pi^1_n(p)$. When $p$ is recursive,
we write $\Sigma^1_n$, $\Pi^1_n$, $\Delta^1_n$. 

Lemma 14. (a) The set BC of all Borel codes is $\Pi^1_1$.
(b) There is a $\Delta^1_1$ relation $R$ such that for Borel codes $c$, $R(a,c)$ iff $a \in A_c$.
(c) The following properties of Borel codes are $\Pi^1_1$:

$$
A_c \subseteq A_d \\
A_c = A_d \\
A_c = \emptyset \\
A_c = A_d \cup A_e \\
A_c = \sim A_d \\
A_c = A_d \cap A_e \\
A_c = A_d \Delta A_e \\
A_c = \bigcup_n A_{c_n}.
$$

Proof. (a) Define the relation $E$ by:

$$xEy \text{ iff either } (y(0) = 0 \text{ and } x = u(y)) \text{ or } (y(0) = 1 \text{ and } x = v_i(y) \text{ for some } i).$$

Then $y$ is a Borel code iff there is no infinite sequence $y = z_0, z_1, \ldots$ with $z_{n+1}Ez_n$ for each $n$. As the relation $E$ is arithmetical, it follows that BC is $\Pi^1_1$.

(b) For any $c \in \mathcal{N}$ there is a smallest countable $T = T_c \subseteq \mathcal{N}$ with the property:

$$(*)_c \text{ } c \in T \text{ and whenever } y \in T, zEy \text{ then } z \in T.$$

And if $c$ is a Borel code, $a \in \mathcal{N}$ then there is a unique function $h = h_{a,c}$ on $T_c$ such that for all $y \in T_c$:

$$(**)_a \text{ If } y(0) > 1 \text{ then } h(y) = 1 \text{ iff } a \in I_n \text{ for some } n \text{ such that } y(n) = 1 \\
\text{If } y(0) = 0 \text{ then } h(y) = 1 \text{ iff } h(u(y)) = 0 \\
\text{If } y(0) = 1 \text{ then } h(y) = 1 \text{ iff } h(v_i(y)) = 1 \text{ for some } i.$$

For $y \in T_c$ and $h$ as above we have $h(y) = 1$ iff $a \in A_y$. Thus for a Borel
code $c$:
\( a \in A_c \) iff
For all countable \( T \) satisfying \((*)_c\) and all \( h \) defined on \( T \) satisfying \((**)_a\), \( h(c) = 1 \) iff
There is a countable \( T \) satisfying \((*)_c\) and an \( h \) defined on \( T \) satisfying \((**)_a\) such that \( h(c) = 1 \).

As \((*)_c\) is \( \Sigma^1_1 \) and \((**)_a\) is arithmetical, this gives the desired result.

(c) This follows easily from (b). \( \square \)

Lemma 15. (Mostowski Absoluteness) Suppose that \( M \) is a transitive model of \( \text{ZFC}^- \).

(a) If \( \varphi(y_1, \ldots, y_n) \) is a \( \Sigma^1_1 \) formula with parameter in \( M \) then for all \( y_1, \ldots, y_n \) in \( M \):
\[
M \models \varphi(y_1, \ldots, y_n) \text{ iff } \varphi(y_1, \ldots, y_n) \text{ is true.}
\]

(b) If \( \varphi(y_1, \ldots, y_n) \) is a \( \Sigma^1_2 \) formula with parameter in \( M \) then for all \( y_1, \ldots, y_n \) in \( M \):
\[
M \models \varphi(y_1, \ldots, y_n) \text{ implies } \varphi(y_1, \ldots, y_n) \text{ is true.}
\]

Proof. (a) Using a recursive homeomorphism between \( \mathcal{N}^n \) and \( \mathcal{N} \) we can assume that \( n = 1 \). In both \( M \) and the universe we have that \( \varphi(y) \) holds iff \( T(y) \) has a branch, where \( T \) is a tree on \( \omega \times \omega \). If \( T(y) \) has a branch in \( M \) then of course it also has one in the universe. If \( T(y) \) has no branch in \( M \) then \( T(y) \) is well-founded in \( M \) and therefore there exists an order-preserving function in \( M \) from \( T(y) \) into the ordinals of \( M \). It follows that there is such a function in the universe and therefore \( T(y) \) has no branch in the universe.

(b) Write \( \varphi(y_1, \ldots, y_n) = \exists z \, \psi(y_1, \ldots, y_n, z) \), where \( \psi \) is \( \Pi^1_1 \). If \( M \) satisfies \( \varphi(y_1, \ldots, y_n) \) then choose \( y \in M \) such that \( \psi(y_1, \ldots, y_n, z) \) holds in \( M \). It follows from (a) that the latter also holds in the universe, and therefore so does \( \varphi(y_1, \ldots, y_n) \). \( \square \)

For a transitive model \( M \) of \( \text{ZFC}^- \), let \( \text{BC}^M \) denote the set of Borel codes, as interpreted in \( M \), and for \( c \in \text{BC}^M \), let \( A^M_c \) denote \( A_c \) as interpreted in \( M \).

Corollary 16. Suppose that \( M \) is a transitive model of \( \text{ZFC}^- \). Then:

(a) \( \text{BC}^M = \text{BC} \cap M \).
(b) If \( c \) belongs to \( BC^M \) then \( A_c^M = A_c \cap M \).

(c) The following properties of Borel codes in \( M \) hold iff they hold in \( M \\
A_c \subseteq A_d \\
A_c = A_d \\
A_c = \emptyset \\
A_c = A_d \cup A_e \\
A_c = \sim A_d \\
A_c = A_d \cap A_e \\
A_c = A_d \Delta A_e \\
A_c = \bigcup_n A_{c_n}.

4. Vorlesung

Lemma 17. The following sets are both \( \Sigma^1_2 \) and \( \Pi^1_2 \) definable:

(a) \( \{ c \mid c \text{ is a Borel code and } A_c \text{ is null} \} \).

(b) \( \{ c \mid c \text{ is a Borel code and } A_c \text{ is meager} \} \).

Proof. For a Borel code \( c \):

\( A_c \) is null iff

For each \( n \) there exists a \( \Sigma_1 \) code \( d \) such that \( A_c \subseteq A_d \) and \( A_d \) has measure less than \( 1/n \) iff

For all \( \Pi_1 \) codes \( e \), if \( A_c \subseteq A_e \) then \( A_e \) has measure 0.

As the properties “\( d \) is a \( \Sigma_1 \) code and \( A_d \) has measure less than \( 1/n \)” and “\( e \) is a \( \Pi_1 \) code and \( A_e \) has measure 0” are arithmetical, the above provides both \( \Sigma^1_2 \) and \( \Pi^1_2 \) definitions for \( \{ c \mid c \text{ is a Borel code and } A_c \text{ is null} \} \).

Also:

\( A_c \) is meager iff

There exist \( \Pi_1 \) codes \( c_n, n \in N \) such that \( A_c \subseteq \bigcup_n A_{c_n} \) and each \( A_{c_n} \) is nowhere dense iff

For all \( \Sigma_1 \) codes \( d \), if \( A_d \) is nonempty then \( A_c \Delta A_d \) is not meager.

As the property “\( c \) is a \( \Pi_1 \) code and \( A_c \) is nowhere dense” is arithmetical, the second line above gives a \( \Sigma^1_2 \) definition of \( \{ c \mid c \text{ is a Borel code and } A_c \text{ is meager} \} \), and using this, the third line above gives a \( \Pi^1_2 \) definition. \( \square \)
Corollary 18. Suppose that $c$ is a Borel code and $c$ belongs to the transitive 
ZFC$^-$ model $M$. Then $A_c$ is null iff $M \models A_c$ is null, and $A_c$ is meager iff $M \models A_c$ is meager.

Proof. Use Lemma 17 and part (b) of Lemma 15. $\square$

We consider $B_m$ and $B_c$, the quotients of the $\sigma$-algebra of Borel sets by 
the ideals $I_m$ of null sets and $I_c$ of meager sets. For each $B \in B$, let $[B]_m$, $[B]_c$ denote the equivalence class of $B$ in $B_m$, $B_c$, respectively. We view $B_m$, $B_c$ as forcing notions by discarding $[\emptyset]_m$, $[\emptyset]_c$ and using the natural order of 
inclusion modulo $I_m$, $I_c$, respectively.

Lemma 19. (a) If $G$ is $B_m$-generic then there is a unique real $x_G$ such that 
for all non-null $B \in B$:

$$[B]_m \in G \leftrightarrow x_G \in B^*,$$

where $B^*$ denotes $A^*_V[G]$ for any Borel code $c$ for $B$ (this definition is independent of the choice of $c$). And a real $x$ (in an outer model of $V$) is of this form iff $x \notin B^*$ for each null $B \in B$. Such reals are called random reals.

(b) If $G$ is $B_c$-generic then there is a unique real $x_G$ such that for all non-meager $B \in B$:

$$[B]_c \in G \leftrightarrow x_G \in B^*,$$

where $B^*$ denotes $A^*_V[G]$ for any Borel code $c$ for $B$. And a real $x$ (in an outer model of $V$) is of this form iff $x \notin B^*$ for each meager $B \in B$. Such reals are called Cohen reals.

Proof. (a) For convenience we work not in Baire space but in the (real) real numbers $\mathcal{R}$. Define $x_G = \sup\{r \mid r \text{ is rational and } [(r, \infty)] \in G\}$. We show that $x_G$ belongs to $A^*_c$ iff $[A_c] \in G$, by induction on $c \in BC$. If $c$ is a $\Sigma_1$ code for a rational interval $(p, q)$ then we have:

$x_G \in A^*_c$ iff

$p < x_G < q$ iff

$p < \sup\{r \in \mathcal{Q} \mid [(r, \infty)] \in G\} < q$ iff

$[(p, \infty)] \in G$ and $[(q, \infty)] \notin G$ iff

$[(p, q)] \in G$

$[A_c] \in G$. 

14
If \( c \) is a \( \Sigma_1 \) code for the union of rational intervals \( A_c = \bigcup_n I_{k_n} \), then:

\[
x \in A_c^* \text{ iff } x \in \bigcup_n I_{k_n}^* \text{ iff } [I_{k_n}] \in G \text{ for some } n \text{ iff } [\bigcup_n I_{k_n}] \in G \text{ iff } [A_c] \in G.
\]

Inductively, if \( \alpha \) is countable and \( c \) is a \( \Sigma_\alpha \) code, then the result holds by induction by the same argument as above. If \( c \) is a \( \Pi_\alpha \) code, we may assume that \( c(0) = 0 \) and therefore \( u(c) \) is a \( \Sigma_\alpha \) code, \( A_{u(c)} = \mathcal{R} - A_c \) and we have:

\[
x \in A_c^* \text{ iff } x \notin A_{u(c)}^* \text{ iff } [A_{u(c)}] \notin G \text{ iff } [A_c] \in G.
\]

This proves the first part of (a), as the uniqueness of \( x_G \) is clear.

Suppose that \( x = x_G \) is random. If \( A_c \) is null then \([A_c] \notin G \) and therefore by the first part of (a), \( x \notin A_c^* \). Conversely, suppose that \( x \notin A_c^* \) whenever \( A_c \) is null. Note that if \([A_c] = [A_d] \) then \( A_c \triangle A_d \) is null, \( A_c^* \triangle A_d^* \) is null and thus \( x \in A_c^* \) iff \( x \in A_d^* \). Now let \( G = \{[A_c] \mid x \in A_c^* \} \). It is easy to check that \( G \) is a filter on \( \mathcal{B}_\omega \). We claim that \( G \) is \( \mathcal{B}_\omega \)-generic: Since \( \mathcal{B}_\omega \) satisfies the countable chain condition, it suffices to show that if \( \{[A_n] \mid n \in N \} \) is a maximal antichain in \( \mathcal{B}_\omega \) then \( x \) belongs to \( A_n^* \) for some \( n \). But the maximality of this antichain implies that \( x \) belongs to \( \bigcup_n A_n^* \), and the latter equals \( \bigcup_n A_n^{\alpha_n} \).

(b) This is proved exactly as part (a), using the fact that \( \mathcal{B}_c \) also satisfies the countable chain condition. \( \square \)

We can now prove Theorem 12. As \( M \) has only countably many reals, it follows that the set of reals which are not random over \( M \) is null. Thus to show that \( A \) is measurable, it suffices to show that \( \{x \in A \mid x \text{ is random over } M \} \) is Borel. Suppose that \( x \) is random over \( M \) and let \( x = x_G \), where \( G \) is \( \mathcal{B}_\omega \)-generic over \( M \). Then \( M[x] \) is a model of \( \text{ZFC}^- \) and we have:

\[
x \in A \text{ iff } x \in p[T] \text{ iff } M[x] \models x \in p[T] \text{ iff }
\]
For some \([B]_m \in G\), \([B]_m \Vdash x_G \in p[T]\) iff

for some \([B]_m\), \(x \in B^*\) and \([B]_m \Vdash x_G \in p[T]\) iff

\(x \in \bigcup\{B^* \mid [B]_m \Vdash x_G \in p[T]\}\),

and the latter is a Borel property of \(x\). The same proof shows that \(A\) has the property of Baire.  \(\Box\)

We next consider the Ramsey property. For an infinite set \(A \subseteq \omega\) we let \([A]^{\omega}\) denote the set of all infinite subsets of \(A\). Is \(S \subseteq [\omega]^{\omega}\) then we say that an infinite \(H \subseteq \omega\) is homogeneous for \(S\) iff either \([H]^{\omega} \subseteq S\) or \([H]^{\omega} \cap S = \emptyset\).

We say that \(S \subseteq [\omega]^{\omega}\) is Ramsey iff there is an infinite homogeneous set \(H\) for \(S\).

Theorem 20. Suppose that \(A\) is the projection of the tree \(T\) on \(\omega \times \kappa\), where \(T\) belongs to the transitive ZFC\(^-\) model \(M\) and \(M\) has only countably many sets of reals. Then \(A\) is Ramsey.

The proof of this result makes use of Mathias forcing. A condition is a pair \((s, A)\) where \(s\) is a finite subset of \(\omega\) and \(A\) is an infinite subset of \(\omega\) such that \(\max s < \min A\). A condition \((s, A)\) extends a condition \((t, B)\) iff

1. \(t\) is an initial segment of \(s\).
2. \(A \subseteq B\).
3. \(s - t \subseteq B\).

If \(G\) is Mathias generic then \(G\) is determined by the real

\[x_G = \bigcup\{s \mid (s, A) \in G\text{ for some }A\},\]

since \(G = \{(s, A) \mid s \subseteq x_G \subseteq s \cup A\}\). The real \(x_G\) is called a Mathias real. We shall prove:

Lemma 21. Let \(\varphi\) be a sentence of the forcing language and \((s, A)\) a condition. Then there exists an infinite \(B \subseteq A\) such that \((s, B)\) decides \(\varphi\) (i.e., forces either \(\varphi\) or \(\sim \varphi\)).

Lemma 22. A real \(x\) is Mathias over the transitive ZFC\(^-\) model \(M\) iff \(x\) is infinite and for each maximal almost disjoint family \(\mathcal{A} \in M\) of subsets of \(\omega\), there is an \(A \in \mathcal{A}\) such that \(x\) is almost contained in \(A\).
Given these Lemmas we prove Theorem 20 as follows: Suppose that $M$ is a transitive ZFC$^-$ model with only countably many sets of reals and that $A$ is the projection of the tree $T \in M$. For any real $x$, if $M[x]$ satisfies ZFC$^-$, then:

$x$ belongs to $A$ iff
$T(x)$ has a branch iff
$M[x] \models T(x)$ has a branch.

Now let $\varphi$ be the sentence “$T(x_G)$ has a branch”. By Lemma 21 there is a condition of the form $(\emptyset, A)$ which decides $\varphi$; assume that $(\emptyset, A) \models \varphi$. As $M$ has only countably many sets of reals, there exists a Mathias generic $G$ over $M$ which contains the condition $(\emptyset, A)$. Thus $M[x_G] \models \varphi$. By Lemma 22 every infinite $y \subseteq x$ is a Mathias real over $M$, and also the generic determined by $y$ contains the condition $(\emptyset, A)$. Thus $T(y)$ has a branch for each infinite $y \subseteq x_G$ and therefore $x_G$ is homogeneous for $A$.

5. Vorlesung

Lemma 21. Let $\varphi$ be a sentence of the forcing language and $(s, A)$ a condition. Then there exists an infinite $B \subseteq A$ such that $(s, B)$ decides $\varphi$ (i.e., forces either $\varphi$ or $\sim \varphi$).

Lemma 22. A real $x$ is Mathias over the transitive ZFC$^-$ model $M$ iff $x$ is infinite and for each maximal almost disjoint family $A \in M$ of subsets of $\omega$, there is an $A \in A$ such that $x$ is almost contained in $A$.

Proof of Lemma 21. For any $A \subseteq \omega$ and $k \in \omega$ we let $A(> k)$ denote $\{n \in A \mid n > k\}$. If $s$ is a finite subset of $\omega$ then $A(> s)$ denotes $A(> \text{max } s)$. We first construct an infinite $B \subseteq A$ such that

(*) If $(t, C)$ extends $(s, B)$ and decides $\varphi$ then so does $(t, B(> t))$.

By induction we define $b_k = \min B_k$ for $k \in \omega$: Set $B_0 = A$. To define $B_{k+1}$ let $\{t_1, \ldots, t_i\}$ be a list of all subsets of $\{b_1, \ldots, b_k\}$. Construct $B_k = B_k^1 \supseteq B_k^2 \supseteq \cdots \supseteq B_{k+1} = B_{k+1}^1$ as follows: Given $B_{k+1}^j$, if there exists $C \subseteq B_{k+1}^j$ such that $(s \cup t_{j+1}, C)$ decides $\varphi$ then set $B_{k+1}^{j+1} = C$; otherwise, $B_{k+1}^{j+1} = B_{k+1}^j$. Then $B = \{b_k \mid k \in \omega\}$ is as desired.
So we can suppose that $A$ satisfies (*). Now define $b_k = \min B_k$ by induction on $k$ as follows: Set $B_0 = A$. Given $B_k$ construct $B_{k+1}$ such that for each $t \subseteq \{b_1, \ldots, b_k\}$ exactly one of the following holds:

$$(s \cup t \cup \{n\}, B_{k+1}(> n)) \models \varphi \text{ for each } n \in B_{k+1}$$
$$(s \cup t \cup \{n\}, B_{k+1}(> n)) \models \sim \varphi \text{ for each } n \in B_{k+1}$$
$$(s \cup t \cup \{n\}, B_{k+1}(> n)) \text{ does not decide } \varphi \text{ for each } n \in B_{k+1}.$$  

We claim that $(s, B)$ decides $\varphi$: Let $(t, C)$ be an extension of $(s, B)$ deciding $\varphi$. Assume that $\text{Length}(t)$ is minimal. If $\text{Length}(t) = \text{Length}(s)$ then we are done since it follows from (*) for $A$ that $(s, B)$ decides $\varphi$. Otherwise let $m = \max t$ and write $t = s \cup t' \cup \{m\}$ where $t' \subseteq \{b_1, \ldots, b_k\}$. By construction $(s \cup t' \cup \{m\}, B_{k+1}(> m))$ either forces $\varphi$ or $\sim \varphi$; assume the former. Then the same holds for each $m \in B_{k+1}$. It follows that $(s \cup t', C)$ decides $\varphi$, contradicting the minimality of $\text{Length}(t)$. So $(s, B)$ is the desired extension of $(s, A)$ which decides $\varphi$. □

Proof of Lemma 22. If $x$ is Mathias over $M$ and $A$ is a maximal almost disjoint family in $M$ then $D = \{(s, B) \mid B \text{ is almost contained in an element of } A\}$ is dense, and hence there exists $(s, B) \in G_x \cap D$; as $x$ is almost contained in $B$ and $B$ is almost contained in an element of $A$, it follows that $x$ is almost contained in an element of $A$.

Conversely, suppose that $x$ is infinite and each maximal almost disjoint family of $M$ has an element which almost contains $x$. Let $D \in M$ be dense open for Mathias forcing; we must show that $D$ has an element $(s, A)$ such that $s \subseteq x \subseteq s \cup A$. We say that an infinite $A$ captures $(s, D)$ iff

$$(***) \text{ For all infinite } B \subseteq A(> s) \text{ there exists a finite initial segment } t \text{ of } B \text{ such that } (s \cup t, A(> (s \cup t))) \text{ belongs to } D.$$

Main Claim. For any infinite $A$, there is an infinite $A^* \subseteq A$ such that $A^*$ captures $(s, D)$ for each $s$ with $\max s \in A^*$.

Proof of Main Claim. It suffices to show that for each infinite $A$ and $s$ there is an infinite $A^* \subseteq A$ such that $A^*$ captures $(s, D)$. For then, we can inductively define $k_n = \min A_n$ by setting $A_0 = A$ and choosing an infinite $A_{n+1} \subseteq A_n$ which captures $(s, D)$ for all $s$ with $\max s \in \{k_0, \ldots, k_n\}$; then $A^* = \{k_0, k_1, \ldots\}$ is as desired.

18
So suppose that $A$ and $s$ are given, with $\max s \in A$. We may assume that for all finite $t \subseteq A (> s)$ if $(s \cup t, B)$ belongs to $D$ for some $B \subseteq A (> (s \cup t))$ then in fact $(s \cup t, A (> (s \cup t)))$ belongs to $D$. Now let $S$ be the set of all $B$ such that either $B$ is not contained in $A (> s)$ or $(s \cup t, A (> (s \cup t)))$ belongs to $D$ for some finite initial segment $t$ of $B$. Then $S$ is an open set (in the space of infinite subsets of $\omega$). We shall show that open sets are completely Ramsey (see below), which implies that there exists an infinite $B \subseteq A (> s)$ all of whose infinite subsets either belong to $S$ or to the complement of $S$. By our assumption about $A$ and the density of $D$, it must be the case that all infinite subsets of $B$ belong to $S$. It follows that $B$ captures $(s, D)$, as desired.

Finally, we establish the complete Ramseyess of open sets: $S$ is completely Ramsey iff for any condition $(s, A)$ there exists $A^* \subseteq A$ such that either $[s, A^*] \subseteq S$ or $[s, A^*] \cap S = \emptyset$, where $[s, A^*] = \{B \mid s \subseteq B \subseteq s \cup A^*\}$. It suffices to show that open sets are Ramsey, as given $(s, A)$ we can consider $S^* = \{B \mid f^*(B) \in S\}$, where $f^*(B) = s \cup f[B]$ and $f$ is the increasing enumeration of $A$; the Ramseyess of the open set $S^*$ implies the complete Ramseyess of $S$ with respect to $(s, A)$. We say that $A$ accepts $s$ iff $[s, A] \subseteq S$ and rejects $s$ iff no $B \subseteq A$ accepts $s$.

(i) There is an $A = \{k_0, k_1, \ldots\}$ which either accepts or rejects each of its finite subsets, obtained by inductively defining $k_n = \min A_n$ where $A_0 = \omega$ and $A_{n+1} \subseteq A_n$ accepts or rejects each subset of $\{k_0, \ldots, k_n\}$.

(ii) In fact there is an $A = \{k_0, k_1, \ldots\}$ that either accepts $\emptyset$ or rejects each of its finite subsets, obtained by choosing $A_0$ as in (i) to reject $\emptyset$ (without loss of generality) and assuming that $A_0$ rejects each subset of $\{k_0, \ldots, k_{n-1}\}$, choosing $k_n$ as follows: For every subset $s$ of $\{k_0, \ldots, k_{n-1}\}$ there are only finitely many $z \in A_0$ such that $A_0$ accepts $s \cup \{z\}$, as otherwise there is an infinite $Z \subseteq A_0$ such that $A_0$ accepts $s \cup \{z\}$ for each $z \in Z$ and therefore $A_0$ accepts $s$, in contradiction to the assumption that $A_0$ rejects $s$. Thus we can choose $k_n \in A_0 - \{k_0, \ldots, k_{n-1}\}$ such that $A_0$ rejects each subset of $\{k_0, \ldots, k_n\}$. $A_0$ rejects each finite subset of $A = \{k_0, k_1, \ldots\}$ and therefore so does $A$ itself.

(iii) Now if the set $A$ constructed in (ii) accepts $\emptyset$ we have established the Ramseyess of $S$. Otherwise $A$ rejects each of its finite subsets. We claim that no infinite subset of $A$ belongs to $S$. Otherwise there is an infinite $B \subseteq A$ which belongs to $S$ and since $S$ is open there is a finite initial segment $s$ of $B$ such that $[s, B (> s)]$ is contained in $S$; this contradicts the fact that $A$ rejects $s$. □ (Main Claim).
Now apply the hypothesis on $x$ to obtain $A$ almost containing $x$ which captures $(s, D)$ for all $s$ with $\max s \in A$. Choose a nonempty finite initial segment $s$ of $x$ such that $\max s \in A$ and $x \subseteq s \cup A(> s)$. Consider the tree $T$ of $t \subseteq A(> s)$ such that $(s \cup t, A(> (s \cup t)))$ does not belong to $D$, ordered by end-extension. Then $T$ is well-founded, as $A$ captures $(s, D)$. By absoluteness, $T$ is well-founded in $V$ and therefore $x(> s)$ is not a branch through $T$. But then for some initial segment $t$ of $x(> s)$, the condition $(s \cup t, A(> (s \cup t)))$ belongs to $D$ and satisfies $s \cup t \subseteq x \subseteq s \cup t \cup A(> (s \cup t))$, as desired. □

6. Vorlesung

Borel Determinacy

A pruned tree $T$ on a set $A$ is a nonempty set of finite sequences of elements of $A$, closed under initial segments, such that each element of $T$ has proper extension in $T$. We let $[T]$ denote the collection of infinite branches through $T$. For any $X \subseteq [T]$, we define the game $G(T, X)$ as follows: Players I and II alternately choose $a_0, a_1, \ldots$ in $A$ so that for each $n$ the sequence $(a_0, \ldots, a_n)$ belongs to $T$. Player I wins the game iff the infinite sequence $(a_0, a_1, \ldots)$ belongs to $X$. A strategy for I assigns an extension of length $n+1$ in $T$ to each element of $T$ of even length $n$; similarly for II with “even” replaced by “odd”. A strategy $\sigma$ for I is a winning strategy iff I always wins the game using $\sigma$, no matter how II plays; we similarly define a winning strategy for II. The game $G(T, X)$ is determined iff either I or II has a winning strategy.

Theorem 23. If $X \subseteq [T]$ is either closed or open, then $G(T, X)$ is determined.

Proof. Suppose that $X$ is closed and that II has no winning strategy. Consider the strategy for I in which he plays in such a way as to guarantee that II still has no winning strategy afterwards. Inductively this is possible, as otherwise II would have had a winning strategy before I plays his next move. We claim that this is in fact a winning strategy for I: Otherwise there is a play of the game $(a_0, a_1, \ldots)$ where I follows the described strategy but I loses. Since $X$ is closed this means that for some $n$, $(a_0, a_1, \ldots, a_{2n})$ has no extension in $X$ (i.e., I has already lost at a finite stage of the game). But this contradicts the definition of I’s strategy!
Similarly, if \( X \) is open and \( I \) has no winning strategy, then \( II \) uses the strategy which always guarantees that \( I \) still has no strategy. This must win for \( II \), as otherwise \( II \) loses at a finite stage, contradicting the definition of his strategy. \( \square \)

Theorem 24. (Martin) If \( X \subseteq [T] \) is Borel then \( G(T, X) \) is determined.

Proof. Let \( T \) be a nonempty pruned tree on a set \( A \). A covering of \( T \) is a triple \((\hat{T}, \pi, \varphi)\) where

i. \( \hat{T} \) is a nonempty pruned tree on some set \( \hat{A} \).

ii. \( \pi : \hat{T} \rightarrow T \) is monotone (i.e., \( s \subseteq t \rightarrow \pi(s) \subseteq \pi(t) \)) with \( \text{Length}(\pi(s)) = \text{Length}(s) \). Thus \( \pi \) gives rise to a continuous function \( \pi : [\hat{T}] \rightarrow [T] \). 

iii. \( \varphi \) maps strategies for \( I \) (\( II \), respectively) in \( \hat{T} \) to strategies for \( I \) (\( II \), respectively) in \( T \) in such a way that \( \varphi(\hat{\sigma}) \) restricted to positions of length \( \leq n \) depends only on \( \hat{\sigma} \) restricted to positions of length \( \leq n \) for each \( n \).

iv. If \( \hat{\sigma} \) is a strategy for \( I \) (\( II \), respectively) in \( \hat{T} \) and \( x \in [T] \) is played according to \( \varphi(\hat{\sigma}) \) then there is \( \hat{x} \in [\hat{T}] \) played according to \( \hat{\sigma} \) such that \( \pi(\hat{x}) = x \).

It follows that if \( X \subseteq [T] \) and \( \hat{\sigma} \) is a winning strategy for \( I \) (\( II \), respectively) in \( G(\hat{T}, \hat{X}) \), where \( \hat{X} = \pi^{-1}[X] \), then \( \varphi(\hat{\sigma}) \) is a winning strategy for \( I \) (\( II \), respectively) in \( G(T, X) \). A \( k \)-covering of \( T \) is a covering \((\hat{T}, \pi, \varphi)\) of \( T \) such that \( \hat{T} \upharpoonright 2k = T \upharpoonright 2k \) and \( \pi \) is the identity on \( \hat{T} \upharpoonright 2k \). For \( X \subseteq [T] \) we say that \((\hat{T}, \pi, \varphi)\) unravels \( X \) iff \( \pi^{-1}[X] = \hat{X} \) is a clopen subset of \([T]\).

Thus Theorem 24 follows from:

Main Lemma 25. If \( T \) is a nonempty pruned tree and \( X \subseteq [T] \) is Borel then for each \( k \) there is a \( k \)-covering of \( T \) which unravels \( X \).

We prove the Main Lemma using:

Lemma 26. The Main Lemma holds for closed \( X \subseteq [T] \).

Lemma 27. Fix \( k \). Let \((T_{i+1}, \pi_{i+1}, \varphi_{i+1})\) be a \((k+i)\)-covering of \( T_i \) for each \( i \). Then there is a pruned tree \( T_\infty \) and \( \pi_\infty, i, \varphi_\infty, i \) such that for each \( i \) \((T_\infty, \pi_\infty, i, \varphi_\infty, i)\) is a \((k+i)\)-covering of \( T_i \) and \( \pi_\infty, i = \pi_{i+1} \circ \pi_\infty, i+1, \varphi_\infty, i = \varphi_{i+1} \circ \varphi_\infty, i+1 \).
Proof of Main Lemma. We show by induction on $\xi > 0$ that for all $T_i$ all $k$ and all $\Sigma_\xi^0$ subsets $X$ of $T$, there is a $k$-covering of $T$ that unravels $X$. Notice that if a $k$-covering unravels $X$ it also unravels $\sim X$, so by Lemma 26 we are done if $\xi = 1$. Assume $\xi > 1$ and that the desired property holds for all $\eta < \xi$. So for each $T$, each $\eta < \xi$, each $\Pi_\eta^0$ subset $Y$ of $T$, and each $k$ there is a $k$-covering that unravels $\sim Y$, and therefore also $Y$. Let $X$ be $\Sigma_k^0$ and $k \in N$. Then $X = \bigcup_i X_i$ with $X_i \in \Pi_\xi^0$, $\xi_i < \xi$. Let $(T_1, \pi_1, \varphi_1)$ be a $k$-covering of $T_0 = T$ that unravels $X_0$. Then $\pi_1^{-1}[X_i]$ is also $\Pi_\xi^0$ on $T_1$ for each $i$. By recursion define $(T_{i+1}, \pi_{i+1}, \varphi_{i+1})$ to be a $(k + i)$-covering of $T_i$ that unravels $\pi_1^{-1} \circ \pi_{i-1}^{-1} \circ \cdots \circ \pi_1^{-1}[X_i]$. Let $(T_\infty, \pi_\infty, \varphi_\infty)$ be as in Lemma 27. Then $(T_\infty, \pi_\infty, \varphi_\infty) \in \bigcup_i \pi_\infty^{-1}[X_i]$ unravels every $X_i$. Thus $\pi_\infty^{-1}_0[X] = \bigcup_i \pi_\infty^{-1}_0[X_i]$ is open in $[T_\infty]$. Finally, let $(\bar{T}, \bar{\pi}, \bar{\varphi})$ be a $k$-covering of $T_\infty$ that unravels $\pi_\infty^{-1}_0[X]$, by Lemma 26. Then $(\bar{T}, \pi_0 \circ \pi, \varphi_0 \circ \varphi)$ is a $k$-covering of $T$ that unravels $X$.

7. Vorlesung

Proof of Lemma 27. Note that for any finite sequence $s$ if $\text{Length}(s) \leq 2(k + i)$ then whether $s \in T_i$ or not is independent of $i$. So we define:

$s \in T_\infty$ iff $s \in T_i$ for any $i$ with $\text{Length}(s) \leq 2(k + i)$.

It is clear that $T_\infty$ is a pruned tree and that $T_\infty, T_i$ have the same first $2(k + i)$ levels.

We define $\pi_\infty$ and $\varphi_\infty$: If $\text{Length}(s) \leq 2(k + i)$ then $\pi_\infty(s) = s$. If $2(k + i) < \text{Length}(s) \leq 2(k + j)$ we put $\pi_\infty(s) = \pi_{i+1} \circ \pi_{i+2} \circ \cdots \circ \pi_j(s)$ (this is independent of $j$).

We similarly define $\varphi_\infty$: If $\sigma_\infty$ is a strategy for $T_\infty$ let $\varphi_\infty(s_{\infty}) \upharpoonright 2(k + i) = \sigma_\infty \upharpoonright 2(k + i)$ and for $j > i$, $\varphi_\infty(s_{\infty}) \upharpoonright 2(k + j) = \varphi_{i+1} \circ \varphi_{i+2} \circ \cdots \circ \varphi_j(s_{\infty}) \upharpoonright 2(k + j)$.

It remains to verify condition (iv) of the definition of covering. Suppose that $\sigma_\infty$ is a strategy for $T_\infty$ and let $x_i \in [\varphi_\infty(s_{\infty})]$ (i.e., $x_i$ is a play according to the strategy $\varphi_\infty(s_{\infty})$). Let $x_{i+1} \in [\varphi_\infty(s_{\infty})]$, $x_{i+2} \in [\varphi_\infty(s_{\infty})]$, . . . . come from (iv) for the coverings $(T_{i+1}, \pi_{i+1}, \varphi_{i+1})$, $(T_{i+2}, \pi_{i+2}, \varphi_{i+2})$, . . . applied to the strategies $\varphi_{i+1}(\varphi_{i+1}(s_{\infty})) = \varphi_{i+1}(s_{\infty})$ for $j \geq i$, so that $\pi_{j+1}(x_{j+1}) = x_j$ for any $j \geq i$. Since $\pi_{j+1}$ is the identity on sequences of length $\leq 2(k + j)$, it follows that $(x_i, x_{i+1}, x_{i+2}, \ldots)$ converges to a sequence.
$x_\infty$ defined by $x_\infty \upharpoonright 2(k+j) = x_j \upharpoonright 2(k+j)$ for $j \geq i$. Now $\sigma_\infty$ and $\varphi_{\infty,j}(\sigma_\infty)$ agree on sequences of length $\leq 2(k+j)$ so as $x_j$ follows the strategy $\varphi_{\infty,j}(\sigma_\infty)$ for $j \geq i$ we have that $x_\infty$ follows the strategy $\sigma_\infty$. Finally it is clear that $\pi_{\infty,i}(x_\infty) = x_i$. □

Proof of Lemma 26. We first introduce quasistrategies. If $T$ is a nonempty pruned tree then a quasistrategy for $I$ in $T$ is a nonempty pruned subtree $\Sigma \subseteq T$ such that if $(a_0, \ldots, a_{2j}) \in \Sigma$ and $(a_0, \ldots, a_{2j}, a_{2j+1}) \in T$ then $(a_0, \ldots, a_{2j}, a_{2j+1}) \in \Sigma$. Similarly we define quasistrategies for $II$ in $T$. If $X \subseteq [T]$ is given we say that a quasistrategy $\Sigma$ for $I$ is winning in $G(T, X)$ iff $[\Sigma] \subseteq X$ (similarly for $II$). Thus a winning strategy for $I$ in $G(T, X)$ can be identified with a winning quasistrategy for $I$ with the additional property that for each $(a_0, \ldots, a_{2j-1}) \in \Sigma$ there is a unique $a_{2j}$ such that $(a_0, \ldots, a_{2j}) \in \Sigma$. If $X$ is closed, then there is a canonical quasistrategy for $I$ in $G(T, X)$, defined by $\Sigma = \{p \in T \mid p$ is not losing for $I\}$. If $I$ has a winning quasistrategy then his canonical quasistrategy is winning.

Fix $k$, $T$ and $X$ as in the Lemma and let $T_X$ be the subtree of $T$ whose branches are the elements of $X$. For a tree $S$, $S_u$ denotes $\{v \mid u * v \in S\}$ and for $Y \subseteq [S]$, $Y_u$ denotes $\{x \mid u * x \in Y\}$. The desired $k$-covering $(\tilde{T}, \pi, \varphi)$ is described via the following auxiliary game:

(i) Players start with moves $x_0, x_1, \ldots, x_{2k-2}, x_{2k-1}$ such that $(x_0, \ldots, x_{2k-1}) \in T$.

(ii) In his next move, $I$ plays $(x_{2k}, \Sigma_I)$ where $(x_0, \ldots, x_{2k}) \in T$ and $\Sigma_I$ is a quasistrategy for $I$ in $T(x_0, \ldots, x_{2k})$ (with the convention that $II$ starts first in games on $T(x_0, \ldots, x_{2k})$).

(iii) $II$ now has two options:
Option 1: $II$ plays $(x_{2k+1}, u)$ where $(x_0, \ldots, x_{2k+1}) \in T$ and $u$ is a sequence of even length such that $u \in T(x_0, \ldots, x_{2k+1})$ and $u \in (\Sigma_I)_{(x_{2k+1})} - (T_X)_{(x_0, \ldots, x_{2k+1})}$. From then on $I$ and $II$ play $x_{2k+2}, x_{2k+3}, \ldots$ so that $(x_0, \ldots, x_j) \in T$ for all $j$ and $u \subseteq (x_{2k+2}, x_{2k+3}, \ldots)$.

Option 2: $II$ plays $(x_{2k+1}, \Sigma_{II})$ where $(x_0, \ldots, x_{2k+1}) \in T$ and $\Sigma_{II}$ is a quasistrategy for $II$ in $(\Sigma_I)_{(x_{2k+1})}$ with $\Sigma_{II} \subseteq (T_X)_{(x_0, \ldots, x_{2k+1})}$. From then on $I$ and $II$ play $x_{2k+2}, x_{2k+3}, \ldots$ so that $(x_{2k+2}, x_{2k+3}, \ldots, x_i) \in \Sigma_{II}$ for all $i \geq 2k+2$.

The map $\pi$ is given by $\pi(x_0, \ldots, x_{2k-1}, (x_{2k}, *), (x_{2k+1}, *), x_{2k+2}, \ldots, x_I) = (x_0, \ldots, x_I)$. As $\tilde{x} \in \pi^{-1}(X)$ iff $\tilde{x}(2k+1)$ is of the form $(x_{2k+1}, \Sigma_{II})$, it follows that $\pi^{-1}(X)$ is clopen.
Finally we must define \( \varphi \) so that given a strategy \( \hat{\sigma} \) on \( \hat{T} \), the strategy 
\( \sigma = \varphi(\hat{\sigma}) \) on \( T \) has the property that for any \( x \in [\sigma] \) there is \( \hat{x} \in [\hat{\sigma}] \) with 
\( \pi(\hat{x}) = x \). We now describe the strategy \( \sigma \). There are two cases.

Case 1. \( \hat{\sigma} \) is a strategy for \( I \) in \( \hat{T} \).

For the first \( 2k \) moves, \( \sigma \) is just \( \hat{\sigma} \). Then \( \hat{\sigma} \) provides \( I \) with \( (x_{2k}, \Sigma_I) \); \( \sigma \) has \( I \) play \( x_{2k} \). Then \( II \) plays \( x_{2k+1} \). There are two subcases.

Subcase 1. \( I \) has a winning strategy in \( G((\Sigma_I)(x_{2k+1}), [(\Sigma_I)(x_{2k+1})] - X(x_0, \ldots, x_{2k+1})) \).

Then \( \sigma \) requires \( I \) to play this winning strategy. After finitely many moves a shortest position \( u \) of even length is reached for which \( u \notin \langle T_X(x_0, \ldots, x_{2k+1}) \rangle \), say \( u = (x_{2k+2}, \ldots, x_{2l-1}) \). Then \( (x_0, \ldots, x_{2k-1}, (x_{2k}, \Sigma_I), (x_{2k+1}, u), x_{2k+2}, \ldots, x_{2l-1}) \) is a legal position in \( \hat{T} \), and \( \sigma \) requires \( I \) from then on to play following \( \hat{\sigma} \).

Subcase 2. \( II \) has a winning strategy in \( G((\Sigma_I)(x_{2k+1}), [(\Sigma_I)(x_{2k+1})] - X(x_0, \ldots, x_{2k+1})) \).

Let \( \Sigma_{II} \) be \( II \)'s canonical quasistrategy in this game. From then on, \( I \) plays following \( \hat{\sigma} \), assuming that in the game on \( \hat{T} \), \( II \) played \( (x_{2k+1}, \Sigma_{II}) \). \( I \) can do this as long as \( II \) collaborates and plays so that \( (x_{2k+2}, \ldots, x_{2l-1}) \in (\Sigma_{II})(x_0, \ldots, x_{2k+1}) \), since then we have legal positions in \( \hat{T} \). But if for some \( l \) with \( 2l - 1 > 2k + 2 \), \( II \) plays so that \( (x_{2k+2}, \ldots, x_{2l-1}) \notin (\Sigma_{II})(x_0, \ldots, x_{2k+1}) \), then by definition of \( \Sigma_{II} \), it follows that \( I \) has a winning strategy in 
\( G((\Sigma_I)(x_{2k+1}, \ldots, x_{2l-1}), [(\Sigma_I)(x_{2k+1}, \ldots, x_{2l-1})] - X(x_0, \ldots, x_{2k+1}, \ldots, x_{2l-1})) \). Then \( I \) continues as in Subcase 1.

Case 2. \( \hat{\sigma} \) is a strategy for \( II \) in \( \hat{T} \).

Again for the first \( 2k \) moves \( \sigma \) is just \( \hat{\sigma} \). Next \( I \) plays \( x_{2k} \). Put \( U = \{(x_{2k+1}) * u \in T_{(x_0, \ldots, x_{2k})} \mid u \text{ has even length and there is a quasistrategy } \Sigma_I \text{ for } I \text{ in } T_{(x_0, \ldots, x_{2k})} \text{ such that } \hat{\sigma} \text{ requires } II \text{ to play } (x_{2k+1}, u) \text{ when } I \text{ plays } (x_{2k}, \Sigma_I) \} \). Then 
\( U = \{x \in T_{(x_0, \ldots, x_{2k})} \mid x \supseteq (x_{2k+1}) * u \text{ for some } (x_{2k+1}) * u \in U \} \)
is an open set in \( T_{(x_0, \ldots, x_{2k})} \).

Consider now the game on \( T_{(x_0, \ldots, x_{2k})} \) where \( II \) plays first, the players produce \( x_{2k+1}, x_{2k+2}, \ldots \) and \( II \) wins iff \( (x_{2k+1}, x_{2k+2}, \ldots) \) belongs to \( U \).
Subcase 1. \( II \) has a winning strategy in this game.

Then \( \sigma \) follows this winning strategy until a position \((x_{2k+1}, \ldots, x_{2l-1}) = u\) is reached which belongs to \( U \). Let \( \Sigma_I \) witness \( u \in U \). \( II \) then follows \( \tilde{\sigma} \) after the position \((x_0, \ldots, x_{2k-1}, (x_{2k}, \Sigma_I), (x_{2k+1}, u), x_{2k+2}, \ldots, x_{2l-1})\).

Subcase 2. \( I \) has a winning strategy in this game.

Let \( \Sigma_I \) be the canonical quistrategy for \( I \). Then if \( I \) plays \((x_{2k}, \Sigma_I)\) in the game on \( \tilde{T} \), \( \tilde{\sigma} \) cannot ask \( II \) to play something of the form \((x_{2k+1}, u)\), because then \((x_{2k+1}) \star u\) belongs to \( U \) and by the rules of \( \tilde{T} \), \((x_{2k+1}) \star u\) belongs to \( \Sigma_I \), contradicting the fact that no sequence in \( \Sigma_I \) can belong to \( U \).

So if \( I \) plays \((x_{2k}, \Sigma_I)\) in the game on \( \tilde{T} \), \( \tilde{\sigma} \) asks \( II \) to play \((x_{2k+1}, \Sigma_{II})\). So \( \sigma \) has \( II \) play \( x_{2k+1} \) and then follow \( \tilde{\sigma} \) as long as \( I \) collaborates so that \((x_{2k+2}, \ldots, x_{2l})\) belongs to \( \Sigma_{II} \). If for some \( l \geq k + 1 \), \( I \) plays \( x_{2l} \) with \((x_{2k+2}, \ldots, x_{2l}) \notin \Sigma_{II} \), then since \( \Sigma_{II} \) is a quistrategy for \( II \) in \((\Sigma_I)(x_{2k+1})\) (and therefore \( I \)'s moves are unrestricted as long as they are in \( \Sigma_I \)) it follows that \((x_{2k+2}, \ldots, x_{2l}) \notin (\Sigma_I)(x_{2k+1}) \) and we are back in Subcase 1 again.

This completes the proof of Lemma 26. \( \Box \)

8. Vorlesung

The Wadge Property

Suppose that \( A, B \) are subsets of Baire space. We say that \( A \) is Wadge reducible to \( B \), written \( A \leq_w B \), if there is a continuous function \( f \) such that \( x \in A \) iff \( f(x) \in B \).

Theorem 28. If \( A, B \) are Borel then either \( A \leq_w B \) or \( B \leq_w \sim A \).

Proof. Consider the Wadge game \( G_w(A, B) \) where players \( I \) and \( II \) alternately choose natural numbers \( x(0), y(0), x(1), y(1), \ldots \) and \( II \) wins iff \((x \in A \leftrightarrow y \in B) \). This is a Borel game and therefore determined. If \( \sigma \) is a winning strategy for \( II \) then \( \sigma \) induces a continuous function \( \sigma^* \) from Baire space to Baire space such that \( x \in A \leftrightarrow \sigma^*(x) \in B \). If \( \sigma \) is a winning strategy for \( I \) then \( \sigma \) induces a continuous function \( \sigma^* \) from Baire space to Baire space such that \( y \in B \leftrightarrow \sigma^*(y) \in \sim A \). \( \Box \)
The Wadge degree of $A$ is its equivalence class $[A]_w$ under the equivalence relation $A \equiv_w B$ iff $(A \leq_w B$ and $B \leq_w A)$. Theorem 28 says that the ordering of Wadge degrees is almost a linear ordering, in the sense that the only incomparable pairs of Wadge degrees are of the form $[A]_w, [\sim A]_w$. It is also possible that $[A]_w = [\sim A]_w$; an example is $A = \{x \mid x(0)$ is even $\}$.

**Theorem 29.** The ordering of Wadge degrees is well-founded.

**Proof.** If not then there is a sequence $A_0, A_1, \ldots$ with $A_n \not\leq_w A_{n+1}$ and $A_n \not\sim A_{n+1}$ for each $n$. Let $\sigma_n^0$ be a winning strategy for $I$ in $G_w(A_n, A_{n+1})$ and $\sigma_n^1$ a winning strategy for $I$ in $G_w(A_n, \sim A_{n+1})$.

Fix $x \in 2^N$. We define plays of games $G_0^w, G_1^w, \ldots$ as follows: In game $G_n^w$, $I$ applies the strategy $\sigma_n^{x(n)}$, and the $k$-th move of $II$ is the $k$-th move of $I$ in $G_{n+1}^w$. Let $y_n(x)$ be the play of $I$ in game $G_n^w$. Then

\[(*) \ y_n(x) \not\in A_n \iff y_{n+1}(x) \in A_{n+1}^{x(n)} , \]

where $A_n^0 = A_{n+1}^1, A_n^1 = \sim A_{n+1}$. Consider now $X = \{x \in 2^N \mid y_0(x) \in A_0 \}$.

**Claim.** If $x, \bar{x}$ differ at exactly one argument, we have $x \in X$ iff $\bar{x} \not\in X$.

**Proof of Claim.** Suppose that $x$ and $\bar{x}$ differ exactly at argument $k$. Then $y_{k+1}(x) = y_{k+1}(\bar{x})$, since these depend only on $x(n)$ for $n > k$. So by $(*)$, $y_k(x) \not\in A_k \iff y_{k+1}(x) \in A_{k+1}^{x(k)} \iff y_{k+1}(\bar{x}) \in A_{k+1}^{\bar{x}(k)} \iff y_k(\bar{x}) \in A_k$. Since $x, \bar{x}$ agree at arguments less than $k$, it follows again by $(*)$ that $y_0(x) \not\in A_0 \iff y_0(\bar{x}) \in A_0$; i.e., $x \not\in X$ iff $\bar{x} \not\in X$. □ (Claim)

$X$ is Borel and therefore has the Baire property. Therefore either $X$ or $\sim X$ is comeager inside some basic open set $N_s = \{x \mid s \subseteq x\}$, $s \in 2^{<\omega}$. But the homeomorphism that switches the value of $x$ at $n = \text{Length}(s)$ sends $X \cap N_s$ to $\sim X \cap N_s$ and therefore gives two disjoint comeager subsets of $N_s$, a contradiction. □

**Uniformisation of $\Pi^1_1, \Sigma^1_2$ Relations**

Let $P \subseteq \mathcal{N} \times \mathcal{N}$ and let $F$ be a function with domain and range contained in $\mathcal{N}$. We say that $F$ uniformises $P$ iff the domain of $F$ equals $\{x \mid \exists y (x, y) \in P\}$ and for $x$ in the domain of $F$, $(x, F(x)) \in P$.  

26
Theorem 30. Every $\Pi^1_1$ relation can be uniformised by a function $F$ whose graph $\{(x, y) \mid y = F(x)\}$ is $\Pi^1_1$.

Proof. First we show how to pick a special element from a given $\Pi^1_1$ set $P$. Let $U$ be a recursive tree on $\omega \times \omega$ such that

$$x \in P \text{ iff } U(x) \text{ is well-founded.}$$

Let $\langle u_n \mid n \in \mathbb{N} \rangle$ be a recursive enumeration of $\text{Seq} = \omega^{<\omega}$ where $u_0 = \emptyset$ and length $u_n \leq n$ for each $n$. Define the tree $T$ on $\omega \times \omega_1$ by:

$$(s, h) \in T \iff \forall m, n < \text{Length}(s)\{ \text{If } u_m \supseteq u_n \text{ and } (s \upharpoonright \text{Length}(u_m), s \upharpoonright u_n) \in U \text{ then } h(m) < h(n) \}. $$

Then $x \in P \iff T(x)$ has a branch. Also, if $x$ belongs to $P$ then $T(x)$ has a least branch, i.e., a branch $g_x$ with the property that $g_x(n) \leq f(n)$ whenever $f$ is a branch through $T(x)$: $g_x$ is defined by $g_x(n) = \text{Rank}(u_n)$ in $T(x)$ if $u_n \in U(x)$; 0 otherwise.

Now define $P_0 = P$, $P_{2n+1} = \{ x \in P_{2n} \mid g_x(n) \text{ is least} \}$, $P_{2n+2} = \{ x \in P_{2n+1} \mid x(n) \text{ is least} \}$. Then the intersection of the $P_n$'s has a unique element $a$.

Claim. $\{a\}$ is $\Pi^1_1$.

Proof of Claim. We have:

$$x \in P_1 \iff \forall y (\text{Rank}(u_0) \text{ in } U(x) \leq \text{Rank}(u_0) \text{ in } U(y))$$

$$x \in P_2 \iff \forall y (\text{Rank}(u_0) \text{ in } U(x) < \text{Rank}(u_0) \text{ in } U(y)) \lor (\text{Rank}(u_0) \text{ in } U(x) = \text{Rank}(u_0) \text{ in } U(y) \land x(0) \leq y(0))$$

$$x \in P_3 \iff \forall y (\text{Rank}(u_0) \text{ in } U(x) < \text{Rank}(u_0) \text{ in } U(y)) \lor (\text{Rank}(u_0) \text{ in } U(x) = \text{Rank}(u_0) \text{ in } U(y) \land x(0) = y(0) \land \text{Rank}(u_1) \text{ in } U(y) \leq \text{Rank}(u_1) \text{ in } U(x))$$

Etc.

The above are $\Pi^1_1$ definitions, using the equivalences:
Rank(u) in U(x) ≤ Rank(u) in U(y) iff
There exists an order-preserving function from (U(x))u into (U(y))u
Rank(u) in U(x) < Rank(u) in U(y) iff
There exists an order-preserving function from (U(x))u into (U(y))u∗ for some u∗ properly extending u.

Now we prove the Uniformisation Theorem. Let P be a Π₁¹ subset of \(\mathcal{N} \times \mathcal{N}\). Let U be a tree in \(\omega \times \omega \times \omega\) such that
\[
(x, y) \in P \text{ iff } U(x, y) \text{ is well-founded.}
\]
For each \(x \in \mathcal{N}\) let \(P^x\) be the Π₁¹ set \(\{y \mid (x, y) \in P\}\) and define \(P^x = P^x_0 \supseteq P^x_1 \supseteq \ldots\) exactly as we defined \(P_n, n \in N\), with \(P, U(y)\) replaced by \(P^x, U(x, y)\). Let \(Q^x\) the intersection of the \(P^x_n\)'s. Then \(Q^x\) has a single element, which we denote by \(F(x)\). The Π₁¹ expression we gave above applies to \(Q^x\), and shows that the graph of \(F, \{(x, y) \mid y \in Q^x\}\), is Π₁¹, as desired. □

Corollary 31. Every \(\Sigma^1_2\) binary relation can be uniformised by a \(\Sigma^1_2\) function.

Proof. Suppose that \(P \subseteq \mathcal{N} \times \mathcal{N}\) is \(\Sigma^1_2\); then there is a Π₁¹ relation \(Q \subseteq \mathcal{N}^3\) such that \(P\) is the projection of \(Q\); i.e., \(P = \{(x, y) \mid \exists z((x, y, z) \in Q)\}\).
Apply Π₁¹ uniformisation to get a Π₁¹ function \(F : \mathcal{N} \to \mathcal{N}^2\) with domain \(\{x \mid \exists y, z((x, y, z) \in Q)\}\) such that for \(x\) in this set, \(F(x) = (F(x)_0, F(x)_1)\) where \((x, F(x)_0, F(x)_1) \in Q\). Then \(F^*\) uniformises \(P\), where \(F^*(x) = F(x)_0\).
And the graph of \(F^*\) is \(\Sigma^1_2\) since
\[
y = F^*(x) \text{ iff } \exists z \ (y, z) = F(x).
\]
Thus \(F^*\) is the desired \(\Sigma^1_2\) uniformising function. □.

9. Vorlesung

Part 2: The Constructible Universe

We showed that the regularity properties measurability, Baire property, perfect set property and Ramsey property hold for \(\Sigma^1_1\) sets, provably in ZFC. However this does not extend to \(\Delta^1_2\) sets:

Theorem 32. In L, there is a wellordering < of the reals of length \(\omega_1\) such that the relation \(\{(x, y) \mid y \text{ codes the set of } <\text{-predecessors of } x\}\) is \(\Delta^1_2\). (Such a wellordering is called a good \(\Delta^1_2\) wellordering.)

28
Proof. $<$ is just the usual canonical wellordering of $L$, restricted to the reals. We have:

$y$ codes the set of $<$-predecessors of $x$ iff
\[ \exists z (z \text{ codes an } L_\alpha \models \text{ZFC}^- \text{ containing } x \text{, and } y \text{ codes the set of } <_{L_\alpha} \text{-predecessors of } x \text{ in } L_\alpha) \text{ iff} \]
\[ \forall z (\text{If } z \text{ codes an } L_\alpha \models \text{ZFC}^- \text{ containing } x \text{ then } y \text{ codes the set of } <_{L_\alpha} \text{-predecessors of } x \text{ in } L_\alpha) \]

Thus we have a good $\Delta^1_2$ wellordering. $\square$

Corollary 33. In $L$, there are $\Delta^1_3$ sets which are not measurable and do not have the Baire, perfect set or Ramsey properties.

Proof. List the reals using a good $\Delta^1_3$ wellordering as $\langle x_\alpha \mid \alpha < \omega_1 \rangle$ and build $X$ by inductively deciding $x_\alpha \in \bar{X}$, diagonalising over reals which might witness one of the four regularity properties. $\square$

We proved determinacy for Borel sets in ZFC. This does not extend to $\Pi^1_1$ sets: A Turing set of reals is a set of reals $X$ with the property: $x \in X$, $x =_T y \rightarrow y \in X$, where $=_T$ denotes Turing equivalence. A cone is a Turing set of the form $\{y \mid x \leq_T y\}$ for some $x$.

Lemma 34. (Martin’s Lemma) Suppose that $X$ is a determined Turing set. Then either $X$ or $\sim X$ contains a cone.

Proof. If $I$ has a winning strategy $\sigma$ in the game for $X$ then for any possible play $y$ for player $II$, $\sigma(y)$ belongs to $X$, where $\sigma(y)$ is the result of the game where $II$ plays $y$ and $I$ follows his strategy $\sigma$. Thus if $\sigma \leq_T y$, it follows that $\sigma(y) =_T y$ belongs to $X$. If $II$ has the winning strategy $\tau$, then we get $\tau(x) =_T x \in \sim X$ for all $x \geq_T \tau$. $\square$

Corollary 35. In $L$, $\Pi^1_1$ determinacy fails.

Proof. It suffices to find a Turing set $X$ such that neither $X$ nor $\sim X$ contains a cone. Consider $X = \{x \mid x$ is the first-order theory of some $L_\alpha \models \text{ZFC}^-\}$. Clearly $\sim X$ cannot contain a cone as every real is recursive in some element of $X$. And if $x$ belongs to $X$, then $x'$ (the Turing jump of $x$) does not, so $\sim X$ does not contain a cone. $\square$

29
We showed that for Borel sets $A$ and $B$ either $A \leq_W B$ or $B \leq_W A$. The same property for $\Pi_1^1$ would imply that any two non-Borel $\Pi_1^1$ sets have the same Wadge degree. The latter fails in $L$:

Theorem 36. In $L$, not all non-Borel $\Pi_1^1$ sets have the same Wadge degree.

Proof. Let $(\alpha_i \mid 0 \leq i < \omega_1)$ be the increasing enumeration of the countable ordinals $\alpha$ such that $L_\alpha$ is admissible. Let $\text{WO}$ be the set of $x$ such that $x$ codes an ordinal. $\text{WO}$ is a complete $\Pi_1^1$ set, and therefore each $\Pi_1^1$ set is Wadge reducible to $\text{WO}$. Let $X$ be the set of $x$ such that $x$ codes an ordinal between $\alpha_i$ and $\alpha_{i+1}$ for an even $i$. $X$ is $\Pi_1^1$ and not Borel. Now suppose that $\text{WO}$ were Wadge reducible to $X$ and let $a$ be a real coding a continuous function $f$ such that $x \in \text{WO} \leftrightarrow f(x) \in X$. Choose an odd $i$ such that $a$ belongs to $L_{\alpha_i}$ and $\alpha_i$ is countable in $L_{\alpha_{i+1}} = M$. Then $x \in \text{WO} \cap M \leftrightarrow f(x) \in X \cap M$, and $X \cap M$ is $\Delta_1(M)$. So $\text{WO} \cap M$ is $\Delta_1(M)$, which is impossible since $M$ is the least admissible set containing some real. $\Box$  

The Uniformisation Property does extend in $L$ beyond $\Sigma^1_2$:

Theorem 37. In $L$: Each $\Sigma^1_n$ binary relation can be uniformised by a $\Sigma^1_n$ function, for each $n \geq 2$.

Proof. Let $<$ be a good $\Delta^1_2$ wellordering of the reals. If $R(x, y) \leftrightarrow \exists z S(x, y, z)$ with $S \Pi^1_{n-1}$, $n \geq 2$, then define:

$S^*(x, y, z) \leftrightarrow (S(x, y, z) \land \forall(y', z') < (y, z)(\sim S(x, y', z')))$.  

Then $S^*$ is $\Sigma^1_n$, as $<$ is a good $\Delta^1_2$ wellordering and $n \geq 2$. So $R$ is uniformised by $R^*(x, y) \leftrightarrow \exists z S^*(x, y, z)$. $\Box$

10. Vorlesung

Part 3: Forcing Extensions of $L$

Is it possible to have regularity properties beyond $\Sigma^1_1$ in a forcing extension of $L$? Solovay constructed, using an $L$-inaccessible, a generic extension of $L$ in which all projective sets are regular.

Theorem 38. (Solovay) Assume that there is an $L$-inaccessible. Then there is a set-generic extension of $L$ in which all projective sets are measurable and have the Baire, perfect set and Ramsey properties.
Proof. Let \( \kappa \) be inaccessible in \( L \) and let \( P \) be the Lévy collapse to make \( \kappa \) equal to \( \omega_1 \): A condition in \( P \) is a finite function \( p : F \to \kappa \), where \( F \subseteq \omega \times \kappa \) and \( p(n, \alpha) < \alpha \) for each \((n, \alpha) \in \text{Dom } p\). \( P \) makes each \( \alpha < \kappa \) countable and has the \( \kappa \)-cc; it follows that in a \( P \)-generic extension \( L[G], \kappa \) is \( \omega_1 \). Also \( P \) is homogeneous: If \( p, q \) are elements of \( P \) then there is an automorphism of \( P \) which sends \( p \) to a condition compatible with \( q \).

We show that in \( L[G] \), each projective set of reals \( X \) is measurable. Suppose that \( x \in X \leftrightarrow L[G] \models \varphi(x, a) \) where \( \varphi \) is a formula of second-order arithmetic and \( a \in L[G] \) is a real parameter. We may choose \( \alpha < \kappa \) such that \( a \) belongs to \( L[G_\alpha] \), where \( G_\alpha = G \cap L_\alpha \) is \( P_\alpha = P \cap L_\alpha \)-generic over \( L \). Then \( L[G] \) is a \( P \)-generic extension of \( L[G_\alpha] \), since \( P \approx P_\alpha \times P \).

Almost every real (in the sense of Lebesgue measure) is random over \( L[G_\alpha] \) and therefore it suffices to show that \( X \) agrees with some Borel set on the reals which are random over \( L[G_\alpha] \). Suppose that \( x \) is such a real; then \( L[G] = L[G_\alpha][G_x][H_x] \), where \( P \approx P_\alpha \ast Q \ast R \). \( G_x \) is the generic corresponding to \( x \) for random real forcing \( Q \) over \( L[G_\alpha] \) and \( H_x \) is generic over \( L[G_\alpha][G_x] \) for some forcing \( R \). It is not difficult to show that \( R \) is equivalent to \( P \), and so we can assume \( R = P \). Thus by the homogeneity of \( P \):

\[
x \in X \iff L[G_\alpha][G_x][H_x] \models \varphi(x, a) \text{ iff } L[G_\alpha][G_x] \models (P \models \varphi(x, a)) \text{ iff } L[G_\alpha][G_x] \models (P \models \varphi(\bar{x}, a))
\]

For some \( B \in G_x \), \( B \models Q (P \models \varphi(\bar{x}, a)) \) iff \( x \) belongs to the union of the \( B^{L[G]} \), \( B \in Q \) such that \( B \models Q (P \models \varphi(\bar{x}, a)) \).

As \( L[G_\alpha] \) has only countably many reals, it follows that \( Q \) is countable and therefore this last condition on \( x \) is a Borel condition. So \( X \) is measurable.

The same argument works for the Baire property, using the Cohen algebra in place of the random algebra.

If \( X \) is an uncountable projective set of reals in \( L[G] \) then again we can write \( x \in X \) iff \( L[G_\alpha][G_x] \models (P \models \psi(x, a)) \), where \( a \) belongs to \( L[G_\alpha] \), \( \alpha < \kappa \) and \( G_x \) is generic over \( L[G_\alpha] \) for some forcing \( Q_x \). As \( X \) is uncountable, we can choose \( x \in X \) to not belong to \( L[G_\alpha] \) and for this \( x \) can assume that \( Q_x \models (\bar{x} \notin L[G_\alpha] \text{ and } (P \models \psi(\bar{x}, a))) \). As the power set of \( Q_x \) in \( L[G_\alpha] \) is countable in \( L[G] \), we can build a perfect set of \( y \)'s with corresponding \( Q_x \)-generics \( G_y \) such that \( L[G_\alpha][G_y] \models (P \models \psi(\bar{y}, a)) \). Thus \( y \) belongs to \( X \) for each such \( y \), showing that \( X \) has a perfect subset.

For the Ramsey property, we again write \( x \in X \) iff \( L[G_\alpha][G_x] \models (P \models \bar{z} \in X \) iff \( L[G_\alpha][G_y] \models (P \models \psi(\bar{y}, a)) \). Thus \( y \) belongs to \( X \) for each such \( y \), showing that \( X \) has a perfect subset.
ψ(x, a)). There is a Mathias condition (∅, A) which forces over \( L[G_α] \) that \( L[G_α][G_x] \models (P ⊩ ψ(\dot{x}, a)) \) where \( \dot{x} \) denotes the Mathias generic real (or the same with \( ψ \) replaced by \( \sim ψ \); assume the former). Then if \( x \) is Mathias generic over \( L[G_α] \), \( x \subseteq A \), it follows that \( L[G_α][G_y] \models (P \models ψ(y, a)) \) for all infinite \( y \subseteq x \), as any such \( y \) is also Mathias generic. Thus \( y \in X \) for such \( y \) and \( x \) is homogeneous for \( X \). \( □ \)

Shelah showed that the use of an \( L \)-inaccessible as above is necessary to obtain the measurability of \( \Sigma^1_3 \) sets. (It is however not necessary for the measurability of \( \Delta^1_3 \) sets.) He also showed that an \( L \)-inaccessible is not needed to obtain the Baire property for all projective sets. It is not difficult to show that an \( L \)-inaccessible is necessary to obtain the perfect set property for \( \Pi^1_1 \) sets. The situation with regard to the Ramsey property is still open.

11. Vorlesung

We showed that both \( \Pi^1_1 \) determinacy and the Wadge property for \( \Pi^1_1 \) sets fail in \( L \). These properties also fail in all set-generic extensions of \( L \), by similar arguments. Are these properties in fact refutable in \( ZFC \)? We will see that they are implied by the existence of large cardinals, which makes this rather unlikely.

We showed that projective relations can be projectively uniformised in \( L \). This can fail in a set-generic extension of \( L \):

Theorem 39. There is a forcing extension of \( L \) in which some \( \Pi^1_2 \) binary relation has no projective uniformisation.

Proof. Add \( ω_1 \) Cohen reals to \( L \). In the extension, we have: For each real \( x \), there is a real \( y \) which is Cohen over \( L[x] \). Consider now the relation \( R(x, y) ↔ y \) is Cohen over \( L[x] \). This is a total \( \Pi^1_2 \) relation. Suppose that \( f \) were a projective function with parameter \( a \) which uniformised \( R \). Then our model can be written as \( L[b, G] \) where \( a \) belongs to \( L[b] \) and \( G \) is generic for adding \( ω_1 \) Cohen reals over \( L[b] \). But by homogeneity, \( f(b) \) must belong to \( L[b] \), which contradicts the fact that \( f(b) \) is Cohen over \( L[b] \). \( □ \)

Part 4: Sharps

Assume that every real has a \( # \). We will show that the regularity, determinacy, Wadge and uniformisation properties that we established in \( ZFC \) can be extended to one more level of the projective hierarchy.
Theorem 40. Suppose that $\omega_1$ is inaccessible to reals, i.e., $L[a]$ contains only countably many reals for each real $a$. Then every $\Sigma^1_2$ set is measurable and has the Baire, perfect set and Ramsey properties.

Proof. By Theorems 10, 12 and 20, it suffices to show that each $\Sigma^1_3$ set $A$ is the projection of a tree $T$ on $\omega \times \kappa$ for some $\kappa$, where $T$ belongs to a transitive ZFC$^-$ model containing only countably many reals. Let $U$ be a tree on $\omega \times \omega \times \omega$ such that

$$x \in A \iff \exists y \ U(x, y) \text{ is well-founded.}$$

Let $\langle u_n \mid n \in N \rangle$ be a recursive enumeration of $\text{Seq}$ such that $u_0 = \emptyset$ and $\text{Length}(u_n) \leq n$ for each $n$. Define the tree $T$ on $\omega \times \omega \times \omega_1$ by:

$$(u, v, h) \in T \iff \forall m, n < \text{Length}(u) = \text{Length}(v) \ [\text{If } u_m \supseteq u_n \text{ and } (u \upharpoonright \text{Length}(u_m), v \upharpoonright \text{Length}(u_m), u_m) \in U \text{ then } h(m) < h(n)].$$

Then $x \in A \iff T(x)$ has a branch. $T$ can be identified with a tree on $\omega \times \omega_1$ and belongs to $L[a]$, where $a$ is a real coding the tree $U$. It follows that $A$ is the projection of a tree which belongs to a transitive ZFC$^-$ model all of whose reals are constructible from $a$. By hypothesis this model contains only countably many reals. $\square$

Theorem 41. Suppose that $x^\#$ exists for every real $x$. Then every $\Sigma^1_4$ set is determined.

Proof. Let $A$ be $\Sigma^1_4$; we want to show that the game $G_A$ (played on $\omega$) is determined. Let $T$ be a tree on $\text{Seq}^2$ such that

$$x \in A \iff T(x) \text{ has a branch.}$$

Let $<^*$ be the Kleene-Brouwer ordering on $\text{Seq}$: $s <^* t$ iff either $s$ properly extends $t$ or $s(n) < t(n)$ where $n$ is least so that $s(n) \neq t(n)$. For $s \in \text{Seq}$ let $T_s = \{ t \mid (u, t) \in T \text{ for some } u \subseteq s \}$. Also let $t_0, t_1, \ldots$ be an enumeration of $\text{Seq}$ and if $s$ has length $2n$, let $K_s$ be the set $\{t_0, \ldots, t_{n-1}\} \cap T_s$, $k_s = \text{Card} \ (K_s)$.

We define an auxiliary game $G^*$ as follows: Player I plays natural numbers $a_0, a_1, \ldots$ and player II plays pairs $(b_0, h_0), (b_1, h_1), \ldots$ where the $b_i$ are natural numbers and for each $n$, $h_n$ is an order-preserving function from $(K_{(a_0, b_0, \ldots, a_n, b_n)}, <^*)$ into $\omega_1$ and $i < j \rightarrow h_i \subseteq h_j$. If Player II can follow
these rules for infinitely many moves then he wins; otherwise player I wins. Clearly $G^*$ is determined, as it is a closed game.

If $II$ has a winning strategy in $G^*$ then of course he also has a winning strategy in $G_A$.

Let $\sigma^*$ be the canonical winning strategy for $I$ in $G^*$. We shall construct a winning strategy $\sigma$ for $I$ in $G_A$. Suppose that $s^* = (a_0, (b_0, h_0), \ldots, a_n, (b_n, h_n))$ is a play in $G^*$, and let $E$ be the range of $h_n$. Then the $h_i$’s are uniquely determined by $E$, as $h_n$ is the unique order-preserving function from the ordering $(K(a_0, b_0, \ldots, a_n, b_n), \prec)$ into $\omega_1$, and the $h_i$ are the appropriate restrictions of $h_n$. So $s^*$ is uniquely determined by $s = (a_0, b_0, \ldots, a_n, b_n)$ and $E$. For each such $s$ let $F_s$ be the function $F_s(E) = \sigma^*(s^*)$, where $s^*$ is uniquely determined as above by $E$. Then as $\sigma^*$ is constructible from a real and this real has a $\#$, it follows that there is an uncountable $H \subseteq \omega_1$ which is homogeneous for $F_s$ for every $s$, i.e., for all $s$, $F_s(E)$ is the same for all $E \subseteq H$ of cardinality $k_s$. Define the strategy $\sigma$ by: $\sigma(s) = F_s(E)$, for any $E \subseteq H$ of cardinality $k_s$.

We claim that this is a winning strategy for $I$. If not, let $x$ be a result of playing this strategy with $x \notin A$. Then there is an order-preserving embedding $h$ of $(T(x), \prec)$ into $H$, as $H$ is uncountable and $(T(x), \prec)$ is a countable well-ordering. But consider now the play of $G^*$ where $II$ plays restrictions of $h$ to the appropriate $K_s$’s. This is a play of $G^*$ according to the strategy $\sigma^*$, in contradiction to the fact that $\sigma^*$ is a winning strategy for $I$ in $G^*$. $\square$

Martin showed that the existence of $\#$’s is also sufficient for the determinacy Boolean combinations of $\Pi^1_1$ sets. Thus from this assumption the Wadge property holds for $\Pi^1_1$ sets.

### 12. Vorlesung

**Theorem 42.** Assume that every real has a $\#$ and $A(x, y)$ is $\Pi^1_2$. Then $A$ can be uniformised by a $\Pi^1_3$ function.

**Proof.** First we show how to choose a canonical element from a nonempty $\Pi^1_2$ set $A$. It will be convenient to work with the space $2^\omega$ instead of Baire space. Write $x \in A \iff \forall y \ T(x, y)$ has a branch, where $T$ is a tree on $2 \times 2 \times \omega$. Then $x \in A \iff U(x)$ is well-founded, where a node in $U$ is of the form $(s, t, f)$ with $s$ and $t$ finite 0,1-sequences of the same length, $f$ an order-preserving function from $(T(s, t), \prec)$ into $\omega_1$ and $\prec$ the Kleene-Brouwer order. Let $L^*$ denote $\bigcup \{L[x] \mid x \text{ a real}\}$. 

34
Now suppose that \( s, t \) are finite 0,1-sequences of the same length and \( F, G \in L^* \) are functions from \( U(s, t) \) into Ord. We write \( F \leq^* G \) iff for some CUB \( C \subseteq \omega_1 \) in \( L^* \), \( F(f) \leq G(f) \) for all \( f \in U(s, t) \) with \( \text{Range}(f) \subseteq C \). For any \( F \) and \( G \), either \( F \leq^* G \) or \( G \leq^* F \), since we have assumed \('#'s and \( F,G \) are constructible from reals. Therefore \( \leq^* \) gives a wellordering if we identify \( F \) with \( G \) when \( F =^* G \).

For \( x \in A \) let \( F^x \) be the canonical ranking function on \( U(x) \). Then \( F^x \) belongs to \( L^* \) since it is constructible from \( T \) and \( x \). Fix \( n \) and let \( t_1, t_2, \ldots, t_n \) list the 0,1-sequences of length \( n \) in lexicographical order. Then set \( \alpha^x_n = (\beta_1, \ldots, \beta_{2^n}) \), where \( \beta_i \) is the rank of \( F^x \upharpoonright U(x \upharpoonright n, t_i) \) in \( \leq^* \).

We can now define a canonical element of \( A \). Choose \( x_1 \) to minimize \( \alpha^x_1, x(0) \) (in the lexicographical ordering of finite sequences of ordinals) for \( x \in A \), and set \( n_0 = x_1(0) \). Then choose \( x_2 \) to minimize \( \alpha^x_2, x(1) \) for \( x \in A \) which minimize \( \alpha^x_1, x(0) \), and set \( n_1 = x_2(1) \). Continue in this way, producing a real \( x^* = (n_0, n_1, \ldots) \).

We claim that \( x^* \) belongs to \( A \). Indeed, for each \( n \) choose \( F^n(t) \) with domain \( U(x^* \upharpoonright n, t) \), \( t \) of length \( n \), whose ranks realize \( \alpha^x_n \). Then for some CUB \( C \), the \( F^n(t) \) restricted to elements of \( U(x^* \upharpoonright n, t) \) with range in \( C \) cohere. It follows that \( U(x^*) \) is well-founded, and therefore \( x^* \) belongs to \( A \).

We show that \( \{ x^* \} \) is \( \Pi^1_3 \) in a real coding \( T \). Indeed:

\[
y = x^* \text{ iff } y \in A \text{ and } \\
\forall z, k \ (\text{If } z \text{ belongs to } A, z \upharpoonright k = y \upharpoonright k \text{ and } \alpha^y_l = \alpha^x_k \text{ for } l < k \text{ then } \alpha^y_l \text{ is greater than } \alpha^x_k \text{ or } z(k) \geq y(k)).
\]

As \( \alpha^y_l \) and \( \alpha^x_k \) can be compared in \( L[(y, z)^\#] \), this gives a \( \Pi^1_3 \) definition. And as in the proof of \( \Pi^1_3 \) uniformisation, the above relativises uniformly to give that a binary \( \Pi^1_2 \) relation can be \( \Pi^1_3 \) uniformised. \( \square \)

**Corollary 43.** Assume \('#'s for reals. Then every binary \( \Sigma^1_3 \) relation can be \( \Sigma^1_4 \) uniformised.

We cannot expect to extend these results to higher projective levels only under the assumption of \('#'s. There is a smallest inner model closed under \('#, and this model has a \( \Delta^1_3 \) wellordering of its reals. Thus in this model regularity must fail for \( \Delta^1_3 \) sets. And in this model there is a \( \Pi^1_2 \) set \( A \) such that both \( A \) and the complement of \( A \) are cofinal in the Turing degrees, showing that \( \Pi^1_2 \) determinacy must fail. (In fact \( \Delta^1_2 \) determinacy fails.) The
Wadge property also fails for $\Pi^1_2$ sets. And in the forcing extension obtained by adding $\omega_1$ Cohen reals, there is a $\Pi^1_3$ binary relation which cannot be projectively uniformised.

To extend descriptive set theory, one must a large cardinal hypothesis which violate the existence of a $\Delta^1_3$ wellorder of the reals. This hypothesis is: There is a Woodin cardinal $\kappa$ such that $V^{\kappa\#}$ exists. With this hypothesis, one obtains $\Sigma^1_3$ regularity, $\Pi^1_3$ determinacy and the uniformisation of $\Pi^1_3$ binary relations by $\Pi^1_3$ functions. More Woodin cardinals carry the theory to the higher levels of the projective hierarchy. Thus with infinitely many Woodin cardinals, one has a very satisfying theory of the projective sets.