Topics in Set Theory, Wintersemester 2007

1. Vorlesung

The singular cardinal hypothesis, SCH

Easton: The continuum function $C(\kappa) = 2^\kappa$ on regular cardinals is subject only to the following conditions:

Monotonicity: $\kappa_0 < \kappa_1 \rightarrow C(\kappa_0) \leq C(\kappa_1)$.

Cofinality jump: $\text{cof } C(\kappa) > \kappa$.

I.e., if GCH holds and $C$ is any class function on the regular cardinals obeying the above conditions, there is a cofinality-preserving forcing extension where $2^\kappa = C(\kappa)$ for all regular $\kappa$.

Generalised continuum problem: What are the possible behaviours of the continuum function on arbitrary cardinals?

Key subproblem: Can the GCH fail at a singular strong limit cardinal?

Initially it was thought that a positive answer to this type of question would be obtained using a forcing method similar to Easton’s. However Silver showed that the situation is not so simple.

**Theorem 1** Suppose that GCH holds below a singular cardinal $\lambda$ of uncountable cofinality. Then the GCH holds at $\lambda$ as well. If $\lambda$ is a singular cardinal of uncountable cofinality and $2^\alpha \leq \alpha^{++}$ for $\alpha < \lambda$, then $2^\lambda \leq \lambda^{++}$. (And more.)

This says nothing about singular cardinals of countable cofinality, and nothing about the possibility of getting the GCH to fail at a singular strong limit cardinal. In the wake of Silver’s result, Jensen proved the following important result:

**Theorem 2** Suppose that $0^#$ does not exist. Then every uncountable set of ordinals is a subset of a constructible set of ordinals of the same cardinality.

**Corollary 3** In forcing extensions of $L$, the GCH holds at singular strong limit cardinals.
Proof. Suppose that $V$ is a forcing extension of $L$. Then $0^\#$ does not exist. By Jensen’s theorem, every uncountable set of ordinals is a subset of a constructible set of the same size. Now suppose that $\lambda$ is a singular strong limit cardinal and let $\alpha$ be the maximum of $\text{cof} \lambda$ and $\omega_1$. There are at most $\lambda^+$ constructible subsets of $\lambda$, as GCH holds in $L$. It follows from Jensen’s theorem that there are at most $\lambda^+ \cdot 2^\alpha = \lambda^+$ subsets of $\lambda$ of size $\alpha$. But as $\lambda$ is a strong limit cardinal, the number of subsets of $\lambda$ is the same as the number of size $\alpha$ subsets of $\lambda$, and therefore $2^\lambda$ is $\lambda^+$. □

Thus if it is possible to violate the GCH at a singular strong limit cardinal, one must use large cardinals. Silver and Prikry achieved such a violation from a supercompact cardinal.

**Theorem 4** Suppose that GCH holds and $\kappa$ is $\kappa^{++}$-supercompact. Then in a forcing extension, $\kappa$ is a measurable cardinal where the GCH fails.

**Theorem 5** Suppose that $\kappa$ is measurable. Then in a cardinal-preserving forcing extension with no new bounded subsets of $\kappa$, $\kappa$ is a strong limit cardinal of cofinality $\omega$.

**Corollary 6** $\text{Con}(\text{ZFC} + \text{there is a } \kappa \text{ which is } \kappa^{++}\text{-supercompact})$ implies $\text{Con}(\text{ZFC} + \text{the GCH fails at a singular, strong limit cardinal})$.

This work was the beginning of the study of the singular cardinal problem, a study which has involved some of the deepest work in large cardinal forcing.

2. Vorlesung

Basic Prikry forcing

Let $\kappa$ be measurable and $U$ a normal ultrafilter on $\kappa$.

**Definition 7** Let $P$ be the set of pairs $(p, A)$ such that:

1. $p$ is a finite subset of $\kappa$.
2. $A$ is an element of $U$.
3. $\min A > \max p$. 

2
(p, A) is an extension of (q, B) ((p, A) ≤ (q, B)) iff p end-extends q, A is a subset of B and p \ q is contained in B. (p, A) is a direct or Prikry extension of (q, B) ((p, A) ≤^* (q, B)) iff p = q and A is a subset of B.

The next lemma is easily verified.

Lemma 8 (a) If G is P-generic then \( \bigcup \{p \mid (p, A) \in G \text{ for some } A\} \) is an \( \omega \)-sequence cofinal in \( \kappa \).
(b) \( P \) is \( \kappa^+\)-\( \omega \).
(c) The direct extension relation \( \leq^* \) is \( \kappa \)-closed.

Lemma 9 (The Prikry property) If \( \sigma \) is a sentence of the forcing language then every condition \( (q, B) \) has a direct extension \( (q, A) \) which decides \( \sigma \) (i.e., either forces \( \sigma \) or \( \neg \sigma \)).

Proof. Define \( h : [B]^\omega \rightarrow 2 \) as follows:

\[
h(s) = 1 \text{ iff } (q \cup s, C) \vDash \sigma \text{ for some } C
\]

\[
h(s) = 0 \text{ otherwise.}
\]

As \( U \) is a normal ultrafilter, there is \( A \in U \) which is homogeneous for \( h \), i.e., for each \( n \in \omega \) and \( s_1, s_2 \in [A]^n \), \( h(s_1) = h(s_2) \). We claim that \( (q, A) \) decides \( \sigma \). Otherwise there would be extensions \( (q \cup s_1, B_1), (q \cup s_2, B_2) \) of \( (q, A) \) which force \( \sigma \) and \( \neg \sigma \), respectively. We can assume that \( s_1 \) and \( s_2 \) have the same size \( n \). Thus both \( s_1 \) and \( s_2 \) belong to \( [A]^n \). But then \( h(s_1) = 0 \), \( h(s_2) = 1 \), contradicting the homogeneity of \( A \).\( \square \)

Corollary 10 \( P \) does not add bounded subsets of \( \kappa \).

Proof. Suppose \((p, A) \vDash \dot{\alpha} \) is a subset of \( \lambda \), where \( \lambda \) is less than \( \kappa \). Set \((p, A_0) = (p, A)\) and by Lemma 9 choose a direct extension \((p, A_1)\) of \((p, A_0)\) which decides \( \dot{\alpha} \). Then choose a direct extension \((p, A_2)\) of \((p, A_1)\) which decides \( 1 \in \dot{\alpha}, \) etc. After \( \lambda \) steps one arrives at a direct extension \((p, A_\lambda)\) of \((p, A)\) which decides which ordinals less than \( \lambda \) belong to \( \dot{\alpha} \), and therefore forces \( \dot{\alpha} \) to belong to the ground model.\( \square \)

Corollary 11 If \( G \) is \( P \)-generic then \( \kappa \) has cofinality \( \omega \) in \( V[G] \) and \( V, \ V[G] \) have the same cardinals and bounded subsets of \( \kappa \). In particular, if \( \text{GCH} \) fails at \( \kappa \) in \( V \), then in \( V[G] \), \( \kappa \) is a singular strong limit cardinal where the \( \text{GCH} \) fails.
Suppose that $G$ is $P$-generic and let $C$ be $\bigcup \{p \mid (p, A) \text{ belongs to } G \text{ for some } A\}$. Then $C$ is called a Prikry sequence for $U$ (over $V$). Note that the entire generic $G$ can be recovered from $C$:

$$G = \{(p, A) \mid p \text{ is an initial segment of } C \text{ and } C \setminus p \text{ is a subset of } A\}.$$ 

The above holds as the set on the right is a compatible set of conditions containing $G$. Thus $V[G] = V[C]$. Also, $C$ generates $U$ in the sense that $A$ belongs to $U$ iff $A$ belongs to $V$ and $C$ is almost contained in $A$. (“Almost” means “with finitely many exceptions”.)

**Theorem 12** Suppose that $M$ is an inner model containing the normal ultrafilter $U$ on $\kappa$ and $C$ is an ordertype $\omega$ subset of $\kappa$ which is almost contained in each element of $U$. Then $C$ is a Prikry sequence for $U$ (over $M$).

**Proof.** We need to show that the set 

$$G(C) = \{(p, A) \mid p \text{ is an initial segment of } C \text{ and } C \setminus p \text{ is contained in } A\}$$

is $P$-generic over $M$. It suffices to check that $G(C)$ intersects all dense subsets of $P$ in $M$. First we show:

**Lemma 13** Suppose that $(q, B)$ is a condition and $D$ is open dense. Then there is a direct extension $(q, A)$ of $(q, B)$ and $m \in \omega$ such that for all $n \geq m$ and $s \in [A]^n$, the condition $(q \cup s, A(> \max s))$ belongs to $D$.

**Proof.** Define $h : |B|^{<\omega} \rightarrow 2$ as follows:

- $h(s) = 1$ iff $(q \cup s, C) \in D$ for some $C$
- $h(s) = 0$ otherwise.

Let $A' \in U$, $A' \subseteq B$, be homogeneous for $h$. As $(q, A')$ has an extension in $D$, there exists an $m$ such that $h(s) = 1$ for all $s \in [A']^n$, all $n \geq m$. For each $s \in [A']^n$, $n \geq m$, choose $A_s \in U$, $A_s \subseteq A'$, so that $(q \cup s, A_s)$ belongs to $D$. Now we take $A$ to be the “diagonal intersection” $\Delta \{A_s \mid s \in [A']^n, n \geq m\}$ of these $A_s$, where

$$\Delta \{A_s \mid s \in [A']^n, n \geq m\} = \{\alpha < \kappa \mid \alpha \in A_s \text{ for all } n \geq m \text{ and for all } s \in [A']^n \text{ with } \max s < \alpha\}.$$
Then $A$ is in $U$ as it contains the usual diagonal intersection of sets in $U$.
The condition $(q, A)$ is as desired, since for each $n \geq m$ and $s \in [A]^n$, we
have $A(>\text{max } s) \subseteq A$, and so $(q \cup s, A(>\text{max } s))$ belongs to $D$. $\square$ (Lemma 13)

Now we show that $G(C)$ intersects all open dense $D$ in $M$. For each
finite $q \subseteq \kappa$, use the previous lemma to choose $m(q) \in \omega$ and $A(q) \in U$
so that $\min A(q) > \text{max } q$ and for $n \geq m(q)$, $s \in [A(q)]^n$, the condition
$(q \cup s, A(q)(>\text{max } s))$ belongs to $D$. Let $A$ be the diagonal intersection
$\Delta\{A(q) \mid q \in [\kappa]^{<\omega}\}$. As $A$ belongs to $U$, there is $\tau < \kappa$ such that $C \setminus \tau$
is contained in $A \setminus \tau$. Consider the condition $(C \cap \tau, A \setminus \tau)$. Then for every
$n \geq m(C \cap \tau)$ and every $s \in [C \setminus \tau]^n$ we have

$((C \cap \tau) \cup s, A(>\text{max } s))$ belongs to $D,$

since $s$ is contained in $A \setminus \tau$ and therefore in $A(C \cap \tau)$, and $A(>\text{max } s))$ is
contained in $A(C \cap \tau)(>\text{max } s)$. Choose $s \in [C \setminus \tau]^n$ for some $n \geq m(C \cap \tau)$.
Then $(C \cap \tau) \cup s$ is contained in $C$ and $C(>\text{max } s)$ is contained in $A(>\text{max } s)$.
So $((C \cap \tau) \cup s, A(>\text{max } s))$ belongs to $G(C) \cap D$. $\square$

3.Vorlesung

Tree Prikry forcing

We eliminate the assumption of normality for the ultrafilter $U$ in basic
Prikry forcing. Assume only that $U$ is a $\kappa$-complete non-principal ultrafilter
on the measurable cardinal $\kappa$.

Definition 14 A set $T$ is called a $U$-tree with trunk $t$ if

1. $T$ consists of finite increasing sequences below $\kappa$.
2. $(T, \preceq)$ is a tree, where $\preceq$ is the initial segment relation.
3. For every $\eta \in T$, $\eta \preceq t$ or $t \preceq \eta$.
4. For every $\eta \in T$, if $t \preceq \eta$ then $\{\alpha < \kappa \mid \eta \ast \alpha \in T\}$ belongs to $U$.

For each $n \in \omega$ we let $\text{Lev}_n(T)$ denote the set of nodes in $T$ of length $n$.

The conditions in Tree Prikry forcing are the pairs $(t, T)$ where $T$ is a
$U$-tree and $t$ is the trunk of $T$. Extension is defined by: $(t, T) \leq (s, S)$ if
$T \subseteq S$. Note that this implies $s \leq t \in S$. If in addition $s = t$, then we say that $(t, T)$ is a direct or Prikry extension of $(s, S)$, written $(t, T) \preceq^* (s, S)$.

The following is an immediate consequence of the $\kappa$-completeness of the ultrafilter $U$.

**Lemma 15** Suppose that $T_\alpha$, $\alpha < \lambda$, are $U$-trees with the same trunk $t$ and $\lambda$ is less than $\kappa$. Then the intersection of the $T_\alpha$’s is also a $U$-tree with trunk $t$.

It now follows, as with basic Prikry forcing, that if $P$ denotes Tree Prikry forcing, then for $P$-generic $G$, $\bigcup\{t \mid (t, T) \in G$ for some $T\}$ is an $\omega$-sequence cofinal in $\kappa$, $P$ is $\kappa^+\text{-cc}$ and the direct extension relation $\preceq^*$ is $\kappa$-closed. We next prove the Prikry property.

**Lemma 16 (The Prikry Property)** If $(t, T)$ is a condition and $\sigma$ is a sentence of the forcing language, then there is a direct extension $(s, S)$ of $(t, T)$ which decides $\sigma$.

**Proof.** Let us say that a finite increasing sequence $s$ is indecisive iff there is no $U$-tree $S$ with trunk $s$ such that $(s, S)$ decides $\sigma$. If the lemma fails, then the node $t$ is indecisive: For, if $(t, S)$ decides $\sigma$ then so does $(t, T \cap S)$, a direct extension of $(t, T)$.

Now note that if $s$ is indecisive, it must be the case that $s * \alpha$ is indecisive for a set of $\alpha$ in $U$: Otherwise we can choose $T(s * \alpha)$ for $U$-measure one $\alpha$ such that $(s * \alpha, T(s * \alpha))$ decides $\sigma$ in the same way for all such $\alpha$, and then form a $U$-tree $S$ with trunk $s$ by gluing together these $T(s * \alpha)$; the resulting condition $(s, S)$ would then decide $\sigma$.

It follows that we can inductively form a $U$-tree $S$ with trunk $t$ consisting entirely of indecisive nodes. But this is impossible, as the condition $(t, S)$ has some extension $(u, R)$ which decides $\sigma$, demonstrating that the node $u$ of $S$ is not indecisive. \[\square\]

As an easy corollary we have:

**Corollary 17** Tree Prikry forcing at $\kappa$ adds no bounded subsets of $\kappa$, preserves cardinals and gives $\kappa$ cofinality $\omega$.  

6
**Prikry forcing at cofinality ω**

Suppose that κ is the supremum of an increasing sequence of measurable cardinals κₙ, n ∈ ω, where κₙ carries the nonprincipal κₙ-complete ultrafilter Uₙ. We describe a cofinality-preserving forcing P for adding an element of \( \prod_n \kappa_n \) which eventually dominates each element of this product in the ground model.

**Definition 18** A condition in P is a sequence \( p = \langle p_n \mid n \in \omega \rangle \) where:

1. For each \( n \), \( p_n \) is either an element of \( U_n \) or an ordinal less than \( \kappa_n \).
2. There is an \( l(p) < \omega \) such that for \( n < l(p) \), \( p_n \) is an ordinal less than \( \kappa_n \) and for \( n \geq l(p) \), \( p_n \) is an element of \( U_n \).

We say that \( p \) extends \( q \), written \( p \leq q \), iff for each \( n \), one of the following holds:

(a) \( p(n) = q(n) \) is an ordinal less than \( \kappa_n \).
(b) \( p(n) \in q(n) \) where \( q(n) \) is an element of \( U_n \).
(c) \( p(n) \subseteq q(n) \) both belong to \( U_n \).

Note that \( p \leq q \) implies that \( l(p) \) is at least \( l(q) \). We say that \( p \) is a direct or Prikry extension of \( q \), written \( p \leq^* q \), iff \( p \leq q \) and \( l(p) = l(q) \).

For each \( n \) and \( p \in P \) we write \( p(< n) \) for \( p \) restricted to \( n \) and \( p(\geq n) \) for \( p \) restricted to \( [n, \omega) \). For each \( n \), \( P \) naturally factors as \( P(< n) \times P(\geq n) \), where \( P(< n) \) consists of all \( p(< n) \), \( p \in P \), and \( P(\geq n) \) consists of all \( p(\geq n) \), \( p \in P \). Note that the direct extension relation on \( P(\geq n) \) is \( \kappa_n \)-closed.

We now prove the important Prikry property.

**Lemma 19 (The Prikry Property)** If \( p = \langle p_n \mid n \in \omega \rangle \) is a condition and \( \sigma \) is a sentence of the forcing language, then \( p \) has a direct extension \( q \) which decides \( \sigma \).

**Proof.** Suppose that \( s \) is a finite sequence from \( \prod_{m < n(s)} \kappa_m \) for some \( n(s) \in \omega \).
We say that \( s \) is indecisive if there is no \( p \) deciding \( \sigma \) with \( p(< n(s)) = s \) and \( l(p) = n(s) \). If the lemma fails, then \( p(< l(p)) \) is indecisive.

Suppose that \( s \) is indecisive. Then for \( U_{n(s)} \)-measure one \( \alpha \), \( s \ast \alpha \) is indecisive: Otherwise, we could choose \( p(s \ast \alpha) \), with \( p(s \ast \alpha)(< n(s) + 1) = s \ast \alpha \).
and $l(p(s \ast \alpha)) = n(s) + 1$, for $U_{n(s)}$-measure one $\alpha$ which decide $\sigma$ in the same way, and then using the $\kappa^+_n$-closure of $U_n$, $n > n(s)$, glue the $p(s \ast \alpha)$ together to a single $p$ with $p(< n(s)) = s$, $l(p) = n(s)$, which would decide $\sigma$.

Now again using the closure properties of the ultrafilters $U_n$, we can build a condition $q \leq^* p$ so that $r(< l(r))$ is indecisive for each $r \leq q$. But this is impossible, as $q$ has some extension $r$ which decides $\sigma$, contradicting the indecisiveness of $r(< l(r))$. □

As an easy corollary we obtain:

**Corollary 20** Let $P$ denote the above forcing. Then $P$ adds no new bounded subsets of $\kappa = \bigcup_{n \in \omega} \kappa_n$, is $\kappa^+ - \omega$, preserves cofinalities and adds an element of $\prod_{n \in \omega} \kappa_n$ which eventually dominates each ground model element of that product.

The last statement of the above corollary holds as if $p_0$ is an element of $\prod_{n \in \omega} \kappa_n$, it is dense for $p \in P$ to have the property that for $n \geq l(p)$, $\min p(n)$ is greater than $p_0(n)$.

**Supercompact Prikry forcing**

For $\kappa \leq \lambda$, $\kappa$ regular, $P_\kappa \lambda$ denotes the set of size $< \kappa$ subsets of $\lambda$. An ultrafilter $U$ on $P_\kappa \lambda$ is fine iff it contains the set $\{x \in P_\kappa \lambda \mid \alpha \in x\}$ for each $\alpha < \lambda$. A function $f : A \to \lambda$, $A \subseteq P_\kappa \lambda$, is regressive iff $f(a) \in a$ for each $a \in A$. An ultrafilter $U$ on $P_\kappa \lambda$ is normal iff it is fine, $\kappa$-complete and any function which is regressive on a set in $U$ is constant on a set in $U$.

**Definition 21** $\kappa$ is $\lambda$-strongly compact iff there is a fine, $\kappa$-complete ultrafilter on $P_\kappa \lambda$. And $\kappa$ is $\lambda$-supercompact iff there is a normal ultrafilter on $P_\kappa \lambda$.

Prikry forcing with a normal ultrafilter on $P_\kappa \lambda$ is analogous to basic Prikry forcing, with $\kappa$ replaced by $P_\kappa \lambda$ and the standard ordering on $\kappa$ replaced with the following ordering on $P_\kappa \lambda$:

**Definition 22** For $a, b$ in $P_\kappa \lambda$ we say that $a$ is strongly included in $b$, written $a < b$, iff $a$ is a subset of $b$ and the ordertype of $a$ is less than the ordertype of $b \cap \kappa$. 
Lemma 23  Suppose that $U$ is a normal ultrafilter on $P_\kappa \lambda$.
(a) If $F$ is a function defined on a set in $U$ such that $F(a) < a$ for each $a$ in
the domain of $F$, then $F$ is constant on a set in $U$.
(b) Suppose that $A_a$ belong to $U$ for each $a \in P_\kappa \lambda$. Then the diagonal inter-
section $\triangle_a A_a = \{b \mid b \in A_a \text{ for each } a < b\}$ belongs to $U$.

Proof. (a) Note that function $a \mapsto$ ordtype $(F(a))$ is regressive on a set in
$U$ and therefore constant with some value $\bar{\kappa} < \kappa$ on a set in $U$. Also, for each
$\alpha < \bar{\kappa}$, the function $a \mapsto \alpha$-th element of $F(a)$ is regressive and therefore
constant on a set in $U$. As there are fewer than $\kappa$ possible $\alpha$’s, it follows that
$F$ is constant on a set in $U$.

(b) If not then there is a function $G$ defined on a set in $U$ such that $G(a) < a$
and $a$ does not belong to $A_{G(a)}$ for each $a$. But then by (a), $G$ is constant on
a set in $U$, which contradicts the fact that each $A_{G(a)}$ belongs to $U$. □

Definition 24  For $A \subseteq P_\kappa \lambda$, $[A]^{[n]}$ denotes the set of $n$-element subsets of
$A$ which are totally ordered by $<$, and $[A]^{[< \omega]}$ denotes the union of the $[A]^{[n]}$,
$n \in \omega$.

The following is a generalisation of the fact that measurable cardinals are
“measure-one Ramsey”. The proof is as in the measurable cardinal case, using
Lemma 23 (b).

5. Vorlesung

Lemma 25  Suppose that $U$ is a normal ultrafilter on $P_\kappa \lambda$. If $F : [A]^{[< \omega]} \to 2$
and $A$ belongs to $U$ then there is $B \subseteq A$ in $U$ such that $F \upharpoonright [B]^{[n]}$ is constant
for each $n \in \omega$.

In our analysis of the effect of supercompact Prikry forcing on the cardinals,
we will need the following important result of Solovay.

Theorem 26  Suppose that $\kappa$ is $\lambda$-strongly compact, $\kappa \leq \lambda$ regular. Then
$\lambda^{<\kappa} = \lambda$.

We need three lemmas.

Lemma 27  Suppose that $\kappa$ is $\lambda$-strongly compact, $\kappa \leq \lambda$ regular. Then there
is an elementary embedding $j : V \rightarrow M$ with critical point $\kappa$ such that every
subset of $M$ of size $\lambda$ is covered by an element of $M$ of $M$-cardinality less
than $j(\kappa)$. In particular, $j(\kappa)$ is greater than $\lambda$ and $j$ is discontinuous at all
regular cardinals in $[\kappa, \lambda]$. 9
Proof. Suppose that $U$ is a $\kappa$-complete fine ultrafilter on $P_\kappa \lambda$ and let $j : V \rightarrow M$ be the ultrapower via $U$. If $[f_\alpha]_U$, $\alpha < \lambda$, are elements of $M$ then define $F(x) = \{ f_\alpha(x) \mid \alpha \in x \}$. Then $[F]_U$ belongs to $[F]_U$ for each $\alpha$ by the fineness of $U$ and $[F]_U$ has $M$-cardinality less than $j(\kappa)$. Thus every subset of $M$ of size at most $\lambda$ is covered by an element of $M$ of $M$-cardinality less than $j(\kappa)$. It follows that $\sup j[\mu]$ is singular in $M$ for each regular $\mu \in (\kappa, \lambda]$ as $\sup j[\mu]$ has cofinality $\mu \leq \lambda$ in $V$ and therefore $M$-cofinality less than $j(\kappa) < \sup j[\mu]$. As $j(\mu)$ is regular in $M$ for regular $\mu$, it follows that $j$ is discontinuous at all regulars in $(\kappa, \lambda]$. The above covering property also implies that $j(\kappa)$ is greater than $\lambda$ and the $\kappa$-completeness of $U$ implies that $\kappa$ is therefore the critical point of $j$. \Box (Lemma 27)

Lemma 28 Suppose that $\kappa$ is $\lambda$-strongly compact, $\kappa \leq \lambda$ regular. Then if $\langle S_i \mid i < \gamma \rangle$, $\gamma < \kappa$, is a sequence of stationary subsets of $\lambda \cap \text{Cof}(\lambda \cap \kappa)$, there exists a $\bar{\lambda} < \lambda$ of cofinality strictly between $\aleph_0$ and $\kappa$ such that $S_i \cap \bar{\lambda}$ is stationary for each $i < \gamma$.

Proof. Let $j : V \rightarrow M$ be as in the previous Lemma. By elementarity, it suffices to show that in $M$ there is $\delta < j(\lambda)$ of $M$-cofinality strictly between $\aleph_0$ and $j(\kappa)$ such that $j(S_i) \cap \delta$ is stationary in $M$ for each $i < \gamma$. Let $\delta$ be $\sup j[\lambda] < \lambda$. Then $\delta$ has cofinality $\lambda$ in $V$ and therefore cofinality strictly between $\aleph_0$ and $j(\kappa)$ in $M$.

Note that $j[\lambda]$ is a $< \kappa$-closed unbounded subset of $\delta$. Now suppose that $C$ is closed unbounded in $\delta$. Then $C' = C \cap j[\lambda]$ is $< \kappa$-closed unbounded in $\delta$. Let $D$ be $j^{-1}[C']$. Then $D$ is $< \kappa$-closed unbounded in $\delta$ and therefore $S_i \cap D$ is nonempty. It follows that $j(S_i) \cap C'$ is nonempty and therefore we have shown that each $j(S_i)$, $i < \gamma$, is stationary in $\delta$ (in $V$). \Box (Lemma 28)

Lemma 29 If $\bar{\lambda} < \lambda$ are regular then there is exist pairwise disjoint $\langle S_i \mid i < \lambda \rangle$ such that each $S_i$ is a stationary subset of $\lambda \cap \text{Cof} \bar{\lambda}$, the set of ordinals less than $\lambda$ of cofinality $\bar{\lambda}$.

Proof. For each $\alpha \in \text{Cof} \bar{\lambda}$ let $f_\alpha : \bar{\lambda} \rightarrow \alpha$ be cofinal and continuous. Now for each $\delta < \lambda$ and $\alpha \in (\delta, \lambda)$ choose $i_{\delta, \alpha}$ such that $f_\alpha(i_{\delta, \alpha})$ is greater than $\delta$. Then there is a stationary set $S_\delta \subseteq \lambda \cap \text{Cof} \bar{\lambda}$ such that $i_{\delta, \alpha} = i_\delta$ is constant for $\alpha$ in $S_\delta$ and also $f_\alpha(i_\delta) = \lambda_\delta$ is constant for $\alpha \in S_\delta$. Now choose $\lambda$-many $S_\delta$'s with the same $i_\delta$ and distinct $\lambda_\delta$; these stationary subsets of $\lambda \cap \text{Cof} \bar{\lambda}$ are pairwise disjoint. \Box (Lemma 29)
Now we prove Theorem 26. Let \( \langle S_i \mid i < \lambda \rangle \) be pairwise disjoint stationary subsets of \( \lambda \cap \text{Cof}\omega \). We may assume that \( \lambda \) is greater than \( \kappa \), as \( \kappa^{<\kappa} = \kappa \) follows from the strong inaccessibility of \( \kappa \). For each \( x \in [\lambda]^{<\kappa} \) choose \( \lambda_x < \lambda \) of cofinality strictly between \( \aleph_0 \) and \( \kappa \) such that \( S_i \cap \lambda_x \) is stationary for each \( i \in x \). Let \( C_x \) be a closed unbounded subset of \( \lambda_x \) of ordertype \( \text{cof} \lambda_x \). Then \( S_i \cap C_x \) is stationary and therefore nonempty for each \( i \in x \). Thus if \( x, y \) are distinct elements of \( [\lambda]^{<\kappa} \) and \( \lambda_x = \lambda_y \) then \( C_x = C_y \) and \( \{ S_i \cap C_x \mid i \in x \} \neq \{ S_i \cap C_y \mid i \in y \} \). Now there are at most \( \lambda \) possibilities for \( \lambda_x \) and for each \( \lambda_x \), there are at most \( [2^{\text{cof} \lambda_x}]^{<\kappa} \leq \kappa \) possibilities for \( \{ S_i \cap C_x \mid i \in x \} \). It follows that \( |\lambda|^{<\kappa} \) is \( \lambda \), as desired. \( \square \)

We can extend Theorem 26 to the case of singular \( \lambda \) using the following.

**Lemma 30** If \( \kappa \) is \( \lambda \)-strongly compact then \( \kappa \) is also \( \lambda^{<\kappa} \)-strongly compact.

**Proof.** Let \( U \) be a fine, \( \kappa \)-complete ultrafilter on \( P_\kappa \lambda \). For \( x \in P_\kappa \lambda \) let \( x^* \in P_\kappa P_\kappa \lambda \) be \( \{ y \in P_\kappa \lambda \mid y < x \} \). Then define \( U^* \) contained in the power set of \( P_\kappa P_\kappa \lambda \) by: \( A^* \in U^* \) iff \( \{ x \mid x^* \in A^* \} \) belongs to \( U \). As \( U \) is a \( \kappa \)-complete ultrafilter on \( P_\kappa \lambda \) it follows that \( U^* \) is a \( \kappa \)-complete ultrafilter on \( P_\kappa P_\kappa \lambda \). If \( y \) belongs to \( P_\kappa \lambda \), then the set of \( x \in P_\kappa \lambda \) such that \( y < x \) belongs to \( U \) by the fineness and \( \kappa \)-completeness of \( U \). It follows that \( \{ a \in P_\kappa P_\kappa \lambda \mid y \in a \} \) belongs to \( U^* \), so \( U^* \) is fine. \( \square \) (Lemma 30)

**Corollary 31** Suppose that \( \kappa \) is \( \lambda \)-strongly compact, \( \kappa \leq \lambda \). If \( \lambda \) has cofinality at least \( \kappa \) then \( \lambda^{<\kappa} = \lambda \) and otherwise \( \lambda^{<\kappa} = \lambda^+ \).

**Proof.** If \( \lambda \) is regular then this follows from Theorem 26. If \( \lambda \) is singular of cofinality at least \( \kappa \) then \( \lambda^{<\kappa} \) is the supremum of \( \lambda^{<\kappa} \), \( \lambda^+ < \lambda \), which by Theorem 26 is \( \lambda \). If \( \lambda \) has cofinality less than \( \kappa \) then by Lemma 30, \( \kappa \) is \( \lambda^{<\kappa} \)-strongly compact and therefore \( \lambda^+ \)-strongly compact. Thus again by Theorem 26, \( \lambda^{<\kappa} \leq (\lambda^+)^{<\kappa} = \lambda^+ \). As \( \lambda^{<\kappa} \) is greater than \( \lambda \) in this case, it follows that \( \lambda^{<\kappa} \) equals \( \lambda^+ \). \( \square \)

6. Vorlesung

We now define supercompact Prikry forcing.

**Definition.** \( P \) consists of all pairs \( \langle (a_1, \ldots, a_n), A \rangle \) such that:

11
(1) $a_1 < a_2 < \cdots < a_n$ belong to $P, \lambda$. 
(2) $A \in U$. 
(3) $a_n < a$ for each $a \in A$. 

$((a_1, \ldots, a_n), A)$ extends $((b_1, \ldots, b_m), B)$, written $((a_1, \ldots, a_n), A) \leq ((b_1, \ldots, b_m), B)$, iff:

(a) $n \geq m$. 
(b) For $k \leq m$, $a_k = b_k$. 
(c) $A \subseteq B$. 
(d) $a_k$ belongs to $B$ for each $k$ in $[m + 1, n]$. 

$((a_1, \ldots, a_n), A)$ directly extends $((b_1, \ldots, b_m), B)$, written $((a_1, \ldots, a_n), A) \leq^* ((b_1, \ldots, b_m), B)$, iff $((a_1, \ldots, a_n), A)$ extends $((b_1, \ldots, b_m), B)$ and $n = m$. 

The relation $\leq^*$ is $\kappa$-closed.

**Lemma 32** If $\sigma$ is a sentence of the forcing language then every condition in supercompact Prikry forcing has a direct extension which decides $\sigma$.

**Proof.** The proof is exactly as in the measurable cardinal case, now using Lemma 25. Suppose that $((a_1, \ldots, a_n), A)$ is a condition and define $h : [A]^{< \omega} \rightarrow 2$ as follows:

$h(b_1, \ldots, b_m) = 1$ iff $((a_1, \ldots, a_n, b_1, \ldots, b_m), C)$ forces $\sigma$ for some $C$ 
$h(b_1, \ldots, b_m) = 0$, otherwise.

By Lemma 25, there is $B \subseteq A$ which is homogeneous for $h$, i.e., for each $n \in \omega$, $h$ is constant on $[B]^n$. We claim that $((a_1, \ldots, a_n), B)$ decides $\sigma$. Otherwise there would be extensions $((a_1, \ldots, a_n, b_1, \ldots, b_m), B_1)$ and $((a_1, \ldots, a_n, c_1, \ldots, c_l), B_2)$ of $((a_1, \ldots, a_n), B)$ which force $\sigma \sim \sigma$, respectively. We can assume that $l$ equals $m$. Thus both $(b_1, \ldots, b_m)$ and $(c_1, \ldots, c_m)$ belong to $[B]^m$. But then $h(b_1, \ldots, b_m) = 1$ and $h(c_1, \ldots, c_m) = 0$, contradicting the homogeneity of $B$. \qed

It follows that $P$ does not add bounded subsets of $\kappa$ and therefore preserves cardinals up to $\kappa$.

Let $G$ be $P$-generic and let $C = (a_1, a_2, \ldots)$ be the limit of the $(a_1, \ldots, a_n)$ such that $((a_1, \ldots, a_n), A) \in G$ for some $A$. An easy density argument shows

12
that if $\delta \leq \lambda$, then $\delta = \bigcup_{n}(a_n \cap \delta)$. Therefore, if $\delta \leq \lambda$ had cofinality at least $\kappa$ in $V$, it will have cofinality $\omega$ in $V[G]$. It follows that $\kappa^+$ in $V[G]$ is at least $\lambda^+$ of $V$.

Now as $\lambda$-supercompact Prikry forcing $P$ is $(\text{Card } P, \lambda)^{+\text{-cc}}$ and $\text{card} P \lambda = \lambda^{< \kappa}$, it follows from Corollary 31 that $P$ is $\lambda^+$-cc when $\lambda$ has cofinality at least $\kappa$ and $\lambda^{++}$-cc when $\lambda$ has cofinality less than $\kappa$. Thus in the former case, cofinalities greater than $\lambda$ are preserved and $\kappa^+$ of $V[G]$ equals $\lambda^+$ of $V$. In the latter case, cofinalities greater than $\lambda^+$ are preserved; the remaining question is what happens to $\lambda^+$ itself.

**Lemma 33** Suppose that $\lambda$ has cofinality less than $\kappa$ in $V$. Then $P$ changes the cofinality of $\lambda^+$ to $\omega$.

**Proof.** Fix in $V$ an increasing sequence $\langle \lambda_i \mid i < \text{cof } \lambda \rangle$ of regular cardinals cofinal in $\lambda$, $\lambda_0 \geq \kappa$. As $\lambda^{< \kappa} = \lambda^+$ the cardinality of $\prod_{i < \text{cof } \lambda} \lambda_i$ is $\lambda^+$. Now inductively build a sequence $\langle f_\alpha \mid \alpha < \lambda^+ \rangle$ of elements of $\prod_{i < \text{cof } \lambda} \lambda_i$ with the following properties, where $<^*$ denotes $<$ on a final segment of cof $\lambda$:

1. $\alpha < \beta \rightarrow f_\alpha <^* f_\beta$.
2. $g \in \prod_{i < \text{cof } \lambda} \lambda_i \rightarrow g <^* f_\alpha$ for some $\alpha < \lambda^+$.

Recall the sequence $C = (a_0, a_1, \ldots)$ derived from $G$. For each $n$ consider $g_n \in \prod_{i < \text{cof } \lambda} \lambda_i$ defined by $g_n(i) = \sup(a_n \cap \lambda_i)$. Then as the range of each $f_\alpha$ is contained in $a_n$ for sufficiently large $n$, it follows that the $g_n$’s are cofinal modulo a final segment of cof $\lambda$ in $\prod_{i < \text{cof } \lambda} \lambda_i$ and therefore $\lambda^+$ has cofinality $\omega$ in $V[G]$. \(\square\) (Lemma 33)

This completes the analysis of supercompact Prikry forcing.

### 7. Vorlesung

**Strongly compact Prikry forcing**

The construction here is entirely analogous to that of Tree Prikry forcing with a (possibly) non-normal measure on $\kappa$.

Let $U$ be a fine $\kappa$-complete ultrafilter on $P_\kappa \lambda$ which may fail to be normal.
Definition 34 A set $T$ is called a $U$-tree with trunk $t$ iff

1) $T$ consists of finite sequences $(x_1, \ldots, x_n)$ from $P, \lambda$ which are increasing in the Magidor relation $<$. 
2) $(T, \preceq)$ is a tree, where $\preceq$ is the initial segment relation.
3) For every $\eta \in T$, $\eta \preceq t$ or $t \preceq \eta$.
4) For every $\eta \in T$, if $t \preceq \eta$ then $\{x | \eta \ast x \in T\}$ belongs to $U$.

The conditions in Tree Prikry forcing are the pairs $(t, T)$ where $T$ is a $U$-tree and $t$ is the trunk of $T$. Extension is defined by: $(t, T) \leq (s, S)$ iff $T \subseteq S$. Note that this implies $s \preceq t \in S$. If in addition $s = t$, then we say that $(t, T)$ is a direct or Prikry extension of $(s, S)$, written $(t, T) \leq^* (s, S)$.

The following is an immediate consequence of the $\kappa$-completeness of the ultrafilter $U$.

Lemma 35 Suppose that $T_\alpha$, $\alpha < \lambda$, are $U$-trees with the same trunk $t$ and $\lambda$ is less than $\kappa$. Then the intersection of the $T_\alpha$’s is also a $U$-tree with trunk $t$.

If $P$ denotes the above Tree Prikry forcing, then for $P$-generic $G$, the limit of the $t = (x_1, \ldots, x_n)$ such that $(t, T)$ belongs to $G$ for some $T$ is an $\omega$-sequence in $P, \lambda$ whose union is all of $\lambda$. It follows that each cardinal in the interval $[\kappa, \lambda]$ of cofinality at least $\kappa$ is forced by $P$ to have cofinality $\omega$. $P$ is $(\lambda^{<\kappa})^+\text{-cc}$ and therefore all cofinalities greater than $\lambda^{<\kappa}$ are preserved. The direct extension relation $\leq^*$ is $\kappa$-closed. We also have:

Lemma 36 (The Prikry Property) If $(t, T)$ is a condition and $\sigma$ is a sentence of the forcing language, then there is a direct extension $(s, S)$ of $(t, T)$ which decides $\sigma$.

The proof of this lemma is just as in the case of a Tree Prikry forcing with a measure on $\kappa$. It follows that $P$ does not add bounded subsets of $\kappa$ and therefore cofinalities less than $\kappa$ are preserved. Now recall the following:

(a) If $\lambda$ has cofinality at least $\kappa$ then $\lambda^{<\kappa} = \lambda$.
(b) If $\lambda$ has cofinality less than $\kappa$ then $\lambda^{<\kappa} = \lambda^+$ and $P$ adds an $\omega$-sequence cofinal in $\lambda^+$.  

14
It follows that cofinalities greater than \( \lambda \) are preserved when \( \lambda \) has cofinality at least \( \kappa \) (and therefore \( \kappa^+ \) becomes \( \lambda^+ \) in the generic extension), and cofinalities greater than \( \lambda^+ \) are preserved when \( \lambda \) has cofinality less than \( \kappa \) (and therefore \( \kappa^+ \) becomes \( \lambda^{++} \) in the generic extension). This completes the analysis of Tree Prikry forcing for a strong compact.

**Extender-based Prikry forcing at cofinality \( \omega \)**

We have seen how to add a new \( \omega \)-sequence to an \( \omega \)-limit \( \kappa \) of measurables \( \langle \kappa_n \mid n \in \omega \rangle \) without adding new bounded subsets of \( \kappa \). Now we wish to add many to obtain a violation of the singular cardinal hypothesis.

Assume GCH and let \( \lambda \) be regular and greater than \( \kappa = \sup_{n \in \omega} \kappa_n \). We wish to add at least \( \lambda \)-many sequences through the product of the \( \kappa_n \)'s without adding bounded subsets of \( \kappa \).

We suppose that each \( \kappa_n \) is \( H(\lambda^+) \)-strong; this means that there is an elementary embedding \( j_n : V \to M_n \) with critical point \( \kappa_n \) such that \( H(\lambda^+) \) belongs to \( M_n \) and \( j_n(\kappa_n) \) is greater than \( \lambda \). We may assume that \( j_n \) is an *ultrapower embedding* which is equivalent to saying that every element of \( M_n \) is of the form \( j_n(f)(\alpha) \) for some \( f : \kappa_n \to \kappa_n \) and \( \alpha < \lambda^+ \). This implies that \( M_n \) is closed under \( \kappa_n \)-sequences. For each \( \alpha < \lambda \) we consider the \( \kappa_n \)-complete ultrafilter \( U_{\alpha n} \) defined by

\[
X \in U_{\alpha n} \text{ iff } X \subseteq \kappa_n \text{ and } \alpha \in j_n(X).
\]

For \( \alpha \leq \beta < \lambda \) we define the following ordering (which depends on the choice of \( j_n \)):

\[
\alpha \leq_n \beta \text{ iff } \alpha \leq \beta \text{ and for some } f : \kappa_n \to \kappa_n, j_n(f)(\beta) = \alpha.
\]

**Remark.** This implies that \( U_{\alpha n} \) is below \( U_{\alpha n+1} \) in the Rudin-Keisler ordering of ultrafilters on \( \kappa_n \). The Rudin-Keisler ordering of ultrafilters on a cardinal \( \kappa \) is defined by: \( U_0 \leq_{\text{RK}} U_1 \) iff for some \( f : \kappa \to \kappa \), \( A \in U_0 \) iff \( f^{-1}[A] \in U_1 \). If \( f \) witnesses \( \alpha \leq_{\text{RK}} \beta \) then \( f \) also witnesses \( U_{\alpha n} \leq_{\text{RK}} U_{\beta n} \). But the converse does not hold in general.

**Lemma 37** The partial ordering \( \leq_n \) is \( \kappa_n \)-directed and in fact each \( x \in [\lambda]^{<\kappa_n} \) has \( \lambda \)-many upper bounds in \( \leq_n \).
Proof. Using GCH, let \( \langle a_\alpha \mid \alpha < \kappa_n \rangle \) be an enumeration of \([\kappa_n]^{<\kappa_n}\) such that for each \( x \in [\kappa_n]^{<\kappa_n}\), the set of \( \alpha \) such that \( x = a_\alpha \) is a cofinal subset of \((\sup x)^+\). Now note that as \( j_n \) is the identity below \( \kappa_n \), \( \langle a_\alpha \mid \alpha < \kappa_n \rangle \) is the restriction to \( \kappa_n \) of \( j_n(\langle a_\alpha \mid \alpha < \kappa_n \rangle) \); let \( \langle a_\alpha \mid \alpha < \lambda \rangle \) denote the restriction of \( j_n(\langle a_\alpha \mid \alpha < \kappa_n \rangle) \) to \( \lambda \). Then for each \( x \in [\lambda]^{<\lambda} \cap M_\alpha \), the set of \( \alpha \) such that \( x = a_\alpha \) is a cofinal subset of \((\sup x)^+\).

Now suppose that \( x \in [\lambda]^{<\kappa_n} \); we find \( \alpha < \lambda \) such that \( \beta < \alpha \) for each \( \beta \in x \). Enumerate \( x \) in increasing order as \( \langle \beta_i \mid i < \gamma \rangle \), where \( \gamma \) is less than \( \kappa_n \) and choose \( \alpha < \lambda \) so that \( a_\alpha \) equals \( x \). If \( \beta = \beta_i \) belongs to \( x \), then \( \beta = j_n(f)(\alpha) \), where \( f \) is defined by: \( f(\bar{\alpha}) = \) the \( i \)-th element (in increasing order) of \( a_\alpha \). So \( \beta < \alpha \). As there are \( \lambda \)-many \( \alpha \) such that \( x \) equals \( a_\alpha \), we are done. \( \square \)

Fix \( \pi_{\alpha\beta} \) witnessing \( \beta \leq \alpha \), setting \( \pi_{\alpha\alpha} \) to be the identity.

**Lemma 38** Suppose that \( \gamma < \beta \leq \alpha \) with \( \gamma \leq \alpha \) and \( \beta \leq \alpha \). Then \( \{ \nu < \kappa_n \mid \pi_{\alpha\beta}(\nu) > \pi_{\alpha\gamma}(\nu) \} \) belongs to \( U_{\alpha\alpha} \).

*Proof.* Let \( X \) denote \( \{ \nu < \kappa_n \mid \pi_{\alpha\beta}(\nu) > \pi_{\alpha\gamma}(\nu) \} \). We wish to show that \( \alpha \) belongs to \( j_n(X) \). But \( j_n(X) \) equals \( \{ \nu < j_n(\kappa_n) \mid j_n(\pi_{\alpha\beta}(\nu)) > j_n(\pi_{\alpha\gamma}(\nu)) \} \), so we must show that \( j_n(\pi_{\alpha\beta})(\alpha) = \beta > j_n(\pi_{\alpha\gamma})(\alpha) = \gamma \), which follows from our hypothesis. \( \square \)

**Lemma 39** Suppose that \( x \) belongs to \([\lambda]^{<\kappa_n}\) and \( \beta \leq \alpha \) for each \( \beta \in x \). Then there is \( A \in U_{\alpha\alpha} \) such that \( \pi_{\alpha\beta_0} \) agrees with \( \pi_{\beta_0\beta_1} \pi_{\alpha\beta_1} \) on \( A \) whenever \( \beta_0 \leq \beta_1 \) belong to \( x \).

*Proof.* We must show that \( \{ \nu \mid \pi_{\alpha\beta_0}(\nu) = \pi_{\beta_0\beta_1} \pi_{\alpha\beta_1}(\nu) \} \) belongs to \( U_{\alpha\alpha} \). By the definition of \( U_{\alpha\alpha} \) this means that \( j_n(\pi_{\alpha\beta_0})(\alpha) = j_n(\pi_{\beta_0\beta_1}) j_n(\pi_{\alpha\beta_1})(\alpha) \), which by the choice of the \( \pi \)'s just says \( \beta_0 = j_n(\pi_{\beta_0\beta_1})(\beta_1) = \beta_0 \), so we are done. \( \square \)

8. Vorlesung

We are now ready to define extender-based Prikry forcing at cofinality \( \omega \). We first define forcings \( Q_n \) for each \( n \), and then put them together to form the desired forcing \( P \). Each \( Q_n \) is the union of \( Q_{n0} \) and \( Q_{n1} \), which we define next.
Definition 40 \( Q_{n1} \) consists of all functions \( f \) from a subset of \( \lambda \) of size at most \( \kappa \) into \( \kappa_n \), ordered by: \( f \leq g \) iff \( f \) extends \( g \) as a function.

Definition 41 \( Q_{n0} \) consists of triples \((a, A, f)\) such that:

1. \( f \) belongs to \( Q_{n1} \).
2. \( a \) is a subset of \( \lambda \) of size less than \( \kappa_n \) with a maximum.
3. \( a \) is disjoint from \( \text{Dom} \ f \).
4. \( \alpha \leq_n \max a \) for each \( \alpha \in a \).
5. \( A \) belongs to the ultrafilter \( U_{\max a} \).
6. Whenever \( \alpha \geq_n \beta \) belong to \( a \) then \( \pi_{\max a, \beta}(\mu) = \pi_{\alpha, \max a, \beta}(\mu) \) for all \( \mu \) in \( A \).
7. Whenever \( \alpha > \beta \) belong to \( a \) then \( \pi_{\max a, \alpha}(\mu) > \pi_{\max a, \beta}(\mu) \) for \( \mu \) in \( A \).

Extension is defined by: \((a, A, f) \leq (b, B, g)\) iff \( f \) extends \( g \), \( a \) contains \( b \) and \( A \subseteq \pi_{\max a, \max b}^{-1}[B] \).

Remark. (4) above implies that whenever \( \alpha \geq_n \beta \geq_n \gamma \) belong to \( a \) then \( \pi_{\alpha, \gamma}(\mu) = \pi_{\beta, \gamma}\pi_{\alpha, \beta}(\mu) \) for all \( \mu \) in \( \pi_{\max a, \alpha}[A] \), as if \( \mu = \pi_{\max a, \alpha}(\nu) \) then by (4), the left side is \( \pi_{\alpha, \gamma}(\mu) = \pi_{\alpha, \max a, \gamma}(\nu) = \pi_{\max a, \gamma}(\nu) \) and also the right side is \( \pi_{\beta, \gamma}\pi_{\alpha, \beta}(\mu) = \pi_{\beta, \gamma}\pi_{\alpha, \max a, \beta}(\nu) = \pi_{\max a, \gamma}(\nu) \).

Let \( Q_n \) be the union of \( Q_{n0} \) and \( Q_{n1} \). The direct extension relation \( \leq^* \) on \( Q_n \) is simply the union of the extension relations on \( Q_{n0} \) and \( Q_{n1} \). The extension relation \( \leq \) on \( Q_n \) is defined by: \( p \leq q \) iff \( p \) is a direct extension of \( q \) or \( p \in Q_{n1}, q = (a, A, f) \in Q_{n0} \) where:

(a) \( p \) extends \( f \).
(b) \( \text{Dom} \ p \) contains \( a \).
(c) \( p(\max a) \in A \).
(d) For \( \beta \) in \( a \), \( p(\beta) = \pi_{\max a, \beta}(p(\max a)) \).

At last we define the desired forcing \( P \):

Definition 42 \( P \) consists of \( p = (p_n \mid n \in \omega) \) such that for each \( n \), \( p_n \) belongs to \( Q_n \) and for some finite \( l(p) \), \( p_n \) belongs to \( Q_{n1} \) for \( n \) less than \( l(p) \) and for \( n \) at least \( l(p) \), \( p_n = (a_n, A_n, f_n) \) belongs to \( Q_{n0} \) with \( a_n \subseteq a_{n+1} \).

\( p \leq q \) iff for each \( n \), \( p_n \leq q_n \) in \( Q_n \). And \( p \leq^* q \) (\( p \) is a direct extension of \( q \)) iff for each \( n \), \( p_n \) is a direct extension of \( q_n \).
9. Vorlesung

Lemma 43 Suppose that $\gamma < \beta \leq \alpha$ with $\gamma \leq n \alpha$ and $\beta \leq n \alpha$. Then 
\{ $\nu < \kappa_n \mid \pi_{\alpha\beta}(\nu) > \pi_{\alpha\gamma}(\nu)$ \} belongs to $U_{\kappa_n}$.

Lemma 44 Suppose that $x$ belongs to $[\lambda]^{<\kappa_n}$ and $\beta \leq n \alpha$ for each $\beta \in x$. Then there is $A \in U_{\kappa_n}$ such that $\pi_{\alpha\beta_0}$ agrees with $\pi_{\beta_1\alpha_0}$ on $A$ whenever $\beta_0 \leq n \beta_1$ belong to $x$.

Lemma 45 $P$ is $\kappa^{++}$-cc.

Proof. Let $p(\alpha)$, $\alpha < \kappa^{++}$, be elements of $P$ and write $p(\alpha)$ as $\langle p(\alpha)_n \mid n \in \omega \rangle$, where for $n \geq l(p(\alpha))$, $p(\alpha)_n = (a(\alpha)_n, A(\alpha)_n, f(\alpha)_n)$. There is a stationary 
$S \subseteq \kappa^{++}$ such that for $\alpha, \beta$ in $S$ we have:

(a) $l(p(\alpha)) = l(p(\beta)) = l$.
(b) For $n$ less than $l$, the collection of Dom $(p(\alpha)_n)$, $\alpha \in S$, forms a $\Delta$ system 
on whose root $p(\alpha)_n$ and $p(\beta)_n$ agree.
(c) For $n$ at least $l$, the collection of $a(\alpha)_n \cup$ Dom $(f(\alpha)_n)$, $\alpha \in S$, forms a 
$\Delta$ system on whose root $f(\alpha)_n$ and $f(\beta)_n$ agree. Moreover $a(\alpha)_n$ and $a(\beta)_n$ 
have the same intersection with this root and therefore $a(\alpha)_n$ is disjoint from 
Dom $(f(\beta)_n)$.

Now we claim that if $\alpha, \beta$ belong to $S$ then $p(\alpha)$ and $p(\beta)$ are compatible. We 
construct $q$ below both of these conditions as follows. For $n$ less than $l$ let $q_n$ be $p(\alpha)_n \cup p(\beta)_n$, which by (b) above is a well-defined function. Now suppose 
n is at least $l$; we define $q_n = (b_n, B_n, g_n)$. We take $g_n$ to be $f(\alpha)_n \cup f(\beta)_n$. 
To define $b_n$, choose $\rho$ above all elements of $a(\alpha)_n \cup a(\beta)_n$ in the ordering $\leq_n$ 
and greater than all elements of Dom $(g_n)$, then set $b_n = a(\alpha)_n \cup a(\beta)_n \cup \{ \rho \}$. 
Finally, to define $B_n$, let $\alpha^*, \beta^*$ be $\max a(\alpha)_n$, $\max a(\beta)_n$, respectively, and 
let $B'_n$ be the intersection of $\pi_{\alpha^*\beta^*}^{-1}[A(\alpha)_n] \cap \pi_{\alpha\beta}^{-1}[A(\beta)_n]$. Now using Lemmas 
43 and 44, choose $B_n \in U_{n,p}$ to be a subset of $B'_n$ such that:

i. Whenever $\alpha \geq_n \beta$ belong to $b_n$ then $\pi_{\rho\beta}(\mu) = \pi_{\alpha\beta}(\mu)$ for all $\mu$ in $B_n$.
ii. Whenever $\alpha > \beta$ belong to $b_n$ then $\pi_{\rho\alpha}(\mu) > \pi_{\rho\beta}(\mu)$ for $\mu$ in $B_n$.

Then $q_n = (b_n, B_n, g_n)$ belongs to $Q_{n0}$ for each $n$, and $q$ is a condition 
extending both $p(\alpha)$ and $p(\beta)$, as desired. \qed

10.-11. Vorlesungen

We wish to prove the following two lemmas.

18
Lemma 46 (The Prikry Property) For any sentence $\sigma$, each condition in $P$ has a direct extension that decides $\sigma$.

Lemma 47 $P$ preserves $\kappa^+$. 

Both of these lemmas will follow rather easily, given a certain fact about “minimal extensions” of conditions, which we now describe. Recall the following notation: If $p = \langle p_n \mid n \in \omega \rangle$ is a condition and $n \geq l(p)$, we write $p_n$ as $(a_n(p), A_n(p), f_n(p))$. Now suppose that $q \leq p$ belong to $P$. Define the condition $q \downarrow p = r$ as follows: For $n$ not in the interval $[l(p), l(q))$, $r_n = p_n$. For $n$ in the interval $[l(p), l(q))$, $r_n$ is the union of $f_n(p)$ and $q_n \upharpoonright a_n(p)$. We say that $q$ is a minimal extension of $p$ iff $q = q \downarrow p$.

Note that minimal extensions can be alternatively described as follows. Suppose that $m$ is at least $l(p)$ and choose $\bar{\nu} = \langle \nu_l(p), \ldots, \nu_{m-1} \rangle$ in $\prod_{l(p) \leq k < m} A_k(p)$. Define the condition $q = p \ast \bar{\nu}$ as follows: $q_n = p_n$ for $n$ not in $[l(p), m)$ and for $n$ in $[l(p), m)$,

$$q_n = f_n(p) \cup \{ (\beta, \pi_{\text{max} a_n(p), \beta}(\nu_n)) \mid \beta \in a_n(p) \}.$$

Then $p \ast \bar{\nu}$ is a minimal extension of $p$ and every minimal extension of $p$ is of this form, as $q \downarrow p$ is just the condition $p \ast \langle \nu_l(p), \ldots, \nu_{l(q)-1} \rangle$ where $\nu_n = q_n(\text{max} a_n(p))$.

The main fact we need is the following.

Sublemma 48 Suppose that $p$ belongs to $P$ and $D$ is open dense. Then there is a direct extension $p^*$ of $p$ such that whenever $q \leq p^*$ belongs to $D$, so does $q \downarrow p^*$.

Proof. For each $n \geq l(p)$ and each $\bar{\nu} = \langle \nu_l(p), \ldots, \nu_{n-1} \rangle$ in $\prod_{l(p) \leq k < n} A_k$, we will define a condition $p^\bar{\nu}$ which directly extends $p$. Let $\langle \bar{\nu}_i \mid i < \kappa \rangle$ be an enumeration of the $\bar{\nu}$. We assume that for $i$ less than $j$, length $(\bar{\nu}_i)$ is at most length $(\bar{\nu}_j)$ and if these lengths are equal, then max $\bar{\nu}_i$ is at most max $\bar{\nu}_j$. We define a $\leq^*$-descending sequence $\langle p^i \mid i < \kappa \rangle$ of direct extensions of $p$ and set $p^\bar{\nu} = p^i$ where $\bar{\nu} = \bar{\nu}_i$.

Note that if $\bar{\rho} = \langle p^i \mid i < \lambda \rangle$, $\lambda$ limit, is a $\leq^*$-descending sequence of direct extensions of $p$ with a $\leq^*$-lower bound, then although $\bar{\rho}$ may not have
a greatest \( \leq^* \)-lower bound, it does have a canonical maximal \( \leq^* \)-lower bound
\( q_i \), defined by:
\[ q_k = \bigcup_{i < \lambda} p^i_k \quad \text{for } k < l(p), \quad \text{and for } k \geq l(p), \quad f_k(q) = \bigcup_{i < \lambda} f_k(p^i), \]
\[ a_k(q) = \bigcup_{i < \lambda} a_k(p^i) \cup \{ \alpha \} \]
where \( \alpha \) is the least \( \leq_k \)-upper bound to the elements
\[ \bigcup_{i < \lambda} a_k(p^i) \] and \( A_k(q) = \bigcap_{i < \lambda} \pi^{-1}_{\alpha, \max} a_k(p^i)[A_k(p^i)]. \]

Suppose that \( p^i \) is defined for all \( i < j \) and we wish to define \( p^j \). Let \( q^j \)
be \( p \) if \( j \) equals 0, and otherwise let \( q^j \) be the canonical maximal \( \leq^* \)-lower
bound to the \( p^i, i < j \). (It will be clear from the construction that the \( p^i, i < j \),
have a \( \leq^* \)-lower bound.) Let \( n \) denote \( l(p) + \text{length}(\bar{v}_j) \). If \( \bar{v}_j \) does not
belong to \( \prod_{l(p) \leq k < n} A_k(q^j) \) or if it does but \( q^j \ast \bar{v}_j \) has no direct extension in
\( D \), then let \( p^j \) be \( q^j \). Otherwise choose some direct extension \( r^j \) of \( q^j \ast \bar{v}_j \) in
\( D \) and define the direct extension \( p^j \) of \( q^j \) as follows:

(a) For \( k \) outside the interval \( [l(p), n) \), \( p^i_k = r_k^j. \)

(b) For \( k \) inside the interval \( [l(p), n) \), set \( a_k(p^i) = a_k(q^j), \ A_k(p^i) = A_k(q^j) \)
and \( f_k(p^i) = r_k^j \upharpoonright (\text{Dom}(r_k^j) \setminus a_k(q^j)). \)

Then note that as \( p^j \ast \bar{v}_j \) is defined and equal to \( r^j \), it follows that \( p^j \ast \bar{v}_j \)
belonges to \( D \).

Let \( p^* \) be a \( \leq^* \)-lower bound to all of these conditions \( p^i, i < \kappa \). Such a
\( \leq^* \)-lower bound exists as the extension relation below \( p \) on \([0, l(p)) \) is \( \kappa^+ \)-
closed and in the above construction, \( a_{l(p)}(p^i) \) and \( A_{l(p)}(p^i) \) never grow and
for \( k \) greater than \( l(p) \), \( a_k(p^i) \) and \( A_k(p^i) \) only grow at most \( \kappa_{k-1} \) times. Now
if \( q \leq p^* \) belongs to \( D \) then choose \( \bar{v} \) so that \( q \) is a direct extension of \( p^* \ast \bar{v} \).
(This \( \bar{v} \) is \( \nu_k \), \( \ldots, \nu_{l(q)-1} \) where \( \nu_k = f_k(q)(\max a_k(p^*) \) for each \( k \)) Choose
\( i \) so that \( \bar{v} \) equals \( \bar{v}_j \). Then as \( A_k(p^i) = A_k(p^*) \) for \( k \) in \([l(p), l(q)) \), \( p^i \ast \bar{v} \) is a
well-defined condition and therefore \( p^i \) was chosen so that \( p^i \ast \bar{v} \) belongs to
\( D \). As \( q \upharpoonright p^* = p^* \ast \bar{v} \) extends \( p^i \ast \bar{v} \), it follows that \( q \upharpoonright p^* \) also belongs to \( D \),
as desired. This proves Sublemma 51.

12.-13. Vorlesungen

We prove the following two lemmas.

**Lemma 49** (The Prikry Property) For any sentence \( \sigma \), each condition in \( P \)
has a direct extension that decides \( \sigma \).

**Lemma 50** \( P \) preserves \( \kappa^+ \).
The main fact we need is the following.

**Sublemma 51** Suppose that $p$ belongs to $P$ and $D$ is open dense. Then there is a direct extension $p^*$ of $p$ such that whenever $q \leq p^*$ belongs to $D$, so does $q \upharpoonright p^*$.

**Proof of Lemma 49.** Suppose that the condition $p$ has no direct extension deciding the sentence $\sigma$. Applying Sublemma 51, we may assume that whenever $q \leq p$ decides $\sigma$, then so does $q \upharpoonright p$. Set $l(p) = n$. We claim that \( \{ \nu_n \in A_n(p) \mid p * \langle \nu_n \rangle \} \) does not decide $\sigma$ must belong to the ultrafilter $U_{n, \text{max} \alpha_n(p)}$. Otherwise, we can thin $A_n(p)$ to $A \in U_{n, \text{max} \alpha_n(p)}$ so that the $p * \langle \nu_n \rangle$ for $\nu_n$ in $A$ decide $\sigma$ in the same way, and form $p^*$ by replacing $A_n(p)$ by $A$. Then $p^*$ is a direct extension of $p$ deciding $\sigma$, contradicting our hypothesis.

Similarly, we have that whenever $p * \langle \nu_i(p), \ldots, \nu_{m-1} \rangle$ is an extension of $p$ which does not decide $\sigma$, the set \( \{ \nu_m \in A_m(p) \mid p * \langle \nu_i(p), \ldots, \nu_{m-1}, \nu_m \rangle \} \) does not decide $\sigma$ belongs to $U_{m, \text{max} \alpha_m(p)}$. Therefore we can form a direct extension $p^*$ of $p$ such that no minimal extension $p * \tilde{v}$ of $p$ compatible with $p^*$ decides $\sigma$. Now choose $q \leq p^*$ deciding $\sigma$. By choice of $p$, $q \upharpoonright p$ also decides $\sigma$. But $q \upharpoonright p$ is a minimal extension of $p$ compatible with $p^*$, contradicting the choice of $p^*$. This proves Lemma 49.

**Proof of Lemma 50.** As $\kappa$ is singular it suffices to show that if $p$ forces $\dot{f}$ to be a function from $\kappa$ into $(\kappa^+)^V$, then some extension $q$ of $p$ forces a bound on the range of $\dot{f}$. Assume that $l(p)$ is greater than $n$. Now using Sublemma 51, build a $\kappa$-sequence of direct extensions of $p$ with lower bound $p^*$ having the property that for each $i < \kappa$, if $q \leq p^*$ forces a value of $\dot{f}$ at $i$ then so does $q \upharpoonright p^*$. But there are only $\kappa$-many conditions of the form $q \upharpoonright p^*$, and therefore $p^*$ forces a bound on the range of $\dot{f}$. This proves Lemma 50.

**Lemma 52** $P$ adds $\lambda$-many $\omega$-sequences to the product of the $\kappa_n$'s.

**Proof.** Let $G$ be $P$-generic. For each $\alpha < \lambda$ define $t_\alpha$ by $t_\alpha(n) = p_n(\alpha)$, where $p$ belongs to $G$, $n < l(p)$ and $\alpha \in \text{Dom } p_n$. For any $\alpha < \lambda$, either $\alpha$ is in the domain of $f_n(p)$ for some $p$ in $G$ and all $n \geq l(p)$, or $\alpha$ is in $a_n(p)$ for some $p$ in $G$ and all $n \geq l(p)$, and both cases occur cofinally in $\lambda$. In the former case, $t_\alpha$ belongs to $V$ and in the latter case an easy density argument shows that $t_\alpha$ eventually dominates each element of the product of the $\kappa_n$'s.
in $V$. We show that in the latter case, $t_\alpha$ also eventually dominates each $t_\beta$, $eta < \alpha$, which does not belong to $V$, and therefore as the latter case must occur unboundedly in $\lambda$, $\lambda$-many new elements of the product of the $\kappa_n$’s have been added.

Suppose $\beta$ is less than $\alpha$, $t_\beta$ does not belong to $V$ and $\alpha$ is in $a_n(p)$ for some $p$ in $G$ and all $n \geq l(p)$. Choose $q$ in $G$ so that $\beta$ belongs to $a_n(q)$ for each $n \geq l(q)$. We may assume that $q$ extends $p$. Then both $\beta$ and $\alpha$ belong to $a_n(q)$ for each $n \geq l(q)$. By the definition of condition, we have $\pi_{\max a_n(q), \beta}(\nu) < \pi_{\max a_n(q), \alpha}(\nu)$ for each $\nu$ in $A_n(q)$. But now choose $r$ in $G$ so that $l(r)$ is greater than $n$ and $r$ extends $q$. Then $t_\beta(n) = r_n(\beta) = \pi_{\max a_n(q), \beta}(\nu) < \pi_{\max a_n(q), \alpha}(\nu) = r_n(\alpha) = t_\alpha(n)$, where $\nu = r_n(\max a_n(q)) \in A_n(q)$. So $t_\alpha$ eventually dominates $t_\beta$. $\square$

Thus after forcing with $P$, the GCH still holds below $\kappa$ and $2^\kappa$ is at least $\lambda$, yielding a dramatic failure of the singular cardinal hypothesis.

**Extender-based Prikry forcing with a single extender**

In the previous section we showed how to violate the singular cardinal hypothesis at an $\omega$-limit of cardinals with a rather high degree of strength. In this section we start with a single cardinal with much less strength and simultaneously singularise it and blow up its power set, without adding bounded subsets.

Assume GCH and suppose that $\kappa$ and $\lambda$ are regular with $\lambda$ at least $\kappa^{++}$. We assume that $\kappa$ is $H(\lambda)$-strong, which means that there is an elementary embedding $j : V \rightarrow M$ with critical point $\kappa$ such that $H(\lambda)$ is contained in $M$ and $j(\kappa)$ is greater than $\lambda$. We also make the following additional assumption:

(*) $\lambda$ is of the form $j(f_\lambda)(\kappa)$ for some function $f_\lambda : \kappa \rightarrow \kappa$.

(*) is clearly the case if there is a formula $\varphi$ such that $\lambda$ is the least regular cardinal with $H(\lambda) \models \varphi(\kappa)$; for then we can take $f_\lambda(\kappa) =$ the least $\lambda$ such that $H(\lambda) \models \varphi(\kappa)$. This applies for example when $\lambda = \kappa^{+n}$ for finite $n$ or $\lambda =$ the least inaccessible greater than $\kappa$. It can also be shown that if $\kappa$ is $H(\lambda)$-strong for all $\lambda$ then this is necessarily witnessed by embeddings which obey (*), and that for a single $\lambda$, if $\kappa$ is $H(\lambda)$-strong then in a generic extension of the universe, $\kappa$ is $H(\lambda)$-strong via an embedding obeying (*). Thus the additional hypothesis (*) should be regarded as harmless.
As in the previous section, for each $\alpha < \lambda$ we consider the $\kappa$-complete ultrafilter $U_\alpha$ defined by:

$$X \in U_\alpha \text{ iff } X \subseteq \kappa \text{ and } \alpha \in j(X).$$

We also define the following ordering:

$$\alpha \leq_j \beta \text{ iff } \kappa \leq \alpha \leq \beta \text{ and for some } f : \kappa \rightarrow \kappa, \ j(f)(\beta) = \alpha.$$

**Lemma 53** For each $\alpha \in [\kappa, \lambda)$, $\kappa \leq_j \alpha$.

**Proof.** Define $g : \kappa \rightarrow \kappa$ by $g(\alpha) = \kappa$ such that $f_\alpha(\kappa) > \alpha$ (if such a $\kappa$ exists, 0 otherwise). \(\square\)

**Lemma 54** The partial ordering $\leq_j$ is $\kappa^{++}$-directed and in fact each $x \in [\lambda]^{\kappa^+}$ has $\lambda$-many upper bounds in $\leq_j$.

**Proof.** Using GCH, let $\langle a_\alpha \mid \alpha < \kappa \rangle$ be an enumeration of $^{<\kappa}\kappa$ such that for each $x \in [\kappa]^{<\kappa}$, the set of $\alpha$ such that $x = a_\alpha$ is a cofinal subset of $(\sup x)^+$. Now note that as $j$ is the identity below $\kappa$, $\langle a_\alpha \mid \alpha < \kappa \rangle$ is the restriction to $\kappa$ of $j(\langle a_\alpha \mid \alpha < \kappa \rangle)$; let $\langle a_\alpha \mid \alpha < \lambda \rangle$ denote the restriction of $j(\langle a_\alpha \mid \alpha < \kappa \rangle)$ to $\lambda$. Then for each $x \in ^{<\lambda}\lambda \cap M = ^{<\lambda}\lambda$, the set of $\alpha$ such that $x = a_\alpha$ is a cofinal subset of $(\sup x)^+$.

Now suppose that $x$ belongs to $^{<\kappa}\lambda$; we find $\alpha < \lambda$ such that $\beta <_j x$ for each $\beta \in \text{Range } (x)$. Using the fact that $\lambda \geq \kappa^{++}$ is regular, choose $\alpha < \lambda$ so that $a_\alpha$ equals $x$. Using Lemma 53, choose $g : \kappa \rightarrow \kappa$ such that $j(g)(\alpha) = \kappa$. Now for each $i < \kappa^+$ we may choose a function $f_i : \kappa \rightarrow \kappa$ such that $j(f_i)(\alpha) = i$; such an $f_i$ can be defined by choosing $A \subseteq \kappa$ to code $i$ and then setting $f_i(\alpha) = \text{the ordinal coded by } A \cap g(\alpha)$. Now suppose that $\beta$ belongs to $\text{Range } (x)$ and choose $i < \kappa^+$ so that $\beta = x(i)$. Then $\beta = j(f)(\alpha)$, where $f$ is defined by: $f(\alpha) = a_\alpha(f_i(\alpha))$, the $f_i(\alpha)$-th element of $a_\alpha$. So $\beta <_j \alpha$. As there are $\lambda$-many $\alpha$ such that $x$ equals $a_\alpha$, we are done. \(\square\)

Fix $\pi_{\alpha\beta}$ witnessing $\beta \leq_j \alpha$. As in the previous section we have:

**Lemma 55** Suppose that $\gamma < \beta \leq \alpha$ with $\gamma \leq_j \alpha$ and $\beta \leq_j \alpha$. Then $\{ \nu < \kappa \mid \pi_{\alpha\beta}(\nu) > \pi_{\alpha\gamma}(\nu) \}$ belongs to $U_\alpha$.
Lemma 56 Suppose that $\beta_0 \leq_j \beta_1 \leq_j \alpha$. Then there is $A \in U_\alpha$ such that $\pi_{\alpha, \beta_0}$ agrees with $\pi_{\beta_1, \beta_0} \pi_{\alpha, \beta_1}$ on $A$.

For technical reasons we make a few further demands of the projection maps $\pi_{\alpha, \beta_1}$, $\beta \leq_j \alpha$:

Fixed projection to $\kappa$: $\pi_{\alpha \kappa}(\tilde{\alpha}) = \pi_{\beta \kappa}(\tilde{\alpha})$ for all $\tilde{\alpha}$.

Total commutativity at $\kappa$: For $\beta \leq_j \alpha$, $\pi_{\alpha \kappa}(\tilde{\alpha}) = \pi_{\beta \kappa}(\pi_{\alpha, \beta}(\tilde{\alpha}))$ for all $\tilde{\alpha}$.

$U_\alpha$ is a $P$-point: If $\{A_i \mid i < \kappa\}$ belong to $U_\alpha$ then for some $A \in U_\alpha$, $A$ is almost contained in $A_i$ for each $i$ (i.e., modulo bounded sets).

Lemma 57 Suppose that $\gamma < \beta \leq \alpha$ with $\gamma \leq_j \alpha$ and $\beta \leq_j \alpha$. Then $\{\nu < \kappa \mid \pi_{\alpha \beta}(\nu) > \pi_{\alpha \gamma}(\nu)\}$ belongs to $U_\alpha$.

14.-15. Vorlesungen

Fixed projection to $\kappa$: $\pi_{\alpha \kappa}(\tilde{\alpha}) = \pi_{\beta \kappa}(\tilde{\alpha})$ for all $\tilde{\alpha}$.

Total commutativity at $\kappa$: For $\beta \leq_j \alpha$, $\pi_{\alpha \kappa}(\tilde{\alpha}) = \pi_{\beta \kappa}(\pi_{\alpha, \beta}(\tilde{\alpha}))$ for all $\tilde{\alpha}$.

$U_\alpha$ is a $P$-point: If $\{A_i \mid i < \kappa\}$ belong to $U_\alpha$ then for some $A \in U_\alpha$, $A$ is almost contained in $A_i$ for each $i$ (i.e., modulo bounded sets).

To achieve the first two of these properties, we define $\tilde{X}$ to be the set of $\tilde{\alpha} < \kappa$ such that for some $\tilde{\kappa} \leq \tilde{\alpha}$, $\tilde{\kappa}$ is closed under $f_\lambda$, $\tilde{\kappa}$ is inaccessible and $f_\lambda(\tilde{\kappa}) > \tilde{\alpha}$. Then $\tilde{X}$ belongs to each of the measures $U_\alpha$, $\kappa \leq \alpha < \lambda$. So we can assume that for all $\alpha \in [\kappa, \lambda)$, the projection $\pi_{\alpha \kappa}$ is defined by: $\pi_{\alpha \kappa}(\tilde{\alpha}) = \tilde{\alpha}$ is the unique $\tilde{\kappa}$ witnessing $\tilde{\alpha} \in \tilde{X}$ (for $\tilde{\alpha}$ in $\tilde{X}$); $\pi_{\alpha \kappa}(\tilde{\alpha}) = 0$ (for $\tilde{\alpha}$ not in $\tilde{X}$). This achieves the first property.

To achieve the second property, we require that $\pi_{\alpha \beta}(\tilde{\alpha}) = 0$ for $\tilde{\alpha}$ not in $\tilde{X}$ and for $\tilde{\alpha}$ in $\tilde{X}$, $\pi_{\alpha \beta}(\tilde{\alpha})$ is in the interval $[\pi_{\alpha \kappa}(\tilde{\alpha}), \tilde{\alpha}]$. As $\kappa \leq_j \beta$ for all $\beta \in [\kappa, \lambda)$, Lemma 57 implies that these requirements are vacuous on a set which belongs to each of the ultrafilters $U_\alpha$, $\kappa \leq \alpha < \lambda$, and therefore can be imposed. We now have that for $\beta \leq_j \alpha$, if $\tilde{\alpha}$ belongs to $\tilde{X}$, then $\pi_{\alpha \beta}(\tilde{\alpha})$ is in the interval $[\pi_{\alpha \kappa}(\tilde{\alpha}), \tilde{\alpha}]$ and therefore $\pi_{\alpha \kappa}(\tilde{\alpha})$ witnesses that $\pi_{\alpha \beta}(\tilde{\alpha})$ belongs to $\tilde{X}$; it follows that $\pi_{\beta \kappa}(\pi_{\alpha \beta}(\tilde{\alpha}))$ equals the witness $\pi_{\alpha \kappa}(\tilde{\alpha})$. And if $\tilde{\alpha}$ does not belong to $\tilde{X}$, then $\pi_{\alpha \kappa}(\tilde{\alpha}) = 0$ and $\pi_{\beta \kappa}(\pi_{\alpha \beta}(\tilde{\alpha})) = \pi_{\beta \kappa}(0) = 0$. This establishes the second property.

To verify the $P$-point property, note that the function $\pi_{\alpha \kappa} : \kappa \to \kappa$ is non-decreasing and cofinal on $\tilde{X} \in U_\alpha$, and $j(\pi_{\alpha \kappa})(\alpha) = \kappa$. Then as each $A_i$,
\[ i < \kappa, \text{ belongs to } U_\alpha, \text{ we have that } \alpha \text{ belongs to } j(A_i) = j((A_i \mid i < \kappa)), \text{ for all } i < \kappa = j(\pi_{\alpha_k})(\alpha). \text{ It follows that } A = \Delta_{i<\kappa} A_i = \{\tilde{\alpha} \in X \mid \tilde{\alpha} \text{ belongs to } A_i \} \text{ for all } i < \pi_{\alpha_k}(\tilde{\alpha}) \text{ belongs to } U_\alpha, \text{ and as } \pi_{\alpha_k} \text{ is cofinal and non-decreasing on } X, \text{ it follows that } A \text{ is almost contained in each } A_i, \ i < \kappa.

For \( \alpha \in [\kappa, \lambda) \text{ and } \nu < \kappa \text{ we denote } \pi_{\alpha_k}(\nu) \text{ as } \kappa(\nu). \text{ (This is independent of the choice of } \alpha. \text{) A sequence } \langle \nu_0, \ldots, \nu_{n-1} \rangle \text{ of ordinals less than } \kappa \text{ is } \kappa-\text{increasing iff } i < j \implies \kappa(\nu_i) < \kappa(\nu_j). \text{ For } \kappa(\nu_0) < \kappa(\nu_1), \text{ we have that the cardinality of } \{\nu \in \tilde{X} \mid \kappa(\nu) = \kappa(\nu_0)\} \text{ is less than } \kappa(\nu_1) \text{ and therefore less than } \nu_1. \text{ If } \tilde{\nu} = \langle \nu_0, \ldots, \nu_{n-1} \rangle \text{ is a sequence of ordinals less than } \kappa, \text{ then } \kappa(\tilde{\nu}) \text{ denotes the max of the } \kappa(\nu_i). \text{ Note that by total commutativity at } \kappa, \text{ if } \alpha \leq_j \beta \text{ then } \kappa(\nu) = \kappa(\pi_{\alpha_k}(\nu)) \text{ for each } \nu < \kappa.

We are at last ready to define the desired forcing.

A condition \( p \) is of the form \( \{\langle \gamma, p^\gamma \rangle \mid \gamma \in g \setminus \{\max g\}\} \cup \{(\max g, p^{\max g}, T)\} \) where:

(a) \( \kappa \in g \subseteq [\kappa, \lambda), g \text{ has cardinality at most } \kappa, \ g \text{ has a maximal element and } \alpha \leq_j \max g \text{ for all } \alpha \in g. \)

(b) For \( \gamma \in g, p^\gamma \text{ is a finite } \kappa-\text{increasing sequence of ordinals in } \tilde{X} \subseteq \kappa. \)

(c) \( T \) is a tree of \( \kappa-\text{increasing sequences from } \tilde{X} \text{ with trunk } p^{\max g}. \) For each \( \eta \geq_T p^{\max g}, \text{ Succ}_T(\eta) = \{\nu < \kappa | \eta * \nu \in T\} \text{ belongs to } U_{\max g} \) and for \( \eta_1 \geq_T \eta_0 \geq_T p^{\max g}, T_{\eta_1} \text{ is a subtree of } T_{\eta_0}, \text{ where } T_\eta \text{ denotes the set of } \sigma \text{ such that } \eta * \sigma \text{ belongs to } T. \)

(d) For \( \gamma \in g, \kappa(p^{\max g}) \leq \kappa(p^\gamma). \)

(e) For \( \nu \in \text{Succ}_T(p^{\max g}), \text{ the cardinality of } \{\gamma \in g \mid \kappa(\nu) > \kappa(p^\gamma)\} \text{ is at most } \kappa(\nu). \)

(f) \( \pi_{\max g, \alpha} \text{ sends } p^{\max g} \text{ to } p^\alpha. \)

We denote \( g \) by \( \text{supp}(p), \max g \) by \( \text{mc}(p) \) (for “maximal coordinate” of \( p \), \( T \) by \( T^p \) and \( p^{\max g} \) by \( p^{\text{mc}} \).

For two conditions \( p, q \) as above, we say that \( p \text{ extends } q \iff \):

1. \( \text{supp}(p) \supseteq \text{supp}(q). \)
2. For \( \gamma \in \text{supp}(q), p^\gamma \text{ is an end-extension of } q^\gamma. \)
3. \( p^{\text{mc}(q)} \text{ belongs to } T^q. \)
4. For \( \gamma \in \text{supp}(q), p^\gamma \setminus q^\gamma \text{ is the range of } \pi_{\text{mc}(q), \gamma} \text{ on } p^{\text{mc}(q)} \setminus q^{\text{mc}(q)} \text{ past } i. \)
where \( i \in \text{Dom} \ (p^{\text{mc}(q)}) \) is largest so that \( \kappa(p^{\text{mc}(q)}(i)) \leq \kappa(q^\gamma) \).

5. \( \pi_{\text{mc}(p),\text{mc}(q)} \) maps \( T^p \) to a subtree of \( T^q \).

6. For \( \gamma \in \text{supp}(q) \) and \( \nu \in \text{Succ}_{T^p}(p^{\text{mc}}) \), if \( \kappa(\nu) \) is greater than \( \kappa(p^\gamma) \) then \( \pi_{\text{mc}(p),\gamma}(\nu) = \pi_{\text{mc}(q),\gamma}(\pi_{\text{mc}(p),\text{mc}(q)}(\nu)) \).

**Remark.** The above properties imply that if \( p \) extends \( q \) then \( \pi_{\text{mc}(p),\text{mc}(q)}(p^{\text{mc}}) = p^{\text{mc}(q)} \). See the proof of the transitivity of the extension relation below.

If in addition \( p^\gamma = q^\gamma \) for each \( \gamma \in \text{supp}(q) \), then we say that \( p \) is a direct extension of \( q \). We write \( p \preceq q \) for \( p \) extends \( q \) and \( p \preceq^* q \) for \( p \) directly extends \( q \).

### 16.-17. Vorlesungen

**Lemma 58** The ordering relation of \( P \) is transitive.

**Proof.** Suppose that \( p \preceq q \) and \( q \preceq r \); we check that \( p \preceq r \). Properties 1 and 2 are clearly satisfied.

For property 3, first note that it follows from property (f) for conditions and property 4 for extensions that if \( p \) extends \( q \), then \( \pi_{\text{mc}(p),\text{mc}(q)}(p^{\text{mc}}) \) equals \( p^{\text{mc}(q)} \). To see this, it suffices to show that \( \pi_{\text{mc}(q),\kappa}(\pi_{\text{mc}(p),\text{mc}(q)}(p^{\text{mc}})) \) equals \( \pi_{\text{mc}(q),\kappa}(p^{\text{mc}(q)}) \), as the map \( \pi_{\text{mc}(q),\kappa} \) is 1-1 on \( \kappa \)-increasing sequences.

Now \( \pi_{\text{mc}(q),\kappa}(\pi_{\text{mc}(p),\text{mc}(q)}(p^{\text{mc}})) \) equals \( \pi_{\text{mc}(p),\kappa}(p^{\text{mc}}) \) by total commutativity to \( \kappa \), and by property (f) for the condition \( p \), the latter is \( p^\kappa \). And \( \pi_{\text{mc}(q),\kappa}(p^{\text{mc}(q)}) \) is the union of \( \pi_{\text{mc}(q),\kappa}(q^{\text{mc}}) \) and \( \pi_{\text{mc}(q),\kappa}(p^{\text{mc}(q)} \setminus q^{\text{mc}}) \). The former is \( q^\kappa \) by property (f) for the condition \( q \). The latter is \( p^\kappa \setminus q^\kappa \) by property 4 for the extension \( p \preceq q \). It follows that \( \pi_{\text{mc}(q),\kappa}(p^{\text{mc}(q)}) \) is also \( p^\kappa \).

Now we check property 3 for the pair \( p, r \); i.e., we check that \( p^{\text{mc}(r)} \) belongs to \( T^r \). As \( q \) extends \( r \), \( \kappa(q^{\text{mc}}) \) equals \( \kappa(q^{\text{mc}(r)}) \) and therefore as \( p \) extends \( q \), \( p^{\text{mc}(r)} \setminus q^{\text{mc}(r)} \) is the range of \( \pi_{\text{mc}(q),\text{mc}(r)} \) on \( p^{\text{mc}(q)} \setminus q^{\text{mc}(q)} \). It follows from property 5 for the extension \( q \preceq r \) that \( q^{\text{mc}(r)} = (p^{\text{mc}(r)} \setminus q^{\text{mc}(r)}) \). \( p^{\text{mc}(r)} \) belongs to \( T^r \), as desired.

Next we check property 4. Suppose that \( \gamma \) belongs to \( \text{supp}(r) \); we must show that \( \kappa(p^\gamma \setminus r^\gamma) \) is the range of \( \pi_{\text{mc}(r),\gamma} \) on \( p^{\text{mc}(r)} \setminus r^{\text{mc}(r)} \) past \( i \), where \( \kappa(p^{\text{mc}(r)}(i)) \leq \kappa(r^\gamma) \). Since \( q \) extends \( r \), \( q^\gamma \setminus r^\gamma \) is the range of \( \pi_{\text{mc}(r),\gamma} \) on \( q^{\text{mc}(r)} \setminus r^{\text{mc}(r)} \) past \( j \), where \( j \in \text{Dom} \ (q^{\text{mc}(r)}) \) is largest so that \( \kappa(q^{\text{mc}(r)}(j)) \leq \kappa(r^\gamma) \).
First suppose that \( \gamma \) is a proper extension of \( r\gamma \), from which it follows that \( \kappa(\gamma) \) equals \( \kappa(q^{mc(r)}) \) and \( j \) equals \( i \). It suffices to show that \( p\gamma \setminus q\gamma \) is the range of \( \pi_{mc(r),\gamma} \) on \( p^{mc(r)} \setminus q^{mc(r)} \), for then \( p\gamma \setminus r\gamma = (p\gamma \setminus q\gamma) \cup (q\gamma \setminus r\gamma) \) is the range of \( \pi_{mc(r),\gamma} \) on \( (p^{mc(r)} \setminus q^{mc(r)}) \cup (q^{mc(r)} \setminus r^{mc(r)}) \) past \( i \), which is \( p^{mc(r)} \setminus r^{mc(r)} \) past \( i \), as desired. Now since \( p \) extends \( q \), \( p\gamma \setminus q\gamma \) is the range of \( \pi_{mc(q),\gamma} \) on \( p^{mc(q)} \setminus q^{mc(q)} \) past \( k \), where \( k \in \text{Dom} (p^{mc(q)}) \) is largest so that \( \kappa(p^{mc(q)}(k)) \leq \kappa(\gamma) \). But as \( q \) extends \( r \), \( \kappa(\gamma) \) equals \( \kappa(q^{mc(r)}) \), from which it follows that \( k \) is just max \( q^{mc(q)} \). Therefore \( p\gamma \setminus q\gamma \) is the range of \( \pi_{mc(q),\gamma} \) on \( p^{mc(q)} \setminus q^{mc(q)} \). Now using property 6 for the extension \( p \leq q \) we have:

\[
\begin{align*}
p\gamma \setminus q\gamma &= \pi_{mc(q),\gamma}[p^{mc(q)} \setminus q^{mc(q)}] = \\
\pi_{mc(r),\gamma}[\pi_{mc(q),mc(r)}[p^{mc(q)} \setminus q^{mc(q)}]] &= \\
\pi_{mc(r),\gamma}[p^{mc(r)} \setminus q^{mc(r)}].
\end{align*}
\]

The last equality holds by property 4 for the extension \( p \leq q \), using the fact that \( \kappa(q^{mc(r)}) \) equals \( \kappa(q^{mc(q)}) \).

If \( q\gamma \) equals \( r\gamma \), then as requirement (d) for the condition \( r \) implies that \( \kappa(r^{mc(r)}) \) is at most \( \kappa(r\gamma) = \kappa(q\gamma) \), it follows that \( \kappa(q^{mc(r)}) \) is at most \( \kappa(q\gamma) \). As \( p \) extends \( q \), \( p\gamma \setminus r\gamma = p\gamma \setminus q\gamma \) is the range of \( \pi_{mc(q),\gamma} \) on \( p^{mc(q)} \setminus q^{mc(q)} \) past \( j \), where \( j \in \text{Dom} (p^{mc(q)}) \) is largest so that \( \kappa(p^{mc(q)}(j)) \leq \kappa(q\gamma) \). Also, \( p^{mc(r)} \setminus q^{mc(r)} \) is the range of \( \pi_{mc(q),mc(r)} \) on \( p^{mc(q)} \setminus q^{mc(q)} \). It follows that \( p^{mc(r)} \setminus q^{mc(r)} \) past \( i \) equals the range of \( \pi_{mc(q),mc(r)} \) on \( p^{mc(q)} \setminus q^{mc(q)} \) past \( j \).

Using property 6 for the extension \( q \leq r \) we therefore have:

\[
\begin{align*}
p\gamma \setminus r\gamma &= p\gamma \setminus q\gamma = \pi_{mc(q),\gamma}[p^{mc(q)} \setminus q^{mc(q)} \text{ past } j] = \\
\pi_{mc(r),\gamma}[\pi_{mc(q),mc(r)}[p^{mc(q)} \setminus q^{mc(q)} \text{ past } j]] &= \\
\pi_{mc(r),\gamma}[p^{mc(r)} \setminus q^{mc(r)} \text{ past } i], \text{ as desired.}
\end{align*}
\]

We check property 5. As \( \pi_{mc(p),mc(r)}(p^{mc}) = p^{mc(r)} \), it suffices to show that \( \pi_{mc(p),mc(r)} \) maps \( T_{p^{mc}}^p \) into \( T_{p^{mc(r)}}^r \). Suppose that \( f \sigma \) belongs to \( T_{p^{mc}}^p \). As \( p \) extends \( q \), it follows that \( \pi_{mc(p),mc(q)}(\sigma) \) belongs to \( T_{p^{mc}}^q \) and therefore \( (p^{mc(q)} \setminus q^{mc}) * \pi_{mc(p),mc(q)}(\sigma) \) belongs to \( T_{p^{mc}}^q \). As in the verification of property 4 above, \( \pi_{mc(q),mc(r)}[p^{mc(q)} \setminus p^{mc}] = p^{mc(r)} \setminus q^{mc(r)} \) and therefore \( \pi_{mc(q),mc(r)} \pi_{mc(p),mc(q)}(\sigma) \) belongs to \( T_{p^{mc(r)}}^r \).

We claim that if \( \nu \) is a component of \( \sigma \) then \( \kappa(\nu) \) is greater than \( \kappa(p^{mc(r)}) \). We have \( \kappa(\nu) = \kappa(\pi_{mc(p),mc(q)}(\nu)) \) is greater than \( \kappa(p^{mc(q)}) \). If \( p^{mc(r)} \) is a proper extension of \( r^{mc} \) then as a final segment of \( p^{mc(r)} \) is the image under
\(\pi_{mc(q), mc(r)}\) of a final segment of \(p^{mc(q)}\), it follows that \(\kappa(p^{mc(r)})\) equals \(\kappa(p^{mc(q)})\) and therefore \(\kappa(\nu)\) is also greater than \(\kappa(p^{mc(r)})\). If \(p^{mc(r)}\) equals \(r^{mc}\) then as \(\pi_{mc(q), mc(r)}\) maps \(T^r_{q^{mc}}\) into \(T^r_{p^{mc(r)}}\), \(\kappa(\nu) = \kappa(\pi_{mc(q), mc(r)}(\pi_{mc(p), mc(q)}(\nu)))\) is greater than \(\kappa(q^{mc(r)}) = \kappa(p^{mc(r)})\).

Now we can apply property 6 for the extension \(p \leq q\) to conclude that \(\pi_{mc(q), mc(r)}(\pi_{mc(p), mc(q)}(\sigma)) = \pi_{mc(p), mc(r)}(\sigma) \in T^r_{p^{mc(r)}}\), as desired.

Finally, we verify property 6 for \(p\) and \(r\). Suppose that \(\gamma\) belongs to \(\text{supp}(r)\), \(\nu\) belongs to \(\text{Succ}_{T^r}(p^{mc})\) and \(\kappa(\nu)\) is greater than \(\kappa(p^\gamma)\). Then applying property 6 for the extension \(p \leq q\) we have \(\pi_{mc(p), \gamma}(\nu) = \pi_{mc(q), \gamma}(\pi_{mc(p), mc(q)}(\nu))\). As \(\nu\) belongs to \(\text{Succ}_{T^r}(p^{mc})\), it follows from property 5 for the extension \(p \leq q\) that \(\pi_{mc(p), mc(q)}(\nu)\) belongs to \(\text{Succ}_{T^r}(p^{mc(q)})\) and therefore to \(\text{Succ}_{T^r}(q^{mc})\). As \(\kappa(\pi_{mc(p), mc(q)}(\nu))\) is also greater than \(\kappa(p^\gamma) \geq \kappa(q^\gamma)\) and \(q \leq r\), we have \(\pi_{mc(p), \gamma}(\nu) = \pi_{mc(q), \gamma}(\pi_{mc(p), mc(q)}(\nu))\). Recall that in the verification of property 5 for \(p\), \(r\) we showed that \(\kappa(\nu)\) is greater than \(\kappa(p^{mc(r)})\); so once again applying property 6 to the extension \(p \leq q\), we conclude that \(\pi_{mc(p), \gamma}(\nu) = \pi_{mc(q), \gamma}(\pi_{mc(p), mc(q)}(\nu))\), as desired. \(\square\)

**Lemma 59** If \(q\) belongs to \(P\) and \(\alpha\) belongs to \([\kappa, \lambda]\) then there is \(p \leq^* q\) with \(\alpha \in \text{supp}(p)\).

*Proof.* If \(\alpha\) belongs to \(\text{supp}(q)\) then this is trivial. Suppose that \(\alpha\) does not belong to \(\text{supp}(q)\) but \(\alpha \leq_j mc(q)\). Then add to \(q\) a \(\kappa\)-increasing sequence \(t^\alpha\) such that \(\kappa(t^\alpha) \geq \kappa(q^{mc(q)})\); the result is a direct extension of \(q\).

Now suppose that \(\alpha \not\leq_j mc(q)\). We may assume that \(mc(q) <_j \alpha\), as otherwise we may choose \(\beta <_j \lambda\) so that \(\alpha, mc(q) \leq_j \beta\), find a direct extension of \(q\) whose support includes \(\beta\) and then by the previous paragraph add \(\alpha\) to the support of that direct extension. The desired \(p\) will be of the form \(q' \cup \{\alpha, t, T\}\), where \(q'\) is obtained from \(q\) by replacing the triple \((mc(q), q^{mc}, T^q)\) by \((mc(q), q^{mc})\) and \(t, T\) are defined below.

We take \(t\) to be any \(\kappa\)-increasing sequence such that \(\pi_{\alpha, \kappa}(t) = q^\alpha\). Recall that \(\pi_{\alpha, \kappa}\) is independent of \(\alpha\); therefore a candidate for \(t\) is \(q^{mc(q)}\), which by definition projects under \(\pi_{mc(q), \kappa}\) to \(q^\alpha\).

A first attempt at defining \(T\) is to take \(T_0 = \text{the preimage of } T^q_{q^{mc}}\) under \(\pi_{\alpha, mc(q)}\), with \(t\) added as its trunk. The resulting \(p_0 = q' \cup \{\alpha, t, T_0\}\) is a condition. The only difficulty with verifying \(p_0 \leq q\) is property 6: It
may be the case that for some \( \gamma \in \text{supp}(q) \setminus \{mc(q)\} \) and some \( \nu \in A = \text{Succ}_{T_0}(t) = \pi^{-1}_{a,mc(q)}[\text{Succ}_{T_0}(q^{mc})] \), \( \kappa(\nu) \) is greater than \( \kappa(\gamma) \) but \( \pi_{a,\gamma}(\nu) \neq \pi_{mc(\gamma)} \gamma \pi_{a,mc(q)}(\nu) \).

To fix this problem, we shrink \( T_0 \). For \( \nu \in A \) we let \( B_\nu \) be the set of \( \gamma \in \text{supp}(q) \setminus \{mc(q)\} \) such that \( \kappa(\nu) \) is greater than \( \kappa(\gamma) \). Then \( B_\nu \) has cardinality at most \( \kappa(\nu) \) as \( \kappa(\nu) = \kappa(\pi_{a,mc(q)}(\nu)) \), \( \pi_{a,mc(q)}(\nu) \in \text{Succ}_{T_0}(q^{mc}) \) and \( q \) is a condition. The union of the \( B_\nu \)'s is all of \( \text{supp}(q) \setminus \{mc(q)\} \). Now for each \( \nu \in A \) choose \( C_\nu \in U_a \) such that for \( \gamma \) in \( B^+_\nu = \{ \gamma \in \text{supp}(q) \setminus \{mc(q)\} \mid \kappa(\nu) \geq \kappa(\gamma) \} \), \( \pi_{a,\gamma} \) agrees with \( \pi_{mc(q),\gamma} \pi_{a,mc(q)} \) on \( C_\nu \). Let \( C \) be the “quasi” diagonal intersection \( \Delta_\alpha C_\nu = \{ \nu \mid \nu \in C_\nu \text{ when } \kappa(\nu') < \kappa(\nu) \} \). Then \( C \) also belongs to \( U_a \) and we let \( T \) consist of all sequences in \( T_0 \) all of whose components (beyond the trunk) belong to \( C \). Then \( p = q' \cup \{(\alpha, t, T)\} \) is a condition which (directly) extends \( q \), as if \( \gamma \) belongs to \( \text{supp}(q) \setminus \{mc(q)\} \), \( \nu \in \text{Succ}_T(t) = A \cap C \) and \( \kappa(\nu) \) is greater than \( \kappa(\gamma) \) then \( \nu \) belongs to \( C_{\kappa(\gamma)} \), \( \gamma \) belongs to \( B^+_{\kappa(\gamma)} \) and therefore \( \pi_{a,\gamma} \) agrees with \( \pi_{mc(q),\gamma} \pi_{a,mc(q)} \) at \( \nu \). As \( \alpha \) belongs to the support of \( q \), we are done. \( \square \)

**Lemma 60** If \( q \) belongs to \( P \) and \( \alpha \) belongs to \( [\kappa, \lambda) \) then there is \( p \leq^* q \) with \( \alpha \in \text{supp}(p) \).

**18.-19. Vorlesungen**

**Lemma 61** \( P \) has the \( \kappa^{++} - \alpha \).

**Proof.** Let \( \{p_\alpha \mid \alpha < \kappa^{++}\} \) belong to \( P \). We can assume that the supports of the \( p_\alpha \)'s form a \( \Delta \)-system, the \( p_\alpha \)'s agree on the root of that \( \Delta \)-system and also \( (p^{mc}_\alpha, T^{p_\alpha}) \) is independent of \( \alpha \). This is because the supports have size at most \( \kappa \) and there are only \( \kappa^+ \) possible pairs \( (p^{mc}_\alpha, T^{p_\alpha}) \). We then show that \( p_\alpha, p_\beta \) are compatible for any pair \( \alpha, \beta \). The techniques for doing this are in the proof of the previous lemma: Our first candidate for a common extension of \( p_\alpha \) and \( p_\beta \) is \( p_\alpha \cup p_\beta \). But the support of this may not have a maximal element. So choose \( \delta \) so that \( mc(p_\alpha), mc(p_\beta) < \delta \) and let \( p^*_\alpha \) be formed from \( p_\alpha \) by adding \( \delta \) to the support, as in the proof of the previous lemma. Then \( p^* = p^*_\alpha \cup p_\beta \) is a condition, using the fact that \( \kappa(p^{mc}_\alpha) \) and \( \kappa(p^{mc}_\beta) \) agree. To obtain a condition extending both \( p_\alpha \) and \( p_\beta \) we shrink \( T^{p^*} \) further to ensure that \( \pi_{\delta,mc(p_\alpha)} \) maps \( T^{p^*}_{(p^*)mc} \) into \( T^{p_\beta}_{p^{mc}_\beta} \) and then shrink \( T^{p^*} \) again using the proof of the previous lemma to guarantee that property 6 holds for the resulting condition relative to \( p_\beta \). \( \square \)
Lemma 62 The direct extension relation for $P$ is $\kappa$-closed.

Proof. Suppose that $\langle p_i \mid i < \delta \rangle$ is a $\leq^*_{q}$-decreasing sequence of length $\delta < \kappa$. We assume that $mc(p_i) > \kappa$ for some $i$, as otherwise the result follows from the analogous result for basic Prikry forcing. Choose $\alpha$ so that $mc(p_i) < j \kappa$ for each $i < \delta$. Let $p'$ be the union of the $p_i$'s, with the trees $T_{p_i}$ removed. Let $T^*$ be the tree consisting of all $\langle \nu_0, \ldots, \nu_{n-1} \rangle$ \in $\cap_{i<\delta} \pi^{-1}_{\alpha,mc(p_i)}[T_{p_i}]$ such that $\kappa(\nu_0) > \delta$, let $t$ be $p_{i_{mc}}$ for some $i$ with $mc(p_i) > \kappa$ and let $T'$ be the tree with trunk $t$, followed by the strings in $T^*$. Then $p' \cup \{(\alpha, t, T')\}$ is a condition: (a), (b) and (c) are easily checked; (d) holds as $\kappa(p_i^\kappa) = \kappa(p_{i_{mc}})$ for each $i < \delta$, (e) holds as $\kappa(\nu)$ is greater than $\delta$ for each $\nu \in \text{Succ}_T(t)$ and (f) holds as for some $i$ with $mc(p_i) > \kappa$, $\pi_{\alpha}(t) = \pi_{mc(p_i)}(p_{i_{mc}}) = p_i^\kappa = (p')^\kappa$. Now as in the proof of Lemma 60, we can thin out $T^*$ to $T_i$ for each $i < \delta$ so that $p' \cup \{(\alpha, t, T')\}$ extends $p_i$; finally take $p$ to be $p' \cup \{(\alpha, t, T')\}$, where $T^*$ is the intersection of the $T_i$'s, and we have a direct extension of each $p_i$.

\[\blacksquare\]

20.-21. Vorlesungen

Lemma 63 $P$ satisfies the Prikry property.

Proof. We consider a strengthening of the notion of direct extension. We say that $p$ is a very direct extension of $q$, $p \leq_{**}^{*} q$, iff $p$ is a direct extension of $q$ and $\text{supp}(p) = \text{supp}(q)$. Now if $p$ extends $q$ then we write $p \Downarrow q$ for the condition $r$ obtained as follows:

(i) $\text{supp}(r) = \text{supp}(q)$, $r^\gamma = p^\gamma$ for $\gamma \in \text{supp}(q)$.
(ii) $T^r$ has trunk $p_{mc(q)}^\gamma$ and $T_{r_{mc(q)}}^q = T_{p_{mc(q)}}^q$.

We also write $p \Downarrow q$ for the condition $r$ defined by:

(i) $\text{supp}(r) = \text{supp}(q)$, $r^\gamma = p^\gamma$ for $\gamma \in \text{supp}(q)$.
(ii) $T^r = \pi_{mc(p),mc(q)}[T^p]$.

Note that $p \Downarrow q$ is uniquely determined by $q$ and $p_{mc(q)}$, due to property 4 for the notion of extension, and therefore we also write $p \Downarrow q$ as $q \times p_{mc(q)}$. Also note that $p \leq_{**} (p \Downarrow q) \leq_{**} (p \Downarrow q) \leq q$.

If $p$ extends $q$ we also write $q \Uparrow p$ for the condition $r$ obtained as follows:

30
(i) $	ext{supp}(r) = \text{supp}(p)$.
(ii) $r^\gamma = q^\gamma$ for $\gamma \in \text{supp}(q)$.
(iii) $r^{mc} = \pi^{-1}_{mc(p),mc}(q^\alpha)$.
(iv) $r^\gamma = p^\gamma$ for $\gamma \in \text{supp}(p) \setminus \text{supp}(q)$, $\gamma \neq mc(p)$.
(v) $T^r = \{ \sigma \in \pi^{-1}_{mc(p),mc}(q^\alpha) | p^{mc} \subseteq \sigma \rightarrow \sigma \in T^p \}$.

Note that $q \upharpoonright p$ is a direct extension of $q$ and $p \downarrow (q \upharpoonright p)$ is equal to $p$.

**Sublemma 64** Suppose that $q_0$ is a condition and $D$ is open dense. Then $q_0$ has a direct extension $q$ such that whenever $p$ belongs to $D$ and extends $q$, the condition $p \downarrow q$ has a very direct extension which also belongs to $D$.

**Proof.** We build a sequence $\langle q_i \mid i < \kappa \rangle$ of direct extensions of $q_0$. This sequence will be taken from $M_\lambda$, an elementary submodel of $H(\lambda^+)$ of size $\kappa^+$ closed under $\kappa$-sequences and containing $D$ as an element. Choose $\alpha$ so that $\beta < j < \alpha$ for each $\beta$ in $M \cap [\kappa, \lambda)$ and fix an enumeration $\langle t_i \mid i < \kappa \rangle \in M$ of all $\kappa$-increasing sequences.

If $q_i$, $t_i$ are defined, choose some $p$ in $D \cap M$ extending $q_i$ with $p^{mc} = \pi^{-1}_{mc(p),mc(q_i)}(t_i)$, if possible, and set $q_i+1$ to be $q_i \upharpoonright p$. Then $p$ equals $p \downarrow (q_i \upharpoonright p) = (q_i \upharpoonright p)^{mc} = q_i+1 \ast \pi^{-1}_{mc(q_i+1),mc(q_i)}(t_i)$ and therefore the condition $q_i+1 \ast \pi^{-1}_{mc(q_i+1),mc(q_i)}(t_i)$ belongs to $D$.

For limit $\lambda < \kappa$, we define $q_\lambda = r$ as follows:

(i) $\text{supp}(r) = \bigcup_{i < \lambda} \text{supp}(q_i)$ together with the least $\alpha_\lambda$ such that $\beta < j < \alpha_\lambda$ for each $\beta \in \bigcup_{i < \lambda} \text{supp}(q_i)$.
(ii) $r^\gamma = q_i^\gamma$ for $\gamma \in \text{supp}(q_i)$.
(iii) $r^{\alpha_\lambda} = \pi^{-1}_{\alpha_\lambda,mc}(q_0^\alpha)$.
(iv) For $\eta \in T^r$ extending $r^{mc} = r^{\alpha_\lambda}$, $\text{Succ}_{T^r}(\eta)$ is the intersection of the $[\pi^{-1}_{\alpha_\lambda,mc(q_i)}(\text{Succ}_{T^{\eta_i}}(\pi_{\alpha_\lambda,mc(q_i)}(\eta)))]$ for $i < \lambda$.

Also define $q = q_\kappa$ just as above with $\lambda = \kappa$ and $\alpha_\lambda = \alpha$, except replace (iv) by:

(v) For $\eta \in T^r$ extending $r^{mc} = r^{\alpha}$, $\text{Succ}_{T^r}(\eta)$ is the quasi diagonal intersection $[\pi^{-1}_{1,mc(q_i)}(\text{Succ}_{T^{\eta_i}}(\pi_{\alpha,mc(q_i)}(\eta)))]$.

Now suppose that $p$ is in $D$ and extends $q$. For each $i < \kappa$ let $p_i$ be the condition $r$ defined by:

31
(i) \( \text{supp}(r) = \text{supp}(p), \ r^\gamma = p^\gamma \) for \( \gamma \in \text{supp}(p) \).
(ii) \( T^r = \pi_{mc(p),mc(q)}^{-1}(T^p) \cap T^p \).

Then \( p_i \) is a very direct extension of \( p \) which extends \( q_i \). Choose \( i < \kappa \) so that \( t_i \) equals \( \pi_{mc(p_i),mc(q_i)}(p^{mc}) = p^\alpha \). Then \( p_i \) is an extension of \( q_i \) in \( D \) such that 
\[ \pi_{mc(p_i),mc(q_i)}^{-1}(\pi_{mc(p_i),mc(q_i)}(t_i)) = \pi_{mc(p_i),mc(q_i)}^{-1}(\pi_{mc(p),mc(q)}(p^\alpha)) \text{ as } p_i \text{ extends } q_i, \text{ this is } \pi_{mc(p_i),mc(q_i)}^{-1}(p_i^{mc(q_i)}). \]

and as \( p_i \) extends \( q_i \), this is \( p_i^{mc} \). So \( q_{i+1} \) was chosen so that 
\[ q_{i+1} * \pi_{mc(q_{i+1}),mc(q_i)}^{-1}(\pi_{mc(q_{i+1}),mc(q_i)}(t_i)) = q_{i+1} * \pi_{mc(q_{i+1}),mc(q_i)}^{-1}(\pi_{mc(q_i),mc(q_i)}(p^\alpha)) \] 
belongs to \( D \). But as \( p \) extends \( q \), this is \( q_{i+1} * \pi_{mc(q_{i+1}),mc(q_i)}^{-1}(p^{mc(q_i)}) \) and as \( p_{i+1} \) extends \( q_{i+1} \), this is \( q_{i+1} * p_{i+1}^{mc(q_{i+1})} = q_{i+1} * p_i^{mc(q_{i+1})} \). As \( p \) extends \( q \), this equals \( q_{i+1} * \pi_{mc(q_{i+1}),mc(q_i)}^{-1}(p^\alpha) \), a condition extended by \( p_{i+1} \downarrow q \). Thus \( p_{i+1} \downarrow q \) is a very direct extension of \( p \downarrow q \) which belongs to \( D \). \( \square \) (Sublemma)

Now suppose that \( \varphi \) is a sentence and \( p \) is a condition. We wish to show that \( p \) has a direct extension deciding \( \varphi \). By the sublemma, we may assume that if \( r \) is an extension of \( p \) which decides \( \varphi \), then so does some very direct extension of \( r \downarrow p = p \uparrow mc \). We claim now that some very direct extension of \( p \) decides \( \varphi \). Suppose not; we say that \( p \) is indescribable.

We claim that for \( U_{mc(p)} \)-measure one \( \nu > \max(p^{mc}) \) in \( \text{Succ}_{T^p}(p^{mc}) \), the condition \( p(\nu) = p * (p^{mc} \uparrow \nu) \) is also indescribable. For, if \( p(\nu) \) were decisive for \( U_{mc(p)} \)-measure one \( \nu \), then by thinning \( T^p \) we obtain a very direct extension of \( p \) which decides \( \varphi \), contradiction. Similarly, if \( \nu_0 \) belongs to \( \text{Succ}_{T^p}(p^{mc}) \) and \( p(\nu_0) \) is indecisive, then for \( U_{mc(p)} \)-measure one \( \nu_1 \) in \( \text{Succ}_{T^p}(p^{mc} \uparrow \nu_0) \), the condition \( p(\nu_0, \nu_1) = p * (p^{mc} \uparrow \nu_0 \uparrow \nu_1) \) is indecisive. Continuing in this way we can form a very direct extension \( q \) of \( p \) such that for each \( \sigma \in T^q \) \( p^{mc} \), the condition \( p(\sigma) \) is indecisive.

Now choose \( r \downarrow q \) which decides \( \varphi \). By choice of \( p_i \), a very direct extension of \( r \downarrow p \) also decides \( \varphi \). But \( r \downarrow p \) is a very direct extension of a condition of the form \( p(\sigma) \) where \( \sigma \) belongs to \( T^q_{p^{mc}} \); this contradicts the choice of \( q \). \( \square \)

**Lemma 65** \( P \) preserves \( \kappa^+ \).

**Proof.** As \( P \) forces \( \kappa \) to be singular, \( \kappa^+ \) is either preserved or given a cofinality less than \( \kappa \). Thus it suffices to show that if \( q_0 \) forces \( f \) to be a function from some \( \alpha < \kappa \) into \( \kappa^+ \) then some extension \( q \) of \( q_0 \) forces a bound on the range of \( f \). Now using Sublemma 65 form a \( \leq^* \)-descending sequence \( \langle q_i \mid i \leq \alpha \rangle \) so
that for each $i < \alpha$, $q_{i+1}$ has the property that if $p$ extends $q_{i+1}$ and decides $\hat{f}$ at $i$, then so does a very direct extension of $p \downarrow q_{i+1}$. Then $q = q_\alpha$ forces a bound on the range of $\hat{f}$, as if $p \leq q$ decides $\hat{f}$ at $i < \alpha$, so does a very direct extension of $p \downarrow q \leq p \downarrow q_{i+1}$, and there are only $\kappa$-many conditions of the form $p \downarrow q$. □

**Lemma 66** For each $\alpha$ in $[\kappa, \lambda)$, $G^\alpha = \bigcup\{p^\alpha \mid p \in G\}$ is a Prikry sequence for $U_\alpha$ and if $\alpha < \beta$ belong to $[\kappa, \lambda)$ then $G^\beta$ eventually strictly dominates $G^\alpha$.

**Proof.** The first conclusion follows easily from the definition of the forcing $P$. Suppose that $\alpha < \beta$ belong to $[\kappa, \lambda)$. Choose $\gamma < \lambda$ so that $\alpha, \beta < \gamma$. Then $\{\nu \mid \pi^\gamma_\beta(\nu) > \pi^\gamma_\alpha(\nu)\}$ belongs to the ultrafilter $U_\gamma$. It now follows easily from the definition of the forcing $P$ that $G^\beta$ eventually strictly dominates $G^\alpha$. □

Therefore by forcing with $P$ we obtain a model where $\kappa$ has cofinality $\omega$, GCH holds below $\kappa$ and $2^\kappa = \lambda$.  

33