Introduction

In ZF, the axiom of choice is equivalent to the assertion that for every infinite cardinal \( \kappa \) there is a wellorder of the power set of \( \kappa \). This is equivalent to saying that \( H(\kappa^+) \), the set of sets whose transitive closure has size at most \( \kappa \), can be wellordered for every infinite cardinal \( \kappa \).

In this course we explore the possibilities for definable wellorders in various set-theoretic contexts. For an infinite cardinal \( \kappa \) we say that \( H(\kappa^+) \) has a \( \Sigma_n \) definable wellorder iff there is a wellorder of \( H(\kappa^+) \) which is \( \Sigma_n \) definable over \((H(\kappa^+), \in)\) with parameter \( \kappa \). It has a \( \Sigma_n \) definable wellorder with parameters if arbitrary parameters from \( H(\kappa^+) \) are allowed.

In Gödel’s universe \( L \), the situation is ideal:

**Theorem 1** Assume \( V = L \). Then for each infinite cardinal \( \kappa \), there is a \( \Sigma_1 \) definable wellorder of \( H(\kappa^+) \).

**Proof.** For \( x, y \) in \( H(\kappa^+) \) define:

\[
x < y \iff \text{There exists a transitive model } M \text{ of } \text{ZFC}^- + V = L \text{ of size at most } \kappa \text{ such that } x, y \text{ belongs to } M \text{ and in } M, x <_L y
\]

This wellorder is \( \Sigma_1 \) over \( H(\kappa^+) \) and in fact uses no parameter. \( \square \)

Now what happens if we consider definable wellorders in the context of large cardinals? First consider the case \( \kappa = \omega \) and make the following observation:

**Proposition 2** \( H(\omega_1) \) has a \( \Sigma_n \) definable wellorder (with/without parameters) iff there is a \( \Sigma_{n+1} \) wellorder of the reals (with/without parameters).

**Proof.** Consider the case with no parameters and \( n = 1 \). (The general case \( n \geq 1 \) (with or without parameters) follows easily from this special case.) If
< is a wellorder of $H(\omega_1)$ defined by the $\Sigma_1$ formula $\varphi(x, y)$ then obtain a $\Sigma_{n+1}$ definable wellorder of the reals as follows:

$$R <^* S \text{ iff }$$

There exists a real $T$ which codes a countable transitive set $M$ such that $R, S$ belong to $M$ and in $M$, $\varphi(R, S)$

This is $\Sigma_2^1$ as to say that $T$ codes a countable transitive set is a $\Pi_1^1$ property.

Conversely, if $<$ is a wellorder of the reals defined by the $\Sigma_2^1$ formula $\varphi(R, S)$ then obtain a $\Sigma_1$ definable wellorder of $H(\omega_1)$ as follows:

$$x <^* y \text{ iff }$$

There exists a countable transitive model $M$ of $\text{ZFC}^-$ such that $x, y$ belong to $M$ and in $M$, for some reals $R, S$: $R$ codes $x$, $S$ codes $y$ and $\varphi(R, S)$

This works as for any transitive model $M$ of $\text{ZFC}^-$, if $\varphi(R, S)$ holds in $M$ for reals $R, S$ in $M$, then in fact $\varphi(R, S)$ holds in $V$. □

Now we have:

**Theorem 3** (Mansfield) If there is a $\Sigma_2^1$ wellorder of the reals then every real is constructible.

**Theorem 4** (Martin-Steel) (a) The existence of a $\Sigma_{n+2}^1$ wellorder of the reals is consistent with the existence of $n$ Woodin cardinals. (b) The existence of a $\Sigma_{n+2}^1$ wellorder of the reals with parameters is inconsistent with the existence of $n$ Woodin cardinals and a measurable cardinal above them.

Now suppose that $\kappa = \omega_1$ and therefore we are considering definable wellorders of $H(\omega_2)$. We say that a forcing is small if it has size less than the least inaccessible cardinal. Note that a small forcing preserves large cardinal properties.

**Theorem 5** (F-Asperó) There is a small forcing which forces CH together with a definable wellorder of $H(\omega_2)$. In particular it is consistent with arbitrary large cardinals and CH that there is a definable wellorder of $H(\omega_2)$.

It is not known if “definable” can be taken to be “$\Sigma_2$ definable” in the previous theorem. However $\Sigma_1$ definability is in general not possible:
Theorem 6 (Woodin) Assume that there is a measurable Woodin cardinal and CH holds. Then there is no $\Sigma_1$ definable wellorder of $H(\omega_2)$; in fact there is no wellorder of the reals which is $\Sigma_1$ definable over $H(\omega_2)$.

Woodin’s result is optimal in the following sense:

Theorem 7 (Avraham-Shelah) There is a small forcing which forces a wellorder of the reals which is $\Sigma_1$ definable over $H(\omega_2)$. Necessarily, CH fails in the forcing extension.

Theorem 16 extends to all regular uncountable $\kappa$:

Theorem 8 (F-Asperó) There is a class forcing which forces GCH, adds a definable wellorder of $H(\kappa^+)$ for all regular uncountable $\kappa$ and preserves all supercompact cardinals as well as a proper class of $n$-huge cardinals for each $n$.

It is not known if “definable” can be taken to be “$\Sigma_1$ definable” in the previous theorem, provided one restricts to regular $\kappa$ greater than $\omega_1$.

For singular $\kappa$ there is a limitation in the presence of very large cardinals.

Proposition 9 Suppose that there is a nontrivial elementary embedding from $L(H(\lambda^+)) \rightarrow L(H(\lambda^+))$ (fixing $\lambda$, with critical point less than $\lambda$). Then there is no definable wellorder of $H(\lambda^+)$ with parameters.

The cardinal $\lambda$ in this proposition has cofinality $\omega$.

Next we consider definable wellorders in the context of forcing axioms. First suppose that $\kappa$ equals $\omega$.

Theorem 10 (a) (Harrington, F) Martin’s axiom is consistent with the existence of a $\Sigma_3^1$ wellorder of the reals. (b) (Caicedo-F) Relative to a reflecting cardinal, BSPFA is consistent with the existence of a $\Sigma_3^1$ wellorder of the reals.

It is not known if BMM is consistent with a projective wellorder of the reals (i.e., a wellorder of the reals which is $\Sigma_n^1$ with parameters for some $n$). Unlike BPFA, the full PFA implies that there is no such wellorder as it implies PD.

For $\kappa = \omega_1$ a surprising thing happens:
**Theorem 11** (Moore) BPFA implies that there is a definable wellorder of $H(\omega_2)$ with parameters.

Concerning wellorders without parameters:

**Theorem 12** (Caicedo-F) Relative to a reflecting cardinal there is a model of BSPFA with a $\Sigma_1$ definable wellorder of $H(\omega_2)$.

**Theorem 13** (Larson) Relative to enough supercompacts, there is a model of MM with a definable wellorder of $H(\omega_2)$.

Forcing axioms have no effect on definable wellorders when $\kappa$ is greater than $\omega_1$.

One can consider definable wellorders in many other contexts. Below is a sample of open questions.

1. Is it consistent that for all infinite regular $\kappa$, GCH fails at $\kappa$ and there is a definable wellorder of $H(\kappa^+)$?
2. Is the tree property at $\omega_2$ consistent with a projective wellorder of the reals?
3. Is it consistent that the nonstationary ideal on $\omega_1$ is saturated and there is a $\Sigma^1_4$ wellorder of the reals?
4. Is it consistent that GCH fails at a measurable cardinal $\kappa$ and there is a definable wellorder of $H(\kappa^+)$?

Now we start to prove some of the results listed earlier.

**Theorem 14** (Mansfield) If there is a $\Sigma^1_2$ wellorder of the reals then every real is constructible.

*Proof.* Assume that there is a nonconstructible real and let $< \in \Sigma^1_2$ wellorder of the reals, which we take to be Cantor space, the set of all paths through the binary branching tree $2^{<\omega}$. For any perfect subtree $T$ of $2^{<\omega}$, let $[T]$ denote the set of infinite paths through $T$, a perfect closed subset of Cantor space. For any order-preserving $f : T \to 2^{<\omega}$ we let $f^*$ denote the induced continuous function from $[T]$ to Cantor space.
Lemma 15 Suppose that $T$ is constructible, $f : T \to 2^{<\omega}$ is constructible and $f^*$ is injective. Then there is a constructible perfect $U \subseteq T$ and constructible, order-preserving $g : U \to 2^{<\omega}$ such that $g^*$ is injective and $g^*(x) < f^*(x)$ for all $x \in [U]$.

Proof of Lemma. As $T$ is a perfect tree, there is a constructible $h : T \to 2^{<\omega}$ such that $h^*$ is a bijection from $[T]$ onto Cantor space. For $s \in 2^{<\omega}$ let $\bar{s}$ be the "flip" of $s$, i.e., if $s = (s(0), s(1), \ldots, s(k))$ then $\bar{s} = (1 - s(0), 1 - s(1), \ldots, 1 - s(k))$. For $x$ in Cantor space, $\bar{x}$ is defined similarly.

Let $A$ be the set of $x \in [T]$ such that $f^*(x) > h^*(x)$ and $B$ the set of $x \in [T]$ such that $f^*(x) > h^*(x)$. We claim that either $A$ or $B$ contains a nonconstructible element: Let $z$ be the $\prec$-least nonconstructible real and choose $x, y \in [T]$ so that $h^*(x) = z$, $h^*(y) = \bar{z}$. As $x, y$ are nonconstructible and $f^*$ is an injective, constructible function, it follows that $f^*(x), f^*(y)$ are nonconstructible and therefore $\geq z$. As $f^*(x), f^*(y)$ are distinct, either $f^*(x) > z$ or $f^*(y) > z$. But then either $f^*(x) > z = h^*(x)$ or $f^*(y) > z = h^*(y)$, as desired.

Without loss of generality, assume that $A$ has a nonconstructible element. Then $A$ is $\Sigma^1_2$ with constructible parameters and therefore has a "constructible" perfect subset, i.e., $[U] \subseteq A$ for some constructible perfect tree $U$. If we let $g$ be $h \upharpoonright U$ then we have satisfied the conclusion of the Lemma. □

(Lemma)

Now given the Lemma we easily reach a contradiction: Let $T_0$ be $2^{<\omega}$ and $f_0 : T_0 \to T_0$ the identity. Successively applying the Lemma we get $T_0 \supseteq T_1 \supseteq \cdots$ and $f_0 \supseteq f_1 \supseteq \cdots$ such that $f^*_n(x) > f^*_{n+1}(x)$ for all $x \in T_{n+1}$. Since the $[T_n]$'s are compact sets, they have a nonempty intersection and if $x$ belongs to this intersection we get $f^*_0(x) > f^*_1(x) > \cdots$, contradicting the hypothesis that $\prec$ is a wellorder. □

3. Vorlesung

We say that a forcing is small if it has size less than the least inaccessible cardinal. Note that a small forcing preserves large cardinal properties.

Theorem 16 (F-Asperó) There is a small forcing which forces CH together with a definable wellorder of $H(\omega_2)$. In particular it is consistent with arbitrary large cardinals and CH that there is a definable wellorder of $H(\omega_2)$.  

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I'll begin with the following easier result.

**Theorem 17** There is a small forcing which forces CH together with a $\Sigma_1$ wellorder of $H(\omega_2)$ with parameters.

*Proof.* First force CH by adding an $\omega_1$-Cohen set. Next add an $\omega_2$-Cohen set $A$. In the resulting model, $H(\omega_2)$ is $L_{\omega_2}[A]$ and CH holds. For technical reasons, we assume that $A \cap \omega_1$ is empty.

The final step is to add $B, C \subseteq \omega_1$ which code $A$ in the sense that $A$ is $\Delta_1$ definable over $L_{\omega_2}[A, B, C]$ (the final $H(\omega_2)$) with $B, C, \omega_1$ as parameters. This gives a $\Sigma_1$ wellorder of $L_{\omega_2}[A, B, C]$ with $B, C, \omega_1$ as parameters: simply take the canonical wellorder with parameters $A, B, C$ and eliminate $A$ in favour of its $\Delta_1$ definition with parameters $B, C, \omega_1$.

The forcing $P$ for adding $B$ is a forcing to code $A$ using "canonical functions". For each uncountable $\beta < \omega_2$ choose a bijection $f_\beta : \omega_1 \to \beta$. The set $B$ codes $A$ in the following way: $\beta$ belongs to $A$ iff $\text{ot}(f_\beta[\gamma])$ belongs to $B$ for a CUB set of $\gamma < \omega_1$, where "ot" stands for "ordertype". Note that if $f_0^\beta, f_1^\beta$ are any two bijections from $\omega_1$ onto $\beta$ then the set of $\gamma < \omega_1$ where $\text{ot}(f_0^\beta[\gamma])$ equals $\text{ot}(f_1^\beta[\gamma])$ contains a CUB set. Thus this coding is independent of the choice of the functions $f_\beta$, $\omega_1 \leq \beta < \omega_2$.

A condition in $P$ is a triple $(p, p^*, p^{**})$ where:

- $p$ is an $\omega_1$-Cohen condition, i.e., a function from a countable ordinal $|p|$ to 2.
- $p^*$ is a countable subset of $\omega_2$.
- $p^{**}$ is a closed, bounded subset of $\omega_1$.

For $\beta$ in $p^*$ and $\gamma$ in $p^{**}$, $\text{ot}(f_\beta[\gamma])$ is at least $\gamma$ and less than $|p|$.

We say that $(q, q^*, q^{**})$ extends $(p, p^*, p^{**})$ iff:

- $q$ end-extends $p$, $q^*$ contains $p^*$, $q^{**}$ end-extends $p^{**}$.
- All elements of $q^{**} \setminus p^{**}$ are at least $|p|$.
- For $\gamma$ in $q^{**} \setminus p^{**}$ and $\beta$ in $p^*$, $q(\text{ot}(f_\beta[\gamma]))$ equals $A(\beta)$.

**Lemma 18** (a) For any $(p, p^*, p^{**})$, $\alpha \in [\omega_1, \omega_2)$ and $\delta < \omega_1$ there is an extension $(q, q^*, q^{**})$ of $(p, p^*, p^{**})$ such that $\alpha$ belongs to $q^*$ and $\text{max}(q^{**})$ is greater than $\delta$.

(b) $P$ is $\omega_2$-cc.

(c) $P$ is $\omega$-distributive.
Proof. (a) Choose $\gamma$ greater than $|p|$, $\delta$ so that for distinct $\beta_0, \beta_1$ in $p^* \cup \{\alpha\}$, $\ot(f_{\beta_0}[\gamma])$, $\ot(f_{\beta_1}[\gamma])$ are distinct. This is possible as the set of such $\gamma$ contains a CUB set. Now set $q^* = p^* \cup \{\alpha\}$, extend $p$ to $q$ so that $q(\ot(f_{\beta}[\gamma]))$ equals $A(\beta)$ for $\beta$ in $p^* \cup \{\alpha\}$ and set $q^{**} = p^{**} \cup \{\gamma\}$.

(b) Note that if $p = q$ and $p^{**} = q^{**}$ then $(p, p^*, p^{**})$ and $(q, q^*, q^{**})$ are compatible, as they are both extended by $(p, p^* \cup q^*, p^{**})$. Therefore CH gives us the $\omega_2$-cc.

4.-5. Vorlesungen

We finish the proof of:

Theorem 19 There is a small forcing which forces CH together with a $\Sigma_1$ wellorder of $H(\omega_2)$ with parameters.

Lemma 20 (a) For any $(p, p^*, p^{**})$, $\alpha \in [\omega_1, \omega_2)$ and $\delta < \omega_1$ there is an extension $(q, q^*, q^{**})$ of $(p, p^*, p^{**})$ such that $\alpha$ belongs to $q^*$ and $\max(q^{**})$ is greater than $\delta$.

(b) $P$ is $\omega_2$-cc.

(c) $P$ is $\omega$-distributive.

Proof of (c). Suppose that $(p_0, p_0^*, p_0^{**}) \geq (p_1, p_1^*, p_1^{**}) \cdots$ is a descending $\omega$-sequence of conditions. To obtain a lower bound $(q, q^*, q^{**})$ we start by taking $q$ to be the union of the $p_n$'s, $q^*$ to be the union of the $p_n^*$'s and $q^{**}$ to be the union of the $p_n^{**}$'s together with the supremum $\gamma$ of the max $p_n^{**}$'s. Then $q$ must be lengthened so that for $\beta$ in $q^*$, $q(\ot(f_{\beta}[\gamma]))$ is defined and equal to $A(\beta)$. The problem with this lengthening is that $\ot(f_{\beta}[\gamma])$ may be the same for two distinct $\beta$'s in $q^*$ at which $A$ differs. To solve this problem, it suffices to know that for each $n$:

(*) $\max(p_{n+1}^*)$ belongs to a CUB set of $\delta$'s on which $\ot(f_{\beta}[\delta])$ is distinct for distinct $\beta$ in $p_n^*$.

Then $\ot(f_{\beta}[\gamma])$ will be distinct for any two distinct $\beta$ in $q^*$, enabling us to lengthen $q$ as desired.

Finally, note that if $D_0, D_1, \ldots$ are open dense sets then we can build an $\omega$-sequence $(p_0, p_0^*, p_0^{**}) \geq (p_1, p_1^*, p_1^{**}) \cdots$ below any given condition so that $(p_{n+1}, p_{n+1}^*, p_{n+1}^{**})$ belongs to $D_n$ and obeys (*). $\Box$
Suppose that $G$ is $P$-generic and let $B$ be the union of the $p$ for $(p, p^*, p^{**})$ in $G$, $C$ the union of the $p^{**}$ for $(p, p^*, p^{**})$ in $G$. Then for any $\beta \in [\omega_1, \omega_2]$ we have:

(**) $\beta$ belongs (does not belong) to $A$ iff $\operatorname{ot}(f_\beta[\gamma])$ belongs (does not belong) to $B$ for sufficiently large $\gamma$ in $C$.

In fact we can write:

(***) $\beta$ belongs (does not belong) to $A$ iff for some bijection $f : \omega_1 \to \beta$, $\operatorname{ot}(f[\gamma])$ belongs (does not belong) to $B$ for sufficiently large $\gamma$ in $C$.

This is because if $\beta$ does not belong to $A$, (***) implies that $\operatorname{ot}(f_\beta[\gamma])$ does not belong to $B$ for sufficiently large $\gamma$ in $C$ and as $\operatorname{ot}(f_\beta[\gamma])$ equals $\operatorname{ot}(f[\gamma])$ for unboundedly many $\gamma$ in $C$, it follows that $\operatorname{ot}(f[\gamma])$ does not belong to $B$ for unboundedly many $\gamma$ in $C$.

This shows that in $V[G]$, the predicate $A$ is $\Delta_1$ over $H(\omega_2)$ in parameters $B, C$ and $\omega_1$. As there is a wellorder of $H(\omega_2) = L_{\omega_2}[A, B, C]$ which is $\Sigma_1$ with parameters $A, B, C$ it follows that there is one which is $\Sigma_1$ with parameters $A, B, \omega_1$. □ (Theorem 27)

**Theorem 21** There is a small forcing which forces $\text{CH}$ together with a definable wellorder of $H(\omega_2)$.

We first prove something easier (although certainly not easy!):

**Theorem 22** Suppose that $A$ is a subset of $\omega_1$. Then there is a small forcing which forces $\text{CH}$, preserves $\omega_1$ and forces $A$ to be definable over $H(\omega_2)$.

The proof uses a “weak club-guessing” property (due to Aspéró, inspired by work of Avraham-Shelah). As we will need these properties later when studying $H(\kappa^+)$ for arbitrary regular uncountable $\kappa$, we present the relevant definitions in a general setting.

A **club-sequence** with length $\lambda$ and domain $D$ is a sequence $\vec{C} = \langle C_\delta | \delta < \lambda \rangle$, where $\lambda$ is an ordinal, such that each $C_\delta$ is a subset of $\delta$ for each $\delta$ and $D$ consists of those $\delta$ such that $C_\delta$ is a club in $\delta$. We write $D$ as $\operatorname{dom}(\vec{C})$. The **range** of $\vec{C}$ is the union of the $C_\delta$, $\delta \in \operatorname{dom}(\vec{C})$. 

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$\vec{C}$ is a coherent club sequence iff there is a club-sequence $\vec{D}$ with $\text{dom}(\vec{D}) \supseteq \text{dom}(\vec{C})$ such that $\vec{D}, \vec{C}$ agree on $\text{dom}(\vec{C})$ and whenever $\delta$ belongs to $\text{dom}(\vec{D})$ and $\gamma$ is a limit point of $D_\delta$, $\gamma$ also belongs to $\text{dom}(\vec{D})$ and $D_\gamma = D_\delta \cap \gamma$. In this case we say that $\vec{D}$ witnesses the coherence of $\vec{C}$.

Suppose that $\vec{C}$ is a club sequence and there exists a fixed $\tau$ such that $\text{ot}(C_\delta) = \tau$ for each $\delta$ in $\text{dom}(\vec{C})$; then we say that $\tau$ is the height of $\vec{C}$.

Suppose that $\lambda$ has uncountable cofinality and $\vec{C}$ is a club sequence of length $\lambda$. We say that $\vec{C}$ is guessing iff for every club $C$ in $\lambda$ there is some $\delta$ in $C \cap \text{dom}(\vec{C})$ such that $C_\delta$ is almost contained in $C$, i.e., $C_\delta \setminus C$ is bounded in $\delta$. We say that $\vec{C}$ is strongly guessing iff for every club $C$ in $\lambda$ there is a club $D$ in $\lambda$ such that $C_\delta$ is almost contained in $C$ for all $\delta$ in $D \cap \text{dom}(\vec{C})$. If $\text{dom}(\vec{C})$ is stationary and $\vec{C}$ is strongly guessing then it is also guessing.

Now we weaken the concepts of guessing and strongly guessing. If $X, Y$ are sets of ordinals then we define $X \cap^* Y$ to consist of all $\delta$ in $X \cap Y$ such that $\delta$ is not a limit point of $X$. (This operation is not symmetric.) Then we say that $\vec{C}$ is type-guessing iff for every club $C$ in $\lambda$ there is $\delta \in C \cap \text{dom}(\vec{C})$ such that $\text{ot}(C_\delta \cap^* C) = \text{ot}(C_\delta)$. And $\vec{C}$ is strongly type-guessing iff for every club $C$ in $\lambda$ there is a club $D$ in $\lambda$ such that $\text{ot}(C_\delta \cap^* C) = \text{ot}(C_\delta)$ for every $\delta \in D \cap \text{dom}(\vec{C})$.

An ordinal $\tau$ is perfect iff $\omega^\tau = \tau$.

**Definition 23** For $\kappa$ uncountable and regular, $I_\kappa$ denotes the set of perfect ordinals $\tau < \kappa$ of countable cofinality for which there is a coherent strongly type-guessing club sequence $\vec{C}$ of length $\kappa$ with stationary domain and of height $\tau$.

To prove Theorem 28 we use:

**Lemma 24 (Main Claim)** Assume GCH at $\aleph_0, \aleph_1$ and suppose that $B \subseteq \omega_1$ is a set of perfect ordinals. Then there is an $\omega$-strategically closed, $\aleph_2$-closed forcing $P$ which forces that $I_{\omega_1}$ equals $B$.

The lemma implies that any subset of $\omega_1$ can be made $\Sigma_2$ definable over $H(\omega_2)$ by a small forcing, a strong version of Theorem 28.
Lemma 25 (Main Claim) Assume GCH at $\aleph_0$, $\aleph_1$ and suppose that $B \subseteq \omega_1$ is an unbounded set of perfect ordinals. Then there is an $\omega$-strategically closed, $\aleph_2$-cc forcing $P$ which forces that $I_{\omega_1}$ equals $B$.

The lemma implies that any subset of $\omega_1$ can be made $\Sigma_2$ definable over $H(\omega_2)$ by a small forcing.

To prove the Main Claim we begin with the following lemma.

Lemma 26 Under the assumptions of the Main Claim, write $B$ in increasing order as $(\tau_\nu)_\nu<\omega_1$. Then there is an $\omega$-closed forcing $P^*$ of size $\omega_1$ which forces that there are sequences $(\vec{C}_\nu)_\nu<\omega_1$, $(\vec{D}_\nu)_\nu<\omega_1$, such that $(\text{dom}(\vec{C}_\nu))_\nu<\omega_1$ forms a sequence of pairwise disjoint stationary subsets of $\omega_1$ and for all $\nu < \omega_1$:

(a) $\vec{C}_\nu$ has height $\tau_\nu$.
(b) $\vec{D}_\nu$ witnesses the coherence of $\vec{C}_\nu$.
(c) The range of $\vec{C}_\nu$ is disjoint from the domain of $\vec{C}_{\nu'}$ for all $\nu' < \omega_1$.
(d) Successor elements of $\vec{C}_\delta$ are limit ordinals for each $\delta$ in $\text{dom}(\vec{C}_\nu)$.
(e) $\vec{C}_\nu$ is a guessing club-sequence.

Proof. $P^*$ consists of all pairs

$$p = ((\vec{C}_{p,\nu} | \nu < \lambda_p), (\vec{D}_{p,\nu} | \nu < \lambda_p))$$

(for some ordinal $\lambda_p < \omega_1$) such that for each $\nu < \lambda_p$:

(1) $\vec{C}_{p,\nu}$ and $\vec{D}_{p,\nu}$ are club sequences of length $\lambda_p + 1$.
(2) $\vec{C}_{p,\nu}$ has height $\tau_\nu$.
(3) The range of $\vec{C}_{p,\nu}$ is disjoint from the domain of $\vec{C}_{p,\nu'}$ for each $\nu' < \lambda_p$.
(4) $\vec{D}_{p,\nu}$ witnesses the coherence of $\vec{C}_{p,\nu}$.
(5) Successor elements of $\vec{C}_{p,\delta}$ are limit ordinals for each $\delta$ in $\text{dom}(\vec{C}_{p,\nu})$.

$p_1$ extends $p_0$ iff $\lambda_{p_0} \leq \lambda_{p_1}$ and for each $\nu < \lambda_{p_0}$, $\vec{C}_{p_1,\nu}$ extends $\vec{C}_{p_0,\nu}$ and $\vec{D}_{p_1,\nu}$ extends $\vec{D}_{p_0,\nu}$.

Clearly $P^*$ has size $\omega_1$, as we have assumed CH. To see that $P^*$ is $\omega$-closed, reason as follows. Suppose that $p_0 \geq p_1 \geq \cdots$ is a descending $\omega$-sequence of conditions and we want to show that this sequence has a lower bound. We may assume that this sequence is strictly decreasing, and therefore the supremum
\( \lambda \) of the \( \lambda_{p_n} \)'s does not belong to the domain of any club-sequence mentioned by any of the \( p_n \)'s. But now we can obtain a lower bound \( p \) by choosing the club-sequences \( \vec{C}^{p,\nu} \) and \( \vec{D}^{p,\nu} \), \( \nu < \lambda \), of length \( \lambda + 1 \) to not include \( \lambda \) in their domain.

Let \( G \) be \( P^* \)-generic and for \( \nu < \omega_1 \) let \( \vec{C}^\nu \), \( \vec{D}^\nu \) respectively denote the union of the \( \vec{C}^{p,\nu} \) for \( p \) in \( G \), the union of the \( \vec{D}^{p,\nu} \) for \( p \) in \( G \).

We claim that each \( \vec{C}^\nu \) is a guessing club-sequence in \( V[G] \) for each \( \nu < \omega_1 \). Let \( \mathcal{C} \) be a \( P^* \)-name for a club in \( \omega_1 \) and let \( p \) be a condition in \( P^* \). Let \( (N_i)_{i \leq \tau_\nu} \), be a continuous chain of countable elementary substructures of some large \( (H(\theta), \in, \Delta) \) (where \( \Delta \) is a wellorder of \( H(\theta) \)) such that \( N_0 \) contains \( \nu, \mathcal{C} \) and \( p \) and for each \( i < \tau_\nu \), \( (N_j)_{j \leq \iota} \) belongs to \( N_{i+1} \). For \( i \leq \tau_\nu \) let \( \delta_i \) be \( N_i \cap \omega_1 \) and let \( (\epsilon^i_n)_{n<\omega} \) be the \( \Delta \)-least \( \omega \)-sequence cofinal in \( \delta_i \).

Now choose \( (q_n)_{n<\omega} \) to form a descending sequence of conditions in \( N_0 \) extending \( p \) such that for all \( n \), \( \lambda_{q_n} \) is greater than \( e^0_n \) and \( q_n \) forces some ordinal greater than \( e^0_n \) into \( \mathcal{C} \). Let \( p_0 \) be the lower bound to the \( q_n \)'s obtained by setting \( \lambda_{p_0} = \delta_0 \) and \( \vec{C}^{p_0,\nu'} = \vec{D}^{p_0,\nu'} = \emptyset \) for all \( \nu' < \delta_0 \). Then form \( p_1 \leq p_0 \) in a similar way, with \( N_0, p, (e^i_n)_{n<\omega} \) and \( \delta_0 \) replaced by \( N_1, p_0, (e^1_n)_{n<\omega} \) and \( \delta_1 \), respectively. Continue this for \( \tau_\nu \) steps to build the \( \tau_\nu \)-sequence \( p_0 \geq p_1 \geq \cdots \), choosing lower bounds \( p_i \) at limit stages \( i < \tau_\nu \) to obey the following:

\[
\vec{D}^{p_i,\nu'}_{\delta_i} = \{ \delta_j | j < i \} \\
\vec{C}^{p_i,\nu'}_{\delta_i} = \{ \delta_j | j < \tau_\nu \}.
\]

Then \( q = p_{\tau_\nu} \) is indeed a condition extending \( p \) which forces that \( \vec{C}^{\nu}_{\delta_{\tau_\nu}} \) is a subset of \( \mathcal{C} \). □

Now to prove the Main Claim we perform an iteration with countable support \( (P_\xi | \xi < \omega_2) \) using names \( (\dot{Q}_\xi | \xi < \omega_2) \). The desired forcing that satisfies the Main Claim is \( P_{\omega_2} \), the direct limit of the \( P_\xi \), \( \xi < \omega_2 \).

If \( \vec{C} \) is a (type-) guessing club sequence of length \( \omega_1 \) and \( C \subseteq \omega_1 \) is a club, then \( P(\vec{C}, C) \) is the natural forcing for adding a club \( D \subseteq \omega_1 \) such that \( \text{ot}(C_\delta \cap^* C) = \text{ot}(C_\delta) \) for \( \delta \) in \( D \cap \text{dom}(\vec{C}) \). A condition in this forcing a closed, bounded subset \( d \) of \( \omega_1 \) such that \( \text{ot}(C_\delta \cap^* C) = \text{ot}(C_\delta) \) for all \( \delta \) in \( d \cap \text{dom}(\vec{C}) \).
At the first stage of our iteration we force with the \( P^* \) of Lemma 26. Let \((\vec{C}^\nu)_{\nu<\omega_1}, (\vec{D}^\nu)_{\nu<\omega_1}\) be the club sequences added by this forcing. Let \( \vec{C} \) denote the amalgamation of the \( \vec{C}^\nu \), i.e., the club sequence with domain \( \bigcup \text{dom}(\vec{C}^\nu) \) whose restriction to each \( \text{dom}(\vec{C}^\nu) \) is \( \vec{C}^\nu \).

At each stage \( \xi > 0 \) of the iteration we pick some \( P_\xi \)-name \( \dot{C}_\xi \) for a club in \( \omega_1 \) and we let \( \dot{Q}_\xi \) be a \( P_\xi \)-name for the forcing \( P(\vec{C}, \dot{C}_\xi) \). As we have assumed CH, each \( P_\xi, \xi < \omega_2 \) has a dense subset of size \( \omega_1 \) and the entire iteration is \( \omega_2 \)-cc. It follows that any club \( C \subseteq \omega_1 \) added by \( P \) has a \( P_\xi \)-name for some \( \xi < \omega_2 \). Moreover as we have assumed \( 2^{\omega_1} = \omega_2 \), we can use a bookkeeping function to choose our names \( \dot{C}_\xi \) so that every club \( C \subseteq \omega_1 \) added by \( P \) is named by some \( \dot{C}_\xi \) and therefore we force with \( P(\vec{C}, C) \) at some stage of the iteration.

**8.-9. Vorlesungen**

The \( \omega_2 \)-iteration \( P \) is \( \omega \)-strategically closed: Recall that the first component of \( P \) is the forcing \( P^* \). Suppose that \( p_0 \geq p_1 \geq \cdots \) is an \( \omega \)-sequence in \( P \) such that for some \( \lambda \), the sup of the lengths of the \( p_n \)'s on each component in the union of the supports of the \( p_n \)'s equals \( \lambda \). Then we can obtain a lower bound \( q \) by taking the first component of \( q \) to have length \( \lambda + 1 \) while assigning the empty set at \( \lambda \) for all of its club-sequences, and including \( \lambda \) into the clubs at all later components of \( q \). The \( \omega \)-strategic closure of \( P \) now follows from the fact that it is easy to form a strategy which generates sequences of \( p_n \)'s as above.

It is also easy to verify that the sets added by the forcings \( P(\vec{C}, C) \) are unbounded and therefore clubs; this is simply because the complement of the domain of \( \vec{C} \) is stationary. It follows that \( P \) forces each \( \vec{C}^\nu \) to be strongly type-guessing, as for each club \( C \subseteq \omega_1 \) in the extension, \( P \) explicitly adds a club \( D \) witnessing strong type-guessing for each \( \vec{C}^\nu \) and \( C \). Of course this is vacuous without knowing that the domain of \( \vec{C}^\nu \) is stationary in the final model. (The positive stages of the iteration are not proper.) An argument as in the proof that \( P^* \) produces club-sequences with stationary domain verifies this last fact, and in fact shows that each \( \vec{C}^\nu \) is a guessing club-sequence.

Our main and final task is now to show that if \( \tau \) is perfect but not one of the desired heights, i.e., does not equal \( \tau_\nu \) for some \( \nu < \omega_1 \), then in the \( P \)-generic extension there is no strongly type-guessing club-sequence of height \( \tau \).
with stationary domain. Let $G$ be $P$-generic and $\vec{E}$ a club-sequence of length $\omega_1$ with stationary domain of perfect height $\tau < \omega_1$. Choose $0 < \xi < \omega_2$ so that $\vec{E}$ belongs to $V[G_0]$ where $G_0 = G \cap P_\xi$. We work in $V[G_0]$. Let $D$ be the club added at stage $\xi$ of the iteration (which witnesses strong type-guessing for the club-sequence $\vec{C}$ with respect to the club $C_\xi$) and let $\vec{D}$ be a $P/G_0\tau$-name for $D$. Our goal is to show that if $\tau$ is not of the form $\tau_\eta$, $\eta < \omega_1$, then any condition $p$ in $P/G_0$ forcing that $\vec{E}$ is a name for a club in $\omega_1$ can be extended to a condition $q$ forcing that for some $\delta$ in $\vec{E} \cap \text{dom}(\vec{E})$, $\ot(E_\delta \cap^* \vec{D})$ is less than $\tau$, the ordertype of $E_\delta$.

Let $\theta$ be large and let $(N_i)_{i < \omega_1}$ be a continuous chain of elementary submodels of $H(\theta)$ such that $N_0$ contains all relevant parameters (such as $p$, $\tau$, $\vec{D}$ and $\vec{E}$). Set $\delta_i = N_i \cap \omega_1$ for each $i < \omega_1$ and let $D_0$ be the club consisting of the $\delta_i$'s. In the final model $V[G]$, the set $\{\delta < \omega_1 \mid \delta \in \text{dom}(\vec{C}) \rightarrow \ot(C_\delta \cap^* D_0) = \ot(C_\delta)\}$ contains a club. As $\text{dom}(\vec{E})$ is stationary in the final model we can choose $i^* = \delta_i < \omega_1$ in $\text{dom}(\vec{E})$ such that $i^* \in \text{dom}(\vec{C}) \rightarrow \ot(C_{i^*} \cap^* D_0) = \ot(C_{i^*})$.

We show that some extension $q$ of $p$ of length $i^*$ (i.e., with all names of clubs assigned by $q$ on the components in its support forced to have length $i^*$) forces that $i^*$ belongs to $\vec{E}$ and that $\ot(E_{i^*} \cap^* \vec{D})$ is less than $\tau$, the ordertype of $E_{i^*}$. There are three cases.

Case 1. $i^*$ does not belong to $\text{dom}(\vec{C})$.

In this case we find an extension $q$ of $p$ which forces $\vec{D}$ to be disjoint from $E_{i^*}$ above $\delta_0$.

As $i^*$ is greater than $\tau$, it follows that we can choose an $\omega$-sequence $i_0 < i_1 < \cdots$ cofinal in $i^*$ such that $E_{i^*} \cap \delta_{i_n}$ is bounded in $\delta_{i_n}$ for each $n$. Now build an $\omega$-sequence $p = p_0 \geq p_1 \geq \cdots$ of conditions such that each $p_{n+1}$ belongs to $N_{i_{n+1}}$, forces some ordinal greater than $\delta_{i_n}$ into $\vec{E}$ and forces that the least ordinal in $\vec{D} \cap [\delta_{i_n}, \delta_{i_{n+1}}]$ is greater than $\max(E_{i^*} \cap \delta_{i_{n+1}})$. Moreover we can assume that all of the components of $p_{n+1}$ in its support are forced to have length at least $\delta_{i_n}$. Then as $i^*$ does not belong to the domain of $\vec{C}$ the sequence of $p_n$'s has a greatest lower bound $q$ which forces that $E_{i^*} \cap \vec{D}$ is bounded in $i^*$; in particular $q$ forces that $\ot(E_{i^*} \cap^* \vec{D})$ is less than $\tau$, as desired.
Case 2. $i^*$ belongs to $\text{dom}(\vec{C})$ and $\tau_0 = \ot(C_{i^*})$ is less than $\tau = \ot(E_{i^*})$.

In this case we find an extension $q$ of $p$ which forces $E_{i^*} \cap \check{D}$ to be included in $C_{i^*}$ above $\delta_0$.

Denote $\ot(C_{i^*})$ by $\tau_0$. The desired $q$ will have length $i^*$ and be obtained as the greatest lower bound of a $\tau_0$-sequence of conditions of shorter length. To guarantee that this lower bound $q$ exists we must ensure that the ordinal $i^*$ can be placed into all of the clubs $\check{D}_\eta$ for $\eta$ in the support of $q$. As $i^*$ now belongs to the domain of $\vec{C}$, this demands that $\ot(C_{i^*} \cap \check{C}_\eta)$ be maximised (i.e., equal to $\tau_0$) for each such $\eta$. In particular, the club $D = \check{D}_\xi$ is of the form $C_\eta$ for some $\eta$ in the support of $q$ and therefore we must ensure that $\ot(C_{i^*} \cap \check{C}_\eta)$ is maximised, while at the same time ensuring that $\ot(E_{i^*} \cap \check{D})$ is less than $\ot(E_{i^*}) = \tau$. In the present case the latter goal can be achieved by simply arranging that $E_{i^*} \cap \check{D}$ be contained in $C_{i^*}$ above $\delta_0$, as $C_{i^*}$ has ordertype $\tau_0$ which by assumption is indeed less than $\tau$.

Let $(\delta_{i, j})_{j < \tau_0}$ increasingly enumerate $D_0 \cap C_{i^*}$. We inductively build the $p_j$, $j < \tau_0$, to meet the following conditions:

1. $p_0$ extends $p$ and $p_j$ belongs to $N_{i, j+1}$ for each $j$.
2. For limit $j$, $p_j$ is the greatest lower bound of $(p_k)_{k < j}$.
3. Each $p_{j+1}$ is the greatest lower bound of an $\omega$-sequence of conditions in $N_{i,j+1}$ and forces that $\delta_{i,j+1}$ belongs to $\check{E}$.
4. For each $\eta$ in the support of $p_j$, $p_{j+1}$ forces that $\delta_{i,j+1}$ belongs to $\check{C}_\eta$ (where $\check{C}_\eta$ is the club considered by the iteration at stage $\eta$).
5. Each $p_{j+1}$ forces that $E_{i^*} \cap \check{D} \cap (\delta_{i,j}, \delta_{i,j+1})$ is empty.

As in Case 1, lower bounds are easily obtained at limit stages $j$ less than $\tau_0$, as $C_{i^*}$ is disjoint from the domain of $\vec{C}$ and therefore $\delta_{i,j}$ does not belong to the domain of $\vec{C}$. Condition 4 implies that the $p_j$’s have a greatest lower bound $q$ at the final stage $\tau_0$, as it implies that for each $\eta$ in the union of the supports of the $p_j$’s, a final segment of $C_{i^*} \cap^*_0 D_0$ is forced inside $\check{C}_\eta$, allowing us to put $i^*$ into $\check{D}_\eta$, the club witnessing strong type-guessing for $\vec{C}$ relative to the club $\check{C}_\eta$. Condition 3 implies that $i^*$ is forced into $\check{E}$. And by condition 5, $q$ forces that $E_{i^*} \cap \check{D}$ above $\delta_0$ is contained in $D_0 \cap C_{i^*}$ and therefore has ordertype at most $\tau_0 < \tau$. 

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The conditions 1, 2 and the first part of 3 are easily arranged; to fulfill the remaining conditions, use the fact that \( \tau = \text{ot}(E_i^*) \) is less than \( \delta_{i,j+1} \) in order to meet the relevant dense sets in \( N_{i,j+1} \) between adjacent elements of \( E_i^* \).

Case 3. \( i^* \) belongs to \( \text{dom}(\vec{C}) \) and \( \tau = \text{ot}(E_i^*) \) is less than \( \tau_0 = \text{ot}(C_i^*) \).

In this case we find an extension \( q \) of \( p \) which forces \( \dot{D} \) to be disjoint from \( E_i^* \) above \( \delta_0 \).

For any \( \gamma \) in \( E_i^* \), let \( \gamma^* \) denote the least element of \( E_i^* \) greater than \( \gamma \). Also let \( (t_k \mid k \in \omega) \) be an increasing sequence cofinal in \( \tau_0 \). As \( \tau \) is less than \( \tau_0 \), for each \( k \) there are unboundedly many \( \gamma_k \) in \( E_i^* \) such that the ordertype of \( C_i^* \cap \gamma_k^* D_0 \) on the interval \( (\gamma_k, \gamma_k^*) \) is greater than \( t_k \). Otherwise \( \tau_0 = \text{ot}(C_i^* \cap \gamma_k^* D_0) \) is bounded by \( t_k \cdot \tau \) for some \( k \), contradicting the assumption that \( \tau_0 \) is a perfect ordinal greater than \( \tau \). Choose an increasing sequence of such \( \gamma_k \)'s, and for each \( k \) let \( D_k^0 \) consist of the first \( t_k + 1 \) elements of \( C_i^* \cap D_0 \) in the interval \( (\gamma_k, \gamma_k^*) \).

Let \( (\delta_{i,j})_{j<\tau_0} \) increasingly enumerate the union of the \( D_k^0 \)'s, a club in \( i^* \). We inductively build the \( p_j \), \( j < \tau_0 \), to meet the following conditions:

1. \( p_0 \) extends \( p \) and \( p_j \) belongs to \( N_{i,j+1} \) for each \( j \).
2. For limit \( j \), \( p_j \) is the greatest lower bound of \( (p_k)_{k<j} \).
3. Each \( p_{j+1} \) is the greatest lower bound of an \( \omega \)-sequence of conditions in \( N_{i,j+1} \) and forces that \( \delta_{i,j+1} \) belongs to \( \dot{E} \).
4. For each \( \eta \) in the support of \( p_j \), \( p_j+1 \) forces that \( \delta_{i,j+1} \) belongs to \( \dot{C}_\eta \) (where \( \dot{C}_\eta \) is the club considered by the iteration at stage \( \eta \)).
5. Each \( p_{j+1} \) forces that \( E_i^* \cap \dot{D} \cap (\delta_{i,j}, \delta_{i,j+1}) \) is empty.

As in Case 2, lower bounds exist at limit stages and \( i^* \) is forced by the final \( q \) into \( \dot{E} \). By condition 5, \( q \) forces that \( E_i^* \) is disjoint from \( \dot{D} \) above the length of \( p_0 \), and therefore has ordertype less than \( \tau \), as desired. Conditions 1-4 are easily arranged; so is condition 5 as each \( D_k^0 \) is a closed set lying entirely in the open interval \( (\gamma_k, \gamma_k^*) \).

This completes the proof that there are no unintended heights of strongly type-guessing club sequences in the \( P \)-generic extension. \( \square \)

10.-11. Vorlesungen
Recall that we have:

**Theorem 27** There is a small forcing which forces CH together with a $\Sigma_1$ wellorder of $H(\omega_2)$ with parameters.

**Theorem 28** Suppose that $A$ is a subset of $\omega_1$. Then there is a small forcing which forces CH, preserves $\omega_1$ and forces $A$ to be definable over $H(\omega_2)$.

We now want to combine these results to get:

**Theorem 29** There is a small forcing which forces CH together with a definable wellorder of $H(\omega_2)$.

Roughly speaking, in Theorem 27 we make a wellorder of $H(\omega_2)$ definable by coding it using “canonical function coding” by a subset of $\omega_1$, and in Theorem 28 we make a subset of $\omega_1$ definable by coding it using “club-guessing” by a subset of $H(\omega_2)$. Now we want to combine these methods to add $B \subseteq \omega_1$ and $G \subseteq H(\omega_2)$ so that:

1. $B$ codes $G$ using canonical function coding.
2. $G$ codes $B$ using club-guessing.

If we first add $B$ and then add $G$ then we have not achieved the desired result, as we will only get a definable wellorder of the $H(\omega_2)$ of the ground model, not of the extension. Note that we can’t do this with a standard $\omega_2$-iteration with the $\omega_2$-cc, as then any subset of $\omega_1$ will have appeared by some initial stage of the iteration, which makes it impossible for it to decode the generic for the entire iteration.

We need to add $B$ and $G$ “simultaneously”. There is feedback: the forcing to add $G$ depends on $B$ and the forcing to add $B$ depends on $G$. A condition in the desired forcing specifies partial information about $B$ as well as partial information about $G$; this information is fully determined and does not depend on the ultimate choice of generic. The resulting generic produces both $B$ and $G$ with the desired feedback: $B$ codes $G$ and $G$ codes $B$. The forcing has features of an iteration as $G$ is added in $\omega_2$ stages, however also has of a product, as conditions are completely determined in the ground model.

We now review the earlier terminology regarding canonical function coding and club guessing that will be needed for the construction.
For uncountable $\gamma < \omega_2$, a canonical function for $\gamma$ is a function $f_\gamma : \omega_1 \to \omega_1$ such that for some surjection $\pi : \omega_1 \to \gamma$, $f_\gamma(\nu) = \text{ot}(\pi[\nu])$ for all $\nu < \omega_1$. Any two canonical functions for $\gamma$ agree on a club.

A club-sequence of length $\lambda$ with domain $D$ is a sequence $\vec{C} = (C_\delta | \delta < \lambda)$ where each $C_\delta$ is a subset of $\delta$, $\lambda \leq \omega_1$ and $D = \text{dom}(\vec{C})$ is the set of limit $\delta < \lambda$ such that $C_\delta$ is a club in $\delta$. The range of $\vec{C}$ is the union of the $C_\delta$, $\delta \in \text{dom}(\vec{C})$. We say that $\vec{C}$ is coherent if there is a club-sequence $\vec{D}$ extending $\vec{C}$ to a possibly larger domain such that $\delta \in \text{dom}(\vec{D})$, $\gamma$ a limit point of $D_\delta$ implies $\gamma \in \text{dom}(\vec{D})$ and $D_\gamma = D_\delta \cap \gamma$. We say that $\vec{D}$ witnesses the coherence of $\vec{C}$.

The height of a club guessing sequence $\vec{C}$, if defined, is the unique $\tau$ such that $\text{ot}(C_\delta) = \tau$ for all $\delta \in \text{dom}(\vec{C})$. An ordinal $\tau$ is perfect iff $\omega^\tau = \tau$. If $X$ is a set of ordinals then we let $X^+$ denote the set of elements of $X$ which are not limit points of $X$. A club sequence $\vec{C}$ of length $\omega_1$ with stationary domain is strongly type guessing iff for every club $C$ in $\omega_1$ there is a club $D$ in $\omega_1$ such that $\text{ot}(C_\delta^+ \cap C) = \text{ot}(C_\delta)$ for every $\delta \in \text{dom}(\vec{C}) \cap D$.

The desired forcing $P$

Assume the GCH at $\aleph_0$ and $\aleph_1$ and fix a bookkeeping function $F$, i.e., a function $F : \omega_2 \to H(\omega_2)$ such that for each $a \in H(\omega_2)$, the set of $\alpha$ such that $F(\alpha) = a$ is unbounded in $\omega_2$.

Choose canonical functions $(f_\gamma | \omega_1 \leq \gamma < \omega_2)$. We assume that $f_\gamma(\delta) \geq \delta$ for all $\gamma$ and all limit $\delta < \omega_1$. Also, for distinct $\gamma_0, \gamma_1$ let $E_{\gamma_0, \gamma_1}$ be a club in $\omega_1$ of limit ordinals on which $f_{\gamma_0}$ and $f_{\gamma_1}$ differ.

Let $A$ be a subset of $\omega_2$ such that $L_{\omega_2}[A] = H(\omega_2)$ and the sequences $(f_\gamma | \gamma < \omega_2)$ and $(E_{\gamma_0, \gamma_1} | \gamma_0, \gamma_1 < \omega_2)$ are definable over $(H(\omega_2), \in, A)$.

Let $(\eta_\xi)_{\xi < \omega_1}$ increasingly enumerate the countable perfect ordinals and let $C$ be the set of nonzero $\alpha \leq \omega_2$ such that $\omega_1 \cdot \alpha' < \alpha$ for all $\alpha' < \alpha$.

We will define an increasing sequence of partial orders $(P_\alpha, \leq_\alpha)$, $\alpha \in C$. The desired forcing $P$ will be $(P_{\omega_2}, \leq_{\omega_2})$. 

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Given \( \alpha \in \mathcal{C} \) and assuming that \( P_{\alpha'} \) has been defined for \( \alpha' < \alpha \) in \( \mathcal{C} \), conditions in \( P_{\alpha} \) are of the form:

\[
p = (b, \mathcal{C}, (c_\gamma \mid \gamma \in a), ((\vec{C}^i, \vec{D}^i) \mid i < \beta), (D_\gamma \mid \gamma \in a))
\]

satisfying the following conditions, where for any ordinal \( \alpha \), \( p \upharpoonright \alpha \) denotes

\[
(b, \mathcal{C}, (c_\gamma \mid \gamma \in a \cap \alpha), ((\vec{C}^i, \vec{D}^i) \mid i < \beta), (D_\gamma \mid \gamma \in a \cap \alpha)):
\]

1. \( a \) is a countable subset of \( \bigcup_{1 \leq \rho < \alpha} [\omega_1 \cdot \rho, \omega_1 \cdot \rho + \beta) \) and \( \gamma' \) belongs to \( a \) whenever \( \gamma' \geq \omega_1 \) and \( \gamma \in a \) is of the form \( \omega_1 \cdot \gamma' + \zeta \) for some countable \( \zeta \).
2. \( \mathcal{C} \) is a club in \( \omega_1 \) contained in \( \bigcap \{ E_{\gamma, \gamma'} : \gamma, \gamma' \in a, \gamma \neq \gamma' \} \).
3. \( \beta \) is a countable ordinal closed under Gödel pairing and \( \beta \) belongs to \( \mathcal{C} \).
4. \( \vec{b} \) is a subset of \( \beta \) of ordertype \( \beta \).
5. For \( \gamma \in a \), \( c_\gamma \) is a closed subset of \( \beta \) and \( f_\gamma (\nu) < \beta \) for \( \nu \) in \( c_\gamma \).
6. Each \( \vec{C}^i \) and \( \vec{D}^i \) (for \( i < \beta \)) is a club-sequence of length \( \beta + 1 \), \( \vec{C}^i \) has a well-defined perfect height and \( \vec{D}^i \) witnesses the coherence of \( \vec{C}^i \).
7. \( \vec{b} \) is the set of \( \xi < \beta \) such that some \( \vec{C}^i, i < \beta \), has height \( \eta_k \). Also, the domain of each \( \vec{D}^i \) is contained in \( [i + 1, \omega_1) \) and for each \( i, j \), \( \text{dom}(\vec{D}^j) \cap \text{dom}(\vec{D}^i) = \text{dom}(\vec{D}^i) \cap \text{range}(\vec{D}^j) = \text{range}(\vec{D}^i) \cap \text{range}(\vec{D}^j) = \emptyset \).
8. For \( \gamma \in a \), \( D_\gamma \) is a closed subset of \( \beta + 1 \).
9. Suppose that \( \gamma \) belongs to \( a \) and there is a least \( \alpha' \) in \( \gamma \cap \mathcal{C} \) such that \( F(\gamma) \) is a \( P_{\alpha'} \)-name for a club in \( \omega_1 \). Then for each \( \nu \) in \( \beta \cap (\text{max}(D_\gamma) + 1) \), \( p \upharpoonright \alpha' \) decides (in the forcing \( P_{\alpha'} \)) whether or not \( \nu \) belongs to \( F(\gamma) \). Let \( C_\gamma \) be the closure of the set of \( \nu \in \beta \cap (\text{max}(D_\gamma) + 1) \) such that \( p \upharpoonright \alpha' \) forces \( \nu \in F(\gamma) \). Then \( \text{ot}((C_\gamma^p)^+ \cap C_\gamma) = \text{height}(\vec{C}^i) \) for each \( i < \beta \) and \( \delta \in D_\gamma \cap \text{dom}(\vec{C}^i) \).

Clause 9 reflects our desire to code using strong type guessing. The canonical function coding is reflected in our notion of extension and makes use of components \( C \) and \( (c_\gamma \mid \gamma \in a) \) above. First, for any condition \( p \) in \( P_{\alpha} \) associate in a canonical way a set \( \mathcal{A}(p) \) contained in \( \bigcup_{1 \leq \rho < \alpha} [\omega_1 \cdot \rho, \omega_1 \cdot \rho + \beta) \) which codes \( A \cap \bigcup_{1 \leq \rho < \alpha} [\omega_1 \cdot \rho, \omega_1 \cdot \rho + \beta) \) on \( \bigcup_{1 \leq \rho < \alpha} [\omega_1 \cdot \rho, \omega_1 \cdot \rho + \beta) \) as well as the components of \( p \) on \( \bigcup_{1 \leq \rho < \alpha} [\omega_1 \cdot \rho, \omega_1 \cdot \rho + \beta) \). Then we say that the condition \( q \) extends \( p \), \( q \leq_\alpha p \), iff the following conditions hold (where if \( q = (b, \mathcal{C}, (c_\gamma \mid \gamma \in a), ((\vec{C}^i, \vec{D}^i) \mid i < \beta), (D_\gamma \mid \gamma \in a)) \) is a condition then \( b^g, C^g, c_\gamma^g, a^g \ldots \) denote \( b, C, c_\gamma, a \ldots \)):

a. \( C^g \subseteq C^p \).
b. \( \beta^p \leq \beta^g, a^p \subseteq a^g \) and \( b^p = b^g \cap \beta^p \).
c. For $\gamma \in a^p$, $c^p_\gamma = c^q_\gamma \cap \beta^p$, $c^q_\gamma \setminus c^p_\gamma \subseteq C^p$ and $D^p_\gamma = D^q_\gamma \cap (\beta^p + 1)$.

d. $\bar{C}^i_{\beta ^p} = \bar{C}^{i,1} \upharpoonright \beta^p + 1$ and $\bar{D}^i_{\beta^p} = \bar{D}^{i,1} \upharpoonright \beta^p + 1$ for all $i < \beta^p$.

e. For $\gamma \in a^p$ and $\nu \in c^1_\gamma \setminus c^p_\gamma$, $f_\gamma(\nu) \in b^p$ iff $\gamma \in A(p)$.

The relation $\leq_\alpha$ is transitive, using the fact that if $q \leq_\alpha p$ and $\gamma$ belongs to $a^p$ then $\gamma$ belongs to $A(p)$ iff $\gamma$ belongs to $A(q)$. (The latter is verified using clause 1 in the definition of condition.)

The $P_\alpha$'s form an increasing sequence of partial orders and $P = P_{\omega_2}$ has size $\omega_2$. The following are straightforward:

Lemma 30 For $\alpha' \leq_\alpha \alpha$ in $C$, $p \upharpoonright \alpha'$ belongs to $P_{\alpha'}$ for each $p$ in $P_\alpha$; furthermore, $P_{\alpha'}$ is a complete suborder of $P_\alpha$.

Lemma 31 $P$ has the $\omega_2$-cc.

Lemma 32 $P$ is $\omega_1$-closed.

If $G$ is $P$-generic then $b^G = \bigcup\{b^p \mid p \in G\}$ codes $G$: Since the canonical function coding is built into the definition of the forcing, we have that $b^G$ codes $A(G) = \bigcup\{A(p) \mid p \in G\}$; from the latter we can define the $\bar{C}^i(G)$, $\bar{D}^i(G)$, $C_\gamma(G)$, $D_\gamma(G)$ (the unions of the corresponding objects associated to $p \in G$), and this is enough to define $G$.

The main lemma states that in $V[G]$, $b^G$ is definable over $(H(\omega_2), \epsilon)$, as the set of $\xi$ such that there is a strong type guessing club-sequence with stationary domain of height $\eta_\xi$. The argument is similar to the one used by Asperó to make any given subset of $\omega_1$ definable over $H(\omega_2)$ in a forcing extension using strong type guessing. \(\square\)

The above gives a $\Sigma_4$ definable wellorder of $H(\omega_2)$ in a small forcing extension. It is not known if this is optimal. However Woodin showed that if there is a measurable Woodin cardinal and CH holds then there is no $\Sigma_4$ definable wellorder of $H(\omega_2)$ with parameter $\omega_1$; in fact there is no wellorder of the reals which is $\Sigma_4$ definable over $H(\omega_2)$ with parameter $\omega_1$.

Definable wellorders of $H(\kappa^+)$, $\kappa$ large

Theorem 29 extends to all regular uncountable $\kappa$: 19
Theorem 33 (F-Asperó) There is a class forcing which forces GCH, adds a definable wellorder of $H(\kappa^+)$ for all regular uncountable $\kappa$ and preserves all supercompact cardinals as well as a proper class of $n$-huge cardinals for each $n$.

For singular $\kappa$ there is a limitation in the presence of very large cardinals.

Proposition 34 Suppose that there is an elementary embedding from $L(H(\lambda^+))$ to itself fixing $\lambda$ with critical point less than $\lambda$. Then there is no definable wellorder of $H(\lambda^+)$ with parameters.

Proof of Proposition. Kunen’s proof that there is no nontrivial elementary embedding $j:V\to V$ goes as follows: Let $\kappa$ be the critical point of $j$ and $\lambda$ the supremum of the $j^n(\kappa)$s for $n \in \omega$. Then $\lambda$ is the first fixed point of $j$ greater than $\kappa$. Let $F$ be an $\omega$-Jonsson function for $\lambda$, i.e., a function $F$ from $[\lambda]^{\omega}$ to $\lambda$ such that whenever $X \subseteq \lambda$ has size $\lambda$ then the range of $F$ on $[X]^{\omega}$ is all of $\lambda$. It is not difficult to construct such a function $F$ using the axiom of choice. Then $j(F)$ has the same property and $j[\lambda] = X$ has size $\lambda$. It follows that $\kappa$ is of the form $j(F)(s)$ for some $s \in [X]^{\omega}$, which is impossible as $s$ belongs to the range of $j$ and $\kappa$ does not.

Now suppose that $j$ were an elementary embedding from $L(H(\lambda^+))$ to itself fixing $\lambda$ with critical point $\kappa$ less than $\lambda$. Then $\lambda$ is at least the supremum $\bar{\lambda}$ of the $j^n(\kappa)$, $n \in \omega$. Kunen’s argument shows that there cannot be an $\omega$-Jonsson function for $\bar{\lambda}$ in $L(H(\lambda^+))$. Thus $\lambda$ must equal $\bar{\lambda}$ and there is no $\omega$-Jonsson function for $\lambda$ in $L(H(\lambda^+))$. In particular, the axiom of choice must fail in $L(H(\lambda^+))$, which implies that there is no definable wellorder of $H(\lambda^+)$. $\square$

It is not known if there is a small forcing that creates a definable wellorder of $H(\aleph_{\omega+1})$.

12.-13. Vorlesungen

Definable wellorders and forcing axioms

We first consider definable wellorders of $H(\omega_1)$, or equivalently, projective wellorders of the reals. As forcing axioms imply the negation of CH, we first show:
Theorem 35  A projective wellorder of the reals is consistent with the negation of CH.

I won’t give the simplest proof of this result, but rather a proof which is amenable to generalisation. I begin with the following easier result:

Theorem 36  It is consistent with the negation of CH that there is a wellorder of the reals definable in $H(\omega_2)$.

Proof. The desired model will be obtained via an $\omega_1$-preserving, $\omega_2$-cc iteration over $L$ of length $\omega_2$ with countable support.

Fix a sequence $(S_\alpha \mid \alpha < \omega_2)$ of pairwise almost disjoint stationary subsets of $\omega_1$. We assume that this sequence is definable over $L_{\omega_2}$. For any pair of reals $x, y$ let $z = x * y$ be defined by $z = \{2n \mid n \in x\} \cup \{2n + 1 \mid n \in y\}$. We will force to kill CH and create a wellorder $<$ of the reals so that:

(*) $x < y$ iff for some limit $\alpha$, $n$ belongs to $x * y$ iff $S_{\alpha+n}$ is not stationary.

For the sake of later applications, we will add reals using Sacks forcing, rather than Cohen forcing. We will need a bookkeeping function, i.e., a function $F : \omega_2 \rightarrow L_{\omega_2}$ (definable over $L_{\omega_2}$) such that for each $a \in L_{\omega_2}$, $F(\alpha) = a$ for unboundedly many $\alpha < \omega_2$.

The iteration uses the names $Q_\alpha$ defined as follows. Let $P_\alpha$ denote the first $\alpha$ stages of the iteration (for $\alpha \leq \omega_2$) and let $G_\alpha$ denote the $P_\alpha$-generic. Order the reals in $L[G_\alpha]$ by: $x <_\alpha y$ iff the $L$-least $P_\alpha$-name for $x$ (i.e., the $L$-least $P_\alpha$-name $\sigma_x$ such that $\sigma_x^{G_\alpha} = x$) is less than the $L$-least $P_\alpha$-name for $y$ in the canonical wellorder of $L$. We assume that this is defined in such a way that if $\alpha < \beta$ are both limits then $<_\alpha$ is an initial segment of $<_\beta$.

For limit $\alpha$, $Q_\alpha$ is trivial unless $F(\alpha)$ is a $P_\alpha$-name for a pair of reals $x <_\alpha y$. In that case, $Q_\alpha$ is the forcing that adds a club to the complement of $S_{\alpha+n}$ for each $n$ in $x * y$. A condition in $Q_\alpha$ is an $\omega$-sequence $(c_0, c_1, \cdots)$ of closed, bounded subsets of $\omega_1$ such that for each $n$ in $x * y$, $c_n$ is disjoint from $S_{\alpha+n}$.

For $\alpha$ equal to 0 or a successor, $Q_\alpha$ is Sacks forcing.

The desired forcing is $P = P_{\omega_2}$.  

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Lemma 37  \( P \) is \( \omega_2\)-cc.

**Proof.** This follows easily, as our ground model satisfies CH, we are using countable support and each \( Q_\alpha \) has size \( \omega_1 \). \( \Box \)

Lemma 38  Suppose that \( G \) is \( P \)-generic and at limit stage \( \alpha < \omega_2 \) either \( Q_\alpha \) is trivial or \( n \) does not belong to the real \( x \star y \) considered at stage \( \alpha \). Then \( S_{\alpha+n} \) is stationary in \( L[G] \). In particular, \( \omega_1 \) is preserved.

**Proof.** Let \( p \) be a condition in \( P \) forcing that \( n \) does not belong to the real \( x \star y \) considered at stage \( \alpha \) of the iteration and forcing that \( \dot{C} \) is a \( P \)-name for a club in \( \omega_1 \). We want to find \( q \leq p \) and \( i \) in \( S_{\alpha+n} \) such that \( q \) forces \( i \) to belong to \( \dot{C} \).

Let \((M_i \mid i < \omega_1)\) be a continuous chain of countable elementary submodels of some large \( L_\theta \) such that \( M_0 \) contains \( p, \alpha, F \) and \( \dot{C} \). For each \( i < \omega_1 \) let \( \gamma_i \) denote \( M_i \cap \omega_1 \). Then \( S^0_{\alpha+n} = \{ i < \omega_1 \mid i = \gamma_i \text{ belongs to } S_{\alpha+n} \} \) is stationary.

**Claim.** There exists \( i \) in \( S^0_{\alpha+n} \) such that \( i \) does not belong to \( S_\beta \) for any \( \beta \) in \( M_i \) which differs from \( \alpha + n \).

**Proof of Claim.** Otherwise for each limit \( i \) in \( S^0_{\alpha+n} \) choose \( f(i) < i \) such that \( i \) belongs to \( S_\beta \) for some \( \beta \) in \( M_{f(i)} \) which differs from \( \alpha + n \). By Fodor, \( f \) has some constant value \( i_0 \) on a stationary subset of \( S^0_{\alpha+n} \). As \( M_{i_0} \) is countable, there is a fixed \( \beta \) in \( M_{i_0} \) different from \( \alpha + n \) such that \( i \) belongs to \( S_\beta \) for stationary-many \( i \) in \( S^0_{\alpha+n} \). But this contradicts the fact that \( S_{\alpha+n} \) and \( S_\beta \) are almost disjoint. \( \Box \) (Claim)

Choose \( i \) as in the Claim. We want to build an \( \omega \)-sequence \( p = p_0 \geq p_1 \geq \cdots \) with a lower bound \( q \) forcing \( i \) to belong to \( \dot{C} \). Let \( i_0 < i_1 < \cdots \) be an \( \omega \)-sequence cofinal in \( i \). To define \( p_{n+1} \), choose a finite subset \( F_n \) of the support of \( p_n \) and extend \( p_n \) inside the model \( M_i \) without thinning the \( n \)-th splitting level of \( p_n(\beta) \) for non-limit \( \beta \in F_n \) so that \( p_{n+1} \) forces some ordinal greater than \( i_n \) to belong to \( \dot{C} \). This can be done by successively considering the \( (2^n)^{|F_n|} \) different choices of nodes on the \( n \)-th splitting levels of the trees specified by \( p_n \) on the non-limit components in \( F_n \). In addition, for limit \( \beta \) in \( F_n \), extend \( p_n(\beta) \) to ensure that the max of this closed set is at least \( i_n \). The
$F_n$’s should be chosen so that their union equals the union of the supports of the $p_n$’s.

Then the sequence of $p_n$’s has a lower bound $q$. For non-limit $\alpha$ in the union $A$ of the supports of the $p_n$’s the $p_n(\alpha)$’s form a fusion sequence, so we obtain a Sacks condition when we intersect the $p_n(\alpha)$’s. As $A$ is a subset of the model $M_i$, we know by the choice of $i$ that $i$ does not belong to $S_\beta$ for any $\beta$ in $A$ which differs from $\alpha + n$. Therefore for limit $\beta$ in $A$ different from $\alpha + n$ we get a condition if we take the union of the $p_n(\beta)$’s (which has supremum $i$) and add $i$ at the top. At component $\alpha + n$ we can also put $i$ at the top as $p = p_0$ forces that $n$ does not belong to the real $x * y$ considered at stage $\alpha$ of the iteration.

Finally, note that $q$ forces $i$ to belong to $\mathcal{C}$ and therefore we have proved the stationarity of $S_{\alpha+n}$. □ (Claim)

**Corollary 39** $P$ forces the negation of CH.

Clearly if $Q_\alpha$ is nontrivial at a limit stage $\alpha$ and $n$ does belong to the real $x * y$ considered at stage $\alpha$ then $S_{\alpha+n}$ is not stationary in $L[G]$. Thus if $<$ denotes the wellorder of the reals in $L[G]$ obtained by taking the union of the $<_\alpha$’s we have:

\[ (*) \quad x < y \text{ iff for some limit } \alpha < \omega_2, \ S_{\alpha+n} \text{ is stationary iff } n \text{ belongs to } x * y. \]

As the sequence $(S_\alpha \mid \alpha < \omega_2)$ is definable over $L_{\omega_2}$, this gives a wellorder in $L[G]$ which is definable over $L_{\omega_2}[G] = H(\omega_2)^{V[G]}$. □

Now we prove the more difficult result:

**Theorem 40** It is consistent with the negation of CH that there is a projective (indeed $\Sigma^1_3$ definable) wellorder of the reals.

*Proof.* We perform an $\omega_2$-iteration as in the previous proof, but do more at limit stages. Recall that in the previous proof we started with $L$ and added a wellorder $< \omega_2$-many reals such that:

\[ x < y \text{ iff for some limit } \alpha < \omega_2, \ n \text{ belongs to } x * y \text{ iff } S_{\alpha+n} \text{ is nonstationary,} \]

where $(S_\beta \mid \beta < \omega_2)$ is an $L_{\omega_2}$-definable sequence of pairwise almost disjoint stationary subsets of $\omega_1$. In the present proof this will be modified slightly:
(1) $x < y$ iff for some limit $\alpha < \omega_2$, $S_{\alpha+2n}$ is nonstationary for $n$ in $x \ast y$ and $S_{\alpha+2n+1}$ is nonstationary for $n$ not in $x \ast y$.

This small change has the advantage that not only membership, but also non-membership in $x \ast y$ is witnessed by the existence, rather than the non-existence, of a club.

Our goal is to express the above nonstationarity in terms of quantification over countable models. Ideally, we would like to have (1) together with the following:

(2) If $x < y$ then there exists a real $R$ such that for any countable transitive ZF$^-$ model $M$ containing $R$ there is a limit ordinal $\alpha < \omega_2^M$ such that $S_{\alpha+2n}^M$ is nonstationary in $M$ for $n$ in $x \ast y$ and $S_{\alpha+2n+1}^M$ is nonstationary in $M$ for $n$ not in $x \ast y$,

where $(S_{\beta}^M \mid \beta < \omega_2^M)$ denotes $M$'s interpretation of the sequence $(S_{\beta} \mid \beta < \omega_2)$. We show now that (1) implies the converse of (2). It follows that (1) and (2) together give a projective wellorder of the reals, as the conclusion of (2) is first-order over $H(\omega_1)$.

Suppose that $R$ is a real such that for any countable transitive ZF$^-$ model $M$ containing $R$ there is a limit ordinal $\alpha < \omega_2^M$ such that $S_{\alpha+2n}^M$ is nonstationary in $M$ for $n$ in $x \ast y$ and $S_{\alpha+2n+1}^M$ is nonstationary in $M$ for $n$ not in $x \ast y$. By Löwenheim-Skolem this holds for arbitrary transitive ZF$^-$ models $M$ containing $R$. Consider then the model $M = L_\theta[R]$ for a large regular $\theta$ and let $\alpha < \omega_2^M = \omega_2$ be the limit ordinal guaranteed the conclusion of (2) for $M$. As $(S_{\beta}^M \mid \beta < \omega_2)$ is definable over $L_{\omega_2}$ and $\theta$ is greater than $\omega_2$, it follows that $S_{\beta}^M$ equals $S_{\beta}$ for each $\beta < \omega_2$. Thus $S_{\alpha+2n}$ is nonstationary in $M$ for $n$ in $x \ast y$ and $S_{\alpha+2n+1}$ is nonstationary in $M$ for $n$ not in $x \ast y$. It follows that these sets are nonstationary in the larger model $L[G]$ and therefore by (1), we have $x < y$.

We will not actually achieve (2) above, but a slight weakening of it. Say that a transitive ZF$^-$ model $M$ is suitable iff $M \models \omega_2 = \omega_2^L$ exists. We will obtain (2) restricted to suitable $M$. Then to establish the converse of the new version of (2), we need only observe that as our forcing preserves cardinals, $L_\theta[R]$ is indeed suitable for any large regular $\theta$ and any real $R$ in the generic extension.
We now begin the proof. To facilitate the argument we need some extra properties of the bookkeeping function $F$ and of the sequence $(S_\beta \mid \beta < \omega_2)$ of almost disjoint stationary subsets of $\omega_1$.

**Lemma 41** Assume $V = L$. There is a bookkeeping function $F : \omega_2 \to L_{\omega_2}$ definable over $L_{\omega_2}$ via a formula $\varphi$ and a sequence $(S_\beta \mid \beta < \omega_2)$ of almost disjoint stationary subsets of $\omega_1$ definable over $L_{\omega_2}$ via a formula $\psi$ such that whenever $M, N$ are suitable transitive ZF models, $F^M, F^N$ denote the interpretations of $\varphi$ in $M, N$, respectively, $\vec{S}^M = (S^M_\beta \mid \beta < \omega^M_2)$, $\vec{S}^N = (S^N_\beta \mid \beta < \omega^N_2)$ denote the interpretations of $\psi$ in $M, N$, respectively, and $\omega^M_1 = \omega^N_1$, then $F^M, F^N$ agree on $\omega^M_2 \cap \omega^N_2$ and $\vec{S}^M, \vec{S}^N$ agree on $\omega^M_2 \cap \omega^N_2$. In particular, if $M$ is suitable and $\omega^M_1 = \omega_1$ then $F^M, \vec{S}^M$ equal the restrictions of $F, \vec{S}$ to the $\omega_2$ of $M$.

*Proof Sketch.* For the bookkeeping function define $F(\alpha) = a$ iff via Gödel pairing $\alpha$ codes a pair $(\alpha_0, \alpha_1)$ where $a$ has rank $\alpha_0$ in the natural wellorder of the sets in $L$. For the almost disjoint stationary sets, let $(D_\gamma \mid \gamma < \omega_1)$ be the canonical $L_{\omega_1}$-definable $\diamond$ sequence, for each $\alpha < \omega_2$ let $A_\alpha$ be the $L$-least subset of $\omega_1$ coding $\alpha$ and define $S_\alpha$ to be the set of $i < \omega_1$ such that $D_i = A_\alpha \cap i$. □ (Lemma 41)

**14.-15. Vorlesungen**

Now we describe stage $\alpha$ of our iteration. For non-limit $\alpha < \omega_2$ we add a Sacks real. For limit $\alpha < \omega_2$, we kill the stationarity of $S_{\alpha+2n}$ for $n$ in $x_\alpha * y_\alpha$ and of $S_{\alpha+2n+1}$ for $n$ not in $x_\alpha * y_\alpha$, where $x_\alpha <_\alpha y_\alpha$ are the reals chosen by the bookkeeping function $F$ at that stage. Call this forcing $Q^0_\alpha$ and let $H_\alpha$ denote the $Q^0_\alpha$-generic. Now let $X_\alpha \in L[G_\alpha \ast H_\alpha]$ be a subset of $\omega_1$ which codes the ordinal $\alpha$, codes a level of $L$ in which $\alpha$ has size at most $\omega_1$ and codes the generic $G_\alpha \ast H_\alpha$, which we can regard as an element of $L_{\omega_2}$. We have:

(*) If $M$ is suitable and $X_\alpha$ belongs to $M$, then the limit ordinal $\alpha$ coded by $X_\alpha$ is less than $\omega^M_2$ and $S^M_{\alpha+2n}$ is not stationary in $M$ for $n$ in $x_\alpha * y_\alpha$, $S^M_{\alpha+2n+1}$ is not stationary in $M$ for $n$ not in $x_\alpha * y_\alpha$.

This is because in any such $M$ we can decode $G_\alpha \ast H_\alpha$ from $X_\alpha$ inside $M$ and $S^M_{\alpha+n}$ equals $S_{\alpha+n}$ for each $n$.  

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Recall that we want to add a real which “reflects” this property into all countable, suitable models that contain it. First we force a subset $Y_\alpha$ of $\omega_1$ which “localises” the above property in the following sense:

\[ (** \) \text{ For any } \gamma < \omega_1 \text{ and countable, suitable } M \text{ containing } Y_\alpha \cap \gamma \text{ as an element: If } \gamma = \omega_1^M \text{ then for some limit ordinal } \bar{\alpha} \text{ less than } \omega_2^M, \bar{S}_{\bar{\alpha}+2n}^M \text{ is not stationary in } M \text{ for } n \text{ in } x_\alpha \ast y_\alpha \text{ and } S_{\bar{\alpha}+2n+1}^M \text{ is not stationary in } M \text{ for } n \text{ not in } x_\alpha \ast y_\alpha. \]

We now describe a forcing $Q_1^1$ to create the witness $Y_\alpha$ to (**). A condition in $Q_1^1$ is an $\omega_1$-Cohen condition $r : |r| \rightarrow 2$ in $L[G_\alpha \ast H_\alpha]$ with the following properties:

1. The domain $|r|$ of $r$ is a countable limit ordinal.
2. $X_\alpha \cap |r|$ is the even part of $r$, i.e., for $\gamma < |r|$, $\gamma$ belongs to $X_\alpha$ iff $r(2\gamma) = 1$.
3. (**) holds for all limit $\gamma \leq |r|$ with $Y_\alpha \cap \gamma$ replaced by $r \upharpoonright \gamma$, i.e.:

\[ (** \) \text{ For any limit } \gamma \leq |r| \text{ and countable, suitable } M \text{ containing } r \upharpoonright \gamma \text{ as an element: If } \gamma = \omega_1^M \text{ then for some limit ordinal } \bar{\alpha} \text{ less than } \omega_2^M, \bar{S}_{\bar{\alpha}+2n}^M \text{ is not stationary in } M \text{ for } n \text{ in } x_\alpha \ast y_\alpha \text{ and } S_{\bar{\alpha}+2n+1}^M \text{ is not stationary in } M \text{ for } n \text{ not in } x_\alpha \ast y_\alpha. \]

As a warmup for a later argument, we pause now to consider the case $\alpha = \omega$, assume that $x_\omega <_\omega y_\omega$ are well-defined and show that the forcing $P_\omega \ast Q_0^0 \ast Q_1^1$ preserves the stationarity of the “untouched” $S_\beta$’s, i.e., of those $S_\beta$’s where $\beta$ is not of the form $\omega+2n$, $n \in x_\omega \ast y_\omega$ or of the form $\omega+2n+1$, $n \notin x_\omega \ast y_\omega$. Later we will show that the entire iteration preserves the stationarity of those $S_\beta$’s untouched by the generic for the full $\omega_2$-iteration $P$.

Suppose that $(p,q^0,r)$ is a condition in $P_\omega \ast Q_0^0 \ast Q_1^1$ forcing that $\beta$ is not of the form $\omega+2n$, $n \in x_\omega \ast y_\omega$, $\beta$ is not of the form $\omega+2n+1$, $n \notin x_\omega \ast y_\omega$ and that $\mathcal{C}$ is a club in $\omega_1$. We will find $(p_\omega,q^0_\omega,r_\omega)$ below $(p,q^0,r)$ forcing $i$ to belong to $\mathcal{C}$ for some $i$ in $S_\beta$.

First note that $Q_1^1$ satisfies the following extendibility property: Given $r$ and a countable limit $\gamma$ greater than $|r|$, we can extend $r$ to $r^*$ of length $\gamma$. This is because we can take the odd part of $r^*$ on the interval $[|r|, |r| + \omega)$ to code $\gamma$ and to consist only of 0’s on $[|r| + \omega, \gamma)$; then there are no new instances of requirement 3 for being a condition to check because no ZF$^-$ model containing $r^* \upharpoonright |r| + \omega$ can have its $\omega_1$ in the interval $(|r|, \gamma]$.  

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Now let $\langle M_i \mid i < \omega_1 \rangle$ be a continuous chain of countable elementary submodels of some large $L_\theta$ such that $M_0$ contains the parameters $(p, q^0, r)$, $\beta, \dot{C}, P_\omega * Q^0_\omega * Q^1_\omega$, and a $P_\omega * Q^0_\omega$-name $\dot{X}_\omega$ for $X_\omega$. Let $i$ be an element of $S_\beta$ such that $i = M_i \cap \omega_1$ and $i$ does not belong to $S_\delta$ for any $\delta$ in $M_i$ which differs from $\beta$. (We argued earlier that there must be such an $i$, using a Fodor argument.) Successively extend $(p, q^0, r)$ to $(p_0, q^0_0, r_0) \geq (p_1, q^0_1, r_1) \geq \cdots$ in $M_i$ so that for each finite $n$ the $p_k(n)$, $k \in \omega$, form a fusion sequence and if $D$ in $M_i$ is a dense set for the forcing $P_\omega * Q^0_\omega * Q^1_\omega$ then for some $k$, $(p_k, q^0_k, r_k)$ reduces $D$ to the $k$-th splitting level of finitely many of the trees $p_k(n)$ (i.e., if finitely many of the trees $p_k(n)$ are restricted to some node on their $k$-th splitting level, then the resulting condition $(p_k', q^0_k, r_k)$ meets $D$). In particular, the condition $(p_k, q^0_k, r_k)$ forces the $P_\omega * Q^0_\omega * Q^1_\omega$-generic to meet $D$ in a condition belonging to $M_i$. By extendibility, the max’s of the $q^0_k$’s and the domains of the $r_k$’s converge to $i$. And the $(p_k, q^0_k, r_k)$’s force arbitrary large ordinals less than $i$ into $C$.

We want to show that the $(p_k, q^0_k, r_k)$’s have a lower bound $(p_\omega, q^0_\omega, r_\omega)$. By fusion the $p_k$’s have a greatest lower bound $p_\omega$. And just as in our earlier argument, the $q^0_k$’s have a greatest lower bound $q^0_\omega$ as $i$ does not belong to $S_\delta$ for any $\delta$ in $M_i$ which differs from $\beta$. We show that the condition $(p_\omega, q^0_\omega)$ in $P_\omega * Q^0_\omega$ forces the union $r_\omega$ of the $r_k$’s to be a condition in $Q^1_\omega$. For this it suffices to force property $(**)$ when $\gamma$ is equal to $i$, the length of $r_\omega$. I.e., $(p_\omega, q^0_\omega)$ must force:

\[(**)\] For any suitable $M$ containing $r_\omega$: If $i = \omega_1^M$ then $S_{\omega+2n}^M$ is not stationary in $M$ for $n$ in $x_\omega * y_\omega$ and $S_{\omega+2n+1}^M$ is not stationary in $M$ for $n$ not in $x_\alpha * y_\alpha$.

Fix a generic $G_\omega * H_\omega$ below the condition $(p_\omega, q^0_\omega)$. Then if $D$ is a dense set for $P_\omega * Q^0_\omega$ belonging to $M_i$, by construction we have that $(G_\omega * H_\omega) \cap M_i$ meets $D$. Thus not only is $M_i$ elementary in $L_\theta$, but also $(M_i[(G_\omega * H_\omega) \cap M_i], (G_\omega * H_\omega) \cap M_i)$ is elementary in $(L_\theta[(G_\omega * H_\omega) \cap M_i], (G_\omega * H_\omega) \cap M_i)$. Let $(M[G \ast H], G \ast H)$ be the transitive collapse of $(M_i[(G_\omega * H_\omega) \cap M_i], (G_\omega * H_\omega) \cap M_i)$. As $X_\omega$ has a name in $M_i$, it follows that $X_\omega$ belongs to $M_i[(G_\omega * H_\omega) \cap M_i]$ and therefore $X_\omega \cap i$ belongs to $M[G \ast H]$. As $X_\omega$ codes the generic $G_\omega * H_\omega$, it ensures the nonstationarity of $S_{\omega+2n}^M$ for $n$ in $x_\omega * y_\omega$ and of $S_{\omega+2n+1}^M$ for $n$ not in $x_\omega * y_\omega$ in all suitable models containing $X_\omega$ as an element; it follows that $X_\omega \cap i$ ensures the nonstationarity of $S_{\omega+2n}^M$ for $n$ in $x_\omega * y_\omega$ and of $S_{\omega+2n+1}^M$ for $n$
not in $x_\omega \ast y_\omega$ in all suitable models containing $X_\omega \cap i$ as an element. Now if $M$ is any suitable model containing $r_\omega$ as an element such that $\omega_1^M = i$, $M$ also contains $X_\omega \cap i$ as an element (as $X_\omega \cap i$ is the even part of $r_\omega$) and as $\omega_1^M = i = \omega_1^\mathbb{L}$, we have $S_{\omega+n}^M = S_{\omega+n}^\mathbb{L}$ for each $n$; it follows that $S_{\omega+2n}^M$ is nonstationary in $M$ for $n$ in $x_\omega \ast y_\omega$ and $S_{\omega+2n+1}^M$ is nonstationary in $M$ for $n$ not in $x_\omega \ast y_\omega$, establishing (**)).

So the $(p_k, q_k^i, r_k)$’s have a lower bound $(p_\omega, q_\omega^0, r_\omega)$. This condition forces unboundedly many ordinals less than $i$ into $\mathbb{C}$ and therefore forces $i$ into $\mathbb{C}$, where $i$ belongs to $S_\beta$. Thus we have shown that the stationarity of $S_\beta$ is preserved by the forcing $P_\omega \ast Q_\omega^0 \ast Q_\omega^1$.

16.-17. Vorlesungen

To complete stage $\alpha$ of the iteration, we code the $Q_\alpha^1$-generic $Y_\alpha$ by a real via the forcing $C_\alpha$ defined below. This can most easily be done using a ccc almost disjoint coding with finite conditions; but for the sake of future applications we use here perfect trees to code. Note that the ground model $L[G_\alpha \ast H_\alpha \ast Y_\alpha]$ is in fact equal to $L[Y_\alpha]$ as the even part of $Y_\alpha$ codes $G_\alpha \ast H_\alpha$.

Inductively define $L$-countable ordinals $\mu$, $i < \omega_1^L$ by: $\mu$ is the least $\mu > \bigcup \{ \mu_j \mid j < i \}$ (this condition is vacuous if $i$ equals 0) such that $L_\mu[Y_\alpha \cap i] \models ZF^-$ and $L_\mu \models \omega$ is the largest cardinal. (There are many $\mu$’s with these properties, for example any $\mu$ such that $L_\mu[Y_\alpha \cap i]$ is an elementary submodel of $L_\omega[Y_\alpha \cap i]$). A real $R$ codes $Y_\alpha$ below $i$ iff for all $j < i$, $j \in Y_\alpha$ iff $L_{\mu_j}[Y_\alpha \cap j, R] \models ZF^-$. For $T \subseteq 2^{<\omega}$ a perfect tree, let $[T]$ denote the least $i$ such that $T \in L_{\mu_i}[Y_\alpha \cap i]$. A condition in $C_\alpha$ is a perfect tree $T$ such that $R$ codes $Y_\alpha$ below $[T]$ whenever $R$ is a branch through $T$. (Note that by absoluteness, if $T$ is a condition then $R$ codes $Y_\alpha$ below $[T]$ even for branches $R$ through $T$ in the generic extension; in particular this holds for the generic branch.) $C_\alpha$ is ordered by: $T_0 \leq T_1$ iff $T_0$ is a subtree of $T_1$. This is equivalent to $[T_0] \subseteq [T_1]$ where $[T]$ denotes the set of infinite branches through $T$.

**Lemma 42** (a) If $T$ belongs to $C_\alpha$ and $|T| \leq i < \omega_1$ then there is a $T^* \leq T$ such that $|T^*| = i$. (b) $C_\alpha$ preserves $\omega_1$.

**Proof.** (a) By induction on $i$. We may assume that $|T|$ is less than $i$. If $i = j + 1$ then we may also assume by induction that $|T|$ equals $j$ and hence that $T$ belongs to $A_j = L_{\mu_j}[Y_\alpha \cap j]$. If $j$ belongs to $Y_\alpha$ then we take $T^* \leq T$ to
have the property that $R$ is $P_T$-generic over $A_j$ for $R \in \{T^*\}$, where $P_T$ is the forcing (isomorphic to Cohen forcing) whose conditions are the elements of $T$, ordered by extension. Note that $T^*$ can be chosen in $A_i = L_{\mu_j}[Y_\alpha \cap i]$ as $A_j$ is a countable element of $A_i$. Also $L_{\mu_j}[Y_\alpha \cap j, R] \models ZF^-$ for $R \in \{T^*\}$, by the $P_T$-genericity of $R \in \{T^*\}$. So $T^*$ is a condition and $|T^*| = i$. If $j$ does not belong to $Y_\alpha$ then choose a real $R_0$ coding a well ordering of $\omega$ of ordertype $\mu_j$, $R_0 \in A_i$, and take $T^* \leq T$ to be the tree whose branches are exactly the branches $R$ through $T$ such that for all $n$, $n \in R_0$ iff $R$ goes right at the $2n$-th splitting level of $T$. Then $T^*$ belongs to $A_i$ and for $R \in \{T^*\}$, $(R, T)$ computes $R_0$ and hence $L_{\mu_j}[Y_\alpha \cap j, R]$ is not a model of $ZF^-$, since it contains $R_0$ as an element.

If $i$ is a limit ordinal then choose $|T| = i_0 < i_1 < \cdots$ to be an $\omega$-sequence cofinal in $i$ which belongs to $A_i = L_{\mu_j}[Y_\alpha \cap i]$. Define $T_0 \leq_n T_1$ iff $T_0 \leq T_1$ and $T_0, T_1$ have the same first $n$ splitting levels. Now let $T_0 = T$ and for each $n$ let $T_{n+1} \in C_\alpha$ be least in $A_{\alpha_{n+1}}$ such that $|T_{n+1}| = i_{n+1}$ and $T_{n+1} \leq_n T_n$. Such $T_n$’s exist by induction. If $T^* = \bigcap_n T_n$ then $T^* \leq T$ belongs to $A_i$ and satisfies the requirement for belonging to $C_\alpha$. So $T^* \leq T$, $|T^*| = i$, as desired.

(b) We say that $D \subseteq C_\alpha$ is $n$-dense iff for all $T \in C_\alpha$ there is $T^* \leq_n T$, $T^* \in D$. We show that if for each $n$, $D_n$ is open and $n$-dense then for all $T \in C_\alpha$ there exists $T^* \leq T$ such that $T^*$ belongs to $D_n$ for each $n$. It follows that $C_\alpha$ preserves “cofinality > $\omega$,” for if $\sigma$ is a name for a function from $\omega$ into $\text{Ord}$ then for each $n$, $D_n = \{T \in C_\alpha \mid \text{For some finite } d, T \models \sigma(n) \in d\}$ is $n$-dense and hence our result implies that the range of $\sigma$ is covered by a set countable in the ground model.

So suppose $T$ belongs to $C_\alpha$ and $D_n$ is open and $n$-dense for each $n$. Let $M$ be a countable elementary submodel of some large $L_\eta[Y_\alpha]$ containing $T$ and $\langle D_n \mid n \in \omega \rangle$ as elements and let $i = M \cap \omega_1$. Also let $i_0 < i_1 < \cdots$ be an $\omega$-sequence cofinal in $i$ belonging to $A_i$. Note that the transitive collapse of $M$ belongs to $A_i$ as it satisfies $i = \omega_1$ whereas $L_{\mu_j} \models i$ is countable. So we can choose $T = T_0 \geq T_1 \geq T_2 \geq \cdots$ in $A_i$ so that $T_{n+1} \in D_n \cap M$ and $|T_{n+1}| \geq \alpha_{n+1}$. Then $T^* = \bigcap_n T_n$ belongs to each $D_n$, $T^* \leq T$ and $T^*$ belongs to $C_\alpha$ as $T^*$ belongs to $A_i$. □

This completes the definition for limit $\alpha < \omega_2$ of $Q_\alpha = Q^0_\alpha \ast Q^1_\alpha \ast C_\alpha$. For non-limit $\alpha < \omega_2$, $Q_\alpha$ is Sacks forcing. The desired forcing $P$ is the iteration with countable support of these $Q_\alpha$'s.
Lemma 43

Suppose that $G$ is P-generic. Then for $\beta < \omega^M_2$ not of the form $\alpha+2n$, $\alpha$ limit, $n \in x^G_\alpha \ast y^G_\alpha$ and not of the form $\alpha+2n+1$, $\alpha$ limit, $n \notin x^G_\alpha \ast y^G_\alpha$, $S_\beta$ is stationary in $L[G]$. Moreover $L$ and $L[G]$ have the same cardinals.

Proof. Let $p$ be a condition forcing that $\beta < \omega^M_2$ is not of the form $\alpha+2n$, $\alpha$ limit, $n \in x^G_\alpha \ast y^G_\alpha$ and not of the form $\alpha+2n+1$, $\alpha$ limit, $n \notin x^G_\alpha \ast y^G_\alpha$, and also forcing that $\dot{C}$ is a club in $\omega^L_1$. We want to find an extension $q$ of $p$ and $i < \omega^L_1$ in $S_\beta$ such that $q$ forces $i$ to belong to $\dot{C}$.

As before let $(M_i \mid i < \omega^L_1)$ be a continuous chain of countable elementary submodels of some large $L_\theta$ such that $M_0$ contains all imaginable parameters, and choose $i < \omega^L_1$ in $S_\beta$ so that $i$ does not belong to $S_\delta$ for any $\delta$ in $M_i$ other than $\beta$. Build an $\omega$-sequence $p = p_0 \geq p_1 \geq \cdots$ of conditions below $p$ such that for any dense set $D$ for the forcing $P$ in $M_i$, some $p_k$ forces the generic to intersect $D \cap M_i$. Moreover ensure that for each non-limit $\alpha$ in the union of the supports of the $p_k$'s, the sequence $p_k(\alpha)$ forms a fusion sequence in Sacks forcing and also that for each limit $\alpha$ in the union of the supports of the $p_k$'s,
if we write \( p_k(\alpha) = (p_k(\alpha)^0, p_k(\alpha)^1, p_k(\alpha)^2) \), then the sequence of \( p_k(\alpha)^2 \)'s is forced to form a fusion sequence in the coding forcing \( C_\alpha \). In addition, choose the sequence of \( p_k \)'s to belong to the least \( L_\mu \) in which \( \bar{M} \), the transitive collapse of \( M_i \), is countable.

We now produce a lower bound \( q \) to the sequence of \( p_k \)'s, whose support \( \text{Supp}(q) \) is the union of the supports of the \( p_k \)'s, by defining \( q(\alpha) \) by induction on \( \alpha \) in \( \text{Supp}(q) \). If \( \alpha \) is a non-limit then we take \( q(\alpha) \) to simply be the fusion of the \( p_k(\alpha) \)'s. Suppose then that \( \alpha \) is a limit and \( q \upharpoonright \alpha \) is already defined as a condition in \( P_\alpha \). We want to define \( q(\alpha) = (q(\alpha)^0, q(\alpha)^1, q(\alpha)^2) \).

For \( q(\alpha)^0 \), a name for a sequence of closed sets, we take the union of the closed sets in the \( p_k(\alpha)^0 \)'s and put \( i \) at the top. This results in a condition because \( i \) is forced to not belong to any of the \( S_{\alpha+2n} \), \( n \in x_\alpha \cap y_\alpha \) or the \( S_{\alpha+2n+1} \), \( n \notin x_\alpha \cap y_\alpha \) (because such \( \alpha + 2n \), \( \alpha + 2n + 1 \) belong to \( M_i \) or equal \( \beta \)) and therefore a condition will indeed result if \( i \) is added at the top. Also note that the closed sets in the \( p_k(\alpha)^0 \)'s have maxima cofinal in \( i \) by the construction of the \( p_k \)'s, so we indeed obtain closed sets when putting \( i \) at the top.

For \( q(\alpha)^1 \) we use the same argument that we used earlier for \( Q_\alpha^1 \). We take \( q(\alpha)^1 \) to be the union of the \( p_k(\alpha)^1 \)'s. Fix a generic \( G_\alpha \ast H_\alpha \) below \( (q \upharpoonright \alpha, q(\alpha)^0) \); we must show that when \( q(\alpha)^1 \) is interpreted by \( G_\alpha \ast H_\alpha \) the result is a condition in \( Q_\alpha^1 \) as interpreted by \( G_\alpha \ast H_\alpha \). By the construction of the \( p_k \)'s, \( M_i \) is not only elementary in \( L_\theta \) but this remains so if we introduce \( G_\alpha \ast H_\alpha \) as a predicate, i.e., \( (M_i[(G_\alpha \ast H_\alpha) \cap M_i], (G_\alpha \ast H_\alpha) \cap M_i) \) is elementary in \( (L_\theta[G_\alpha \ast H_\alpha], G_\alpha \ast H_\alpha) \). As \( X_\alpha \subseteq \omega_1 \) codes the generic \( G_\alpha \ast H_\alpha \) and has a name in \( M_i \), it follows that \( X_\alpha \cap i \) belongs to the transitive collapse \( M[G \ast H] \) of \( M_i[(G_\alpha \ast H_\alpha) \cap M_i] \). Moreover, just as \( X_\alpha \) ensures the nonstationarity of the appropriate \( S_{\alpha+n} \)'s, \( X_\alpha \cap i \) ensures the nonstationarity of the appropriate \( S_{\alpha+n}^M \)'s in any suitable \( M \) containing \( X_\alpha \cap i \) such that \( \omega_1^M = i \). This implies that \( q(\alpha)^1 \), which has \( X_\alpha \cap i \) as its even part, ensures the same nonstationarity and therefore is a condition in \( Q_\alpha^1 \).

Finally, we take \( q(\alpha)^2 \) to be the fusion of the \( p_k(\alpha)^2 \)'s. To verify that this is a condition in \( C_\alpha \) we need to verify that it is forced to belong to the structure \( \mathcal{A}_i = L_{\mu_i}[Y_\alpha \cap i] \). Recall that the sequence of \( p_k \)'s belongs to the least \( L_\mu \) in which \( \bar{M} \), the transitive collapse of \( M_i \), is countable. It follows
that \( q(\alpha)^2 \) is forced to belong to \( L_{\mu}[Y_\alpha \cap i] \) for this \( \mu \) and by the definition of \( \mu_i \), we have \( \mu < \mu_i \). Thus \( q(\alpha)^2 \) is indeed forced to belong to \( \mathcal{A}_i \), as desired.

The fact that \( L \) and \( L[G] \) have the same cardinals now follows from \( \omega_1 \)-preservation and the \( \omega_2 \)-cc. \( \square \)

18.-20.Vorlesungen

Our next goal is to prove the following.

**Theorem 44** Relative to the consistency of a reflecting cardinal, BPFA is consistent with the existence of a \( \Sigma^1_3 \) wellorder of the reals.

BPFA is the bounded forcing axiom for proper forcings. It is equivalent to the statement that any \( \Sigma_1 \) sentence with an element of \( H(\omega_2) \) as parameter which is true in a proper forcing extension of the universe is already true. A cardinal \( \kappa \) is reflecting iff it is regular and \( H(\kappa) \) is \( \Sigma_2 \) elementary in \( V \).

Goldstern and Shelah showed that BPFA is consistent relative to a reflecting cardinal by starting with a reflecting cardinal in \( L \) and performing a countable support \( \kappa \)-iteration of proper forcings of size \( < \kappa \). At each stage a proper forcing is chosen to witness a new \( \Sigma_1 \) fact with parameter in (the current) \( H(\omega_2) \). The fact that \( \kappa \) is reflecting is used to show that these proper forcings can in fact be taken to have size \( < \kappa \) and therefore \( \kappa \) will remain reflecting throughout the iteration (until the final stage). As the forcing is proper and \( \kappa \)-cc, it follows that \( \omega_1 \) is preserved and that BPFA holds in the resulting forcing extension.

We first show:

**Theorem 45** Relative to the consistency of a reflecting cardinal, BPFA is consistent with the existence of a wellorder of the reals which is definable over \( H(\omega_2) \).

To prove Theorem 45 we start in the same way as Goldstern-Shelah, with a reflecting cardinal \( \kappa \) in \( L \), and perform a countable support iteration of length \( \kappa \). A possible strategy is to code a wellorder of the reals using stationary subsets of \( \omega_1 \), as in our previous proof. However this will destroy the properness of the iteration, so we take another approach, based on controlling which of certain constructible trees \( T \) have \( T \)-generic branches over \( L \) in the final model.
Lemma 46 Assume $V = L$. Suppose that $\beta$ is regular and uncountable and consider the tree $T(\beta)$ of sequences through $\beta^+$ of length less than $\beta$. Suppose that $Q$ is a forcing such that $2^{2^{Q_1}}$ is less than $\beta$ and $G$ is $Q$-generic over $L$. Then:

(a) $T(\beta)$, viewed as a forcing, is proper in $L[G]$.
(b) There is a proper forcing $R$ in $L[G]$ of size $\beta^+$ which destroys the properness of $T(\beta)$; in fact, if $H$ is $R$-generic over $L[G]$ then in any $\omega_1$-preserving outer model of $L[G][H]$ there is no branch through $T(\beta)$ which is $T(\beta)$-generic over $L$.

Proof. (a) It suffices to show that $Q$ is proper in $T(\beta)$-generic extensions of $L$. But the forcing $T(\beta)$ is $\beta$-closed and therefore does not add subsets of $2^{2^{Q_1}}$; it follows that any witness to the properness of $Q$ in $L$ is still a witness to its properness in any $T(\beta)$-generic extension of $L$.

(b) First add $\beta^{++}$ Cohen reals with finite support product over $L[G]$, producing $L[G][H_0]$. Then Lévy collapse $\beta^+$ to $\omega_1$ with countable conditions, producing $L[G][H_0][H_1]$. As ccc and $\omega$-closed forcings are proper, this is a proper forcing extension of $L[G]$. Now note that in $L[G][H_0][H_1]$, any $\beta$-branch through $T(\beta)$ in fact belongs to $L[G][H_0]$: Otherwise we choose a $L[G][H_0]$-name $\dot{b}$ for the new branch and build a binary $\omega$-tree $U$ of conditions in the Lévy collapse, each branch of which has a lower bound, such that distinct cofinal branches through $U$ force different interpretations of the name $\dot{b}$. It follows that in $L[G][H_0]$, $T(\beta)$ has $2^{\aleph_0} = \beta^+$ nodes on a fixed level, which is impossible because GCH holds in $L$. Thus the tree $T(\beta)$ has at most $\omega_1$-many branches in $L[G][H_0][H_1]$, none of which contains ordinals cofinal in $\beta^+$ and therefore none of which is $T(\beta)$-generic over $L$. Also, every node of $T(\beta)$ belongs to a $\beta$-branch.

Now we use Baumgartner’s general method of “specialising a tree off a small set of branches”.

Fact. If $T$ is a tree of height $\omega_1$ with at most $\aleph_1$ cofinal branches (and every node of $T$ belongs to a cofinal branch of $T$) then there is a ccc forcing $P$ such that if $G$ is $P$-generic over $V$ then in any $\omega_1$-preserving outer model of $V[G]$, all cofinal branches through $T$ belong to $V$.

Proof sketch. List the branches as $(b_i \mid i < \omega_1)$ and write $T$ as the disjoint union of $b_i(x_i)$, where each $x_i$ is a node on $b_i$ and $b_i(x_i)$ denotes the tail of $b_i$. 

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starting at \( x_i \). Now add a function \( f \) with finite conditions from \( \{ x_i \mid i < \omega_1 \} \) into \( \omega \) such that if \( x_i \) is below \( x_j \) in \( T \) then \( f(x_i) \) is different from \( f(x_j) \). Baumgartner shows that this forcing is ccc. Now if \( b \) is a cofinal branch through \( T \) distinct from the \( b_i \)'s in an outer model of \( V[f] \), then \( b \) must intersect uncountably many of the \( b_i(x_i) \)'s and therefore contains uncountably many \( x_i \)'s. But then the \( f(x_i) \)'s are distinct for these uncountably many \( x_i \)'s, contradicting the fact that \( f \) maps into \( \omega \). □ (Fact)

Now use the Fact to create a ccc extension \( L[G][H_0][H_1][H_2] \) of \( L[G][H_0][H_1] \) to ensure that \( T(\beta) \) (viewed as a tree of height \( \omega_1 \) using a cofinal \( \omega_1 \)-sequence through \((\beta^+)\uparrow)\) will have no new branches in any \( \omega_1 \)-preserving outer model. As no \( \beta \)-branch through \( T(\beta) \) in \( L[G][H_0] \) is \( T(\beta) \)-generic over \( L \) and all cofinal branches through \( T(\beta) \) in an \( \omega_1 \)-preserving outer model of \( L[G][H_0][H_1][H_2] = L[G][H] \) belong to \( L[G][H_0] \), we are done. □

Proof of Theorem 45. Let \( \kappa \) be reflecting in \( L \) and let \( C \) enumerate the closed unbounded subset of \( \kappa \) consisting of those \( \alpha \) such that \( L_\alpha \) is \( \Sigma_2 \) elementary in \( L_\kappa \). (As \( \kappa \) is inaccessible, \( C \) is indeed unbounded in \( \kappa \).) We perform a proper iteration of length \( \kappa \) with countable support which is nontrivial at stages \( \alpha \) in \( C \). The iteration \( P_\alpha \ast Q(\alpha) \) up to and including stage \( \alpha \) will belong to \( L_\beta \) where \( \beta \) is the least element of \( C \) greater than \( \alpha \). In particular, \( P_\alpha \) has size less than \( \kappa \) for each \( \alpha < \kappa \) and therefore \( \kappa \) remains reflecting throughout the iteration.

Suppose that \( \alpha \) belongs to \( C \); we describe the forcing \( Q(\alpha) \), which is a two-step iteration \( Q^0(\alpha) \ast Q^1(\alpha) \).

As \( P_\alpha \) has size at most \((\alpha^+)\uparrow\), we know that the forcing \( T(\beta) \), consisting of \(< \beta \) sequences through \( \beta^+ \), is proper in \( L[G_\alpha] \) when \( \beta \) is regular and at least \((\alpha^+++\)\). In addition there is a forcing \( R(\beta) \) of size \( \beta^+ \) which guarantees that there is no \( T(\beta) \)-generic over \( L \). Now let \( \alpha_n \) be \( (\alpha^{+4(n+1)})^L \) for each finite \( n \), and let \( T(n) \) denote \( T(\alpha_n) \), \( R(n) \) denote \( R(\alpha_n) \). Then both \( T(n) \) and \( R(n) \) are proper in any extension of \( L[G_\alpha] \) obtained by forcing with \( U(0) \ast U(1) \ast \cdots \ast U(n-1) \) where each \( U(i) \) is either \( T(i) \) or \( R(i) \).

As in the earlier proofs, let \( x_\alpha < y_\alpha \) be a pair of reals in \( L[G_\alpha] \) provided by the bookkeeping function and now take \( Q^0(\alpha) \) to be the \( \omega \)-iteration \( U(0) \ast U(1) \ast \cdots \ast U(n) \), where \( U(n) \) equals \( T(n) \) if \( n \) belongs to \( x_\alpha \ast y_\alpha \) and equals \( R(n) \)
otherwise. This is a proper forcing and \( P_\alpha \ast Q^0(\alpha) \) belongs to \( L_\beta \), where \( \beta \) is the least element of \( C \) greater than \( \alpha \).

Now we choose a \( \Sigma_1 \) sentence with parameter from \( L[G_\alpha] \), provided by the bookkeeping function, and ask if it holds in a proper forcing extension of \( L[G_\alpha][H^0] \), where \( H^0 \) is our \( Q^0(\alpha) \)-generic. If so, then as \( \kappa \) is reflecting in \( L[G_\alpha][H^0] \), there is such a proper forcing in \( L_\kappa[G_\alpha][H^0] \), and also the witness to the \( \Sigma_1 \) sentence can be assumed to have a name in \( L_\kappa[G_\alpha][H^0] \). Let \( \beta \) be the least element of \( C \) greater than \( \alpha \); then \( L_\beta \) is \( \leq \Sigma_2 \) elementary in \( L_\kappa \) and therefore \( L_\beta[G_\alpha][H^0] \) is \( \leq \Sigma_2 \) elementary in \( L_\kappa[G_\alpha][H^0] \). It follows that we can choose our proper forcing \( Q^1(\alpha) \) witnessing the \( \Sigma_1 \) sentence to be an element of \( L_\beta[G_\alpha][H^0] \), maintaining the requirement that \( P_\alpha \ast Q(\alpha) \) belong to \( L_\beta \). This completes the construction.

The iteration is proper, forces \( \kappa \) to be at most \( \omega_2 \) and is \( \kappa \)-cc. It follows that \( \kappa \) equals \( \omega_2 \) in the generic extension \( L[G] \) and BPFA holds there. The desired wellorder of the reals is defined by:

\[
x < y \text{ iff }
\]
For some \( \alpha \) in \( C \), \( (x, y) = (x^G_\alpha, y^G_\alpha) \) iff

There exists \( \alpha \) in \( C \) such that for all \( n \), \( n \) belongs to \( x \ast y \) iff there is a \( T(\alpha_n) \)-generic over \( L \) in \( L[G] \).

This works because at each stage \( \alpha \) in \( C \) and for each \( n \), we either forced with \( T(\alpha_n) \), thereby producing a \( T(\alpha_n) \)-generic over (more than) \( L \) in \( L[G] \), or we forced with \( R(\alpha_n) \), which guaranteed that there can be no \( T(\alpha_n) \)-generic over \( L \) without collapsing \( \omega_1 \); as \( \omega_1 \) is not collapsed, there is in the latter case no \( T(\alpha_n) \)-generic over \( L \) in \( L[G] \).

Finally, note that as \( C \) is definable over \( L_\kappa \), it follows that the above gives a wellorder definable (indeed \( \Sigma_3 \)) over the \( H(\omega_2) \) of \( L[G] \). \( \Box \)